Chapter 4 of: Social and Economic Network / Matthew O. Jackson.

Properties of Random Networks

Thresholds and Phase Transitions

Erdös-Renyi model is completely specified by the link formation probability $p(n)$.

For a given monotone property A, we define a threshold function $t(n)$ as a function that satisfies:

$$P\,(\text{property } A)\begin{cases} \to 0 & : \frac{p(n)}{t(n)} \to 0 \\ \to 1 & : \frac{p(n)}{t(n)} \to \infty \end{cases}$$

- What is a *property*?
- What is a *monotone* property?

Properties are generally specified as a set of networks for each n, and then a property is satisfied if the realized network is in the set.

The property that a network has no isolated nodes:

$$A(N) = \{g : N_i(g) \neq \emptyset, \forall i \in N\}$$

Monotone property are properties such that if a given network satisfies the property, then any supernetwork (in the sense of set inclusion) satisfies it.

$$g \in A(N) \text{ and } g \subset g' \implies g' \in A(N)$$

- Example of a monotone property: being connected.
- Example of a non-monotone property: having an even number of links.

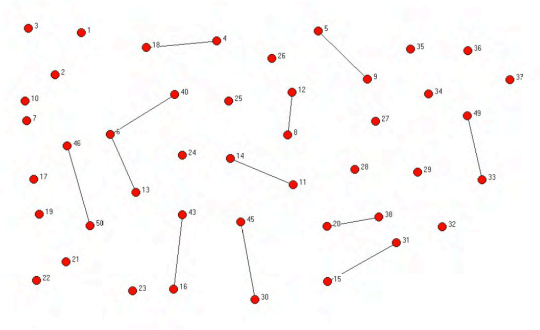
Recall:

For a given monotone property A, we define a threshold function $t(n)$ as a function that satisfies:

$$P\,(\text{property } A) \begin{cases} \to 0 & : \frac{p(n)}{t(n)} \to 0 \\ \to 1 & : \frac{p(n)}{t(n)} \to \infty \end{cases}$$

Property: the network has some links.

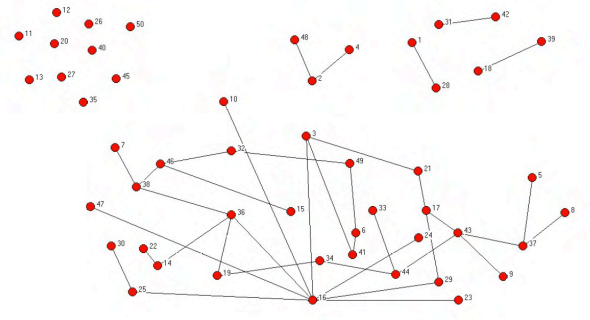
$$t(n) = \frac{1}{n^2}$$



