Diffusion through Networks

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7.1 The Bass Model

- *p* is rate of innovation, *q* is rate of imitation
- *F(t)* is the fraction of agents that have adopted by time *t*

$$F(t) = F(t-1) + p(1 - F(t-1)) + q(1 - F(t-1))F(t-1)$$

$$dF(t) / dt = (p + qF(t))(1 - F(t))$$

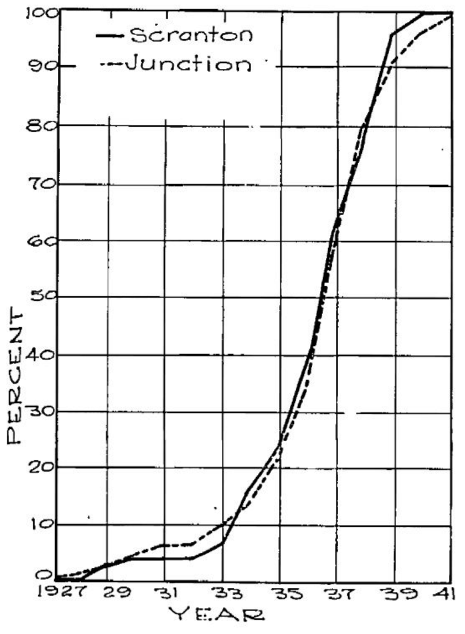


Fig. 1. Cumulative percentages of operators accepting hybrid seed in the two communities during each year of the diffusion process.

7.2.1 Percolation, Component Size, Immunity and Diffusion

- Percolation asks if there is a path across the network
- Immunity corresponds to percolation with a fraction π of nodes removed uniformly at random
- Giant component emerges at the threshold

$$\langle d^2 \rangle_\pi = 2\langle d \rangle_\pi$$

7.2.1 Percolation, Component Size, Immunity and Diffusion

Degree distribution after removing nodes is

$$P_\pi(d) = \sum_{d' \geq d} P(d') \binom{d'}{d} (1-\pi)^d \pi^{d'-d}$$

Giant component emerges when

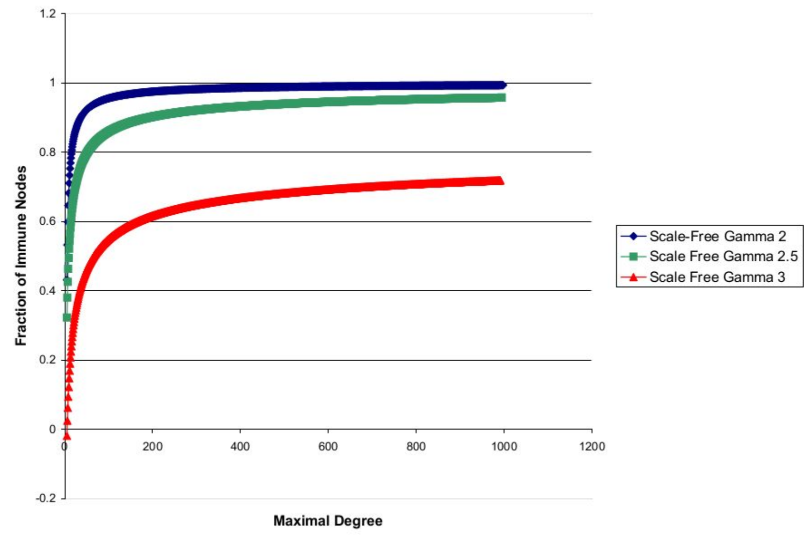
$$\pi = \frac{\langle d^2 \rangle - 2\langle d \rangle}{\langle d^2 \rangle - \langle d \rangle}$$

7.2.1 Percolation, Component Size, Immunity and Diffusion

Regular network of degree $\overline{d}$: $\pi = (\bar{d} - 2)/(\bar{d} - 1)$

Poisson random network: $\pi = 1 - \frac{1}{(n-1)p}$

Scale free network has threshold 0 when γ < 3



7.2.2 Breakdown, Attack and Failure of Networks, and Immunization

- Removing the π nodes with highest degree will remove more than π links
- Proportion of removed links is:

$$f(\pi) = \frac{\sum_{d=\bar{d}(\pi)+1}^{\infty} P(d)d}{\langle d \rangle}$$

Threshold for a giant component to exist becomes:

$$\langle d^2 | d \leq \bar{d}(\pi) \rangle (1 - f(\pi)) = \langle d | d \leq \bar{d}(\pi) \rangle (2 - f(\pi))$$

7.2.2 Breakdown, Attack and Failure of Networks, and Immunization

- In a scale-free distribution with density $(\gamma - 1)d^{-\gamma}$, $\pi = 0.056$
- Uniform immunization leads to threshold of 0
- When $\gamma = 2.5$ immunizing nodes with degrees in highest 5% leads to eliminating ⅓ of links and all nodes with degree 4 or higher

7.2.4 The SIR Model

- Susceptible, Infected, Removed model
    - Infected nodes are eventually removed from the system or become immune (chicken pox)
- Model duration of infection as t, where neighbours have a probability t chance of being infected
- Equivalent to percolation case with $\pi = 1 - t$

7.2.5 The SIS Model

- Susceptible-Infected-Susceptible model
- Match model variant where probability of meeting a node with degree d_i is given by:

$$\frac{P(d)d}{\langle d \rangle}$$

- Average infection rate, ρ, given by:

$$\rho = \sum P(d)\rho(d)$$

7.2.5 The SIS Model

- Chance interaction with infected individual, θ, given by:

$$\theta = \frac{\sum P(d)\rho(d)d}{\langle d \rangle}$$

- Let v be the rate of transmission and δ be the rate of recovery.
- Chance of infection for individual with degree d given by:

$$\nu\theta d$$

Thresholds and Steady-State Infection Rates

- If there is a finite set of agents, the long-run steady-state will approach zero when the infection dies out.
- If there is an infinite set of agents, then v among the unaffected will equal δ among the infected:

$$0 = (1 - \rho(d))\nu\theta d - \rho(d)\delta$$

Thresholds and Steady-State Infection Rates

- Let λ = ν / δ, then solving for ρ(d), we get:

$$\rho(d) = \frac{\lambda\theta d}{\lambda\theta d + 1}$$

- Combining this equation with the equation for ρ, we get:

$$\theta = \sum_d \frac{P(d)\lambda\theta d^2}{\langle d\rangle\,(\lambda\theta d + 1)}$$

Non-Zero Steady State Infection Rates

- Let H(θ) be the number of people infected given that we start at θ.
- H'(θ) describes if an infection can be sustained in the steady state.

$$H(\theta) = \sum \frac{P(d)d}{\langle d \rangle} \left(\frac{\lambda d \theta}{\lambda d \theta + 1} \right)$$

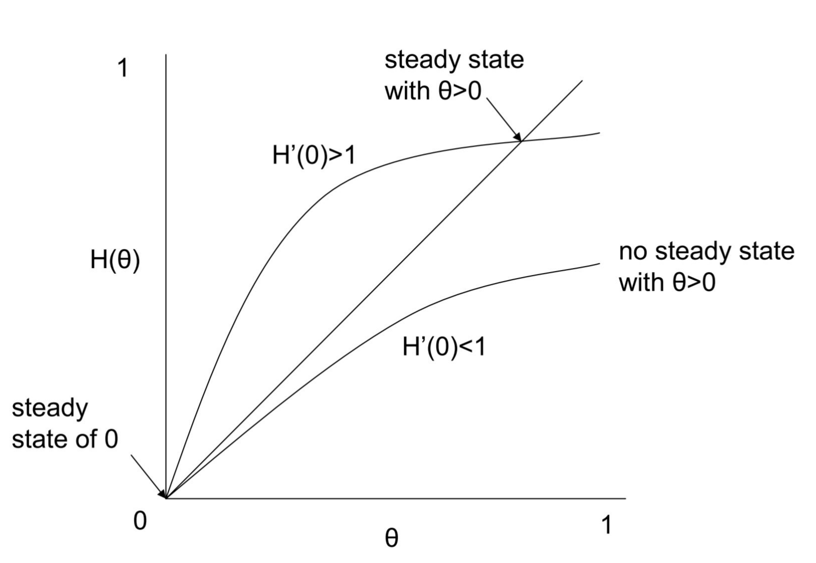
$$H'(\theta) = \sum \frac{P(d)d}{\langle d \rangle} \left(\frac{\lambda d}{(\lambda \theta d + 1)^2} \right)$$

Non-Zero Steady State Infection Rates

- H values for various infections.
- Steady-state at H(0) = 0
- Able to derive equation from H'(0):

$$\lambda > \frac{\langle d \rangle}{\langle d^2 \rangle}$$

- Individuals with high degrees serve as conduits for infection.



Comparisons of Infections Across Network Structure

- How does infection change as network structure is varied?
- First order stochastic domination: One network outperforms another network as its degree distribution is right-shifted.
- Strict mean-preserving spread: Shift some weight to higher degree nodes and some weight to lower degree nodes
- Proposition 7.2.1: Steady-state infection rates depend on network structure differently based on high and low infection rates.

PROPOSITION **7.2.1** *[Jackson and Rogers [336]] Consider two distributions P' and P, with corresponding highest steady-state average neighbor infection rates $\overline{\theta}'$ and $\overline{\theta}$, and largest steady-state overall average infection rates $\overline{\rho}'$ and $\overline{\rho}$; and suppose that $\overline{\theta} > 0$.*

(I) If P' and $\widetilde{P}'$ strictly first order stochastic dominate P and $\widetilde{P}$, respectively, then the infection rates are higher under P' than P (so $\overline{\theta}' > \overline{\theta}$ and $\overline{\rho}' > \overline{\rho}$).

(II) If P' is a strict mean-preserving spread of P, then the average neighbor infection rate increases $\overline{\theta}' > \overline{\theta}$. Moreover, there exist bounds on the relative infection to recovery rate, $\underline{\lambda} \leq \overline{\lambda}$, such that

- *If the infection to recovery rate is below the lower bound, so that $\frac{\nu}{\delta} < \underline{\lambda}$, then the steady-state average infection rate is higher under P', so $\overline{\rho}' > \overline{\rho}$.*
- *If the infection to recovery rate is above the upper bound, so that $\frac{\nu}{\delta} > \overline{\lambda}$, then the steady-state average infection rate is higher under P', so $\overline{\rho}' < \overline{\rho}$.*

7.2.6 Remarks on Models of Diffusion

- Higher variance in degree distribution lead to lower infection thresholds
- Higher degree density increases infection rates, lowers thresholds

- Analyses did not study the effect of loops or cycles, always assumed neighbour's degrees are independent
- No study of how a network might react to a process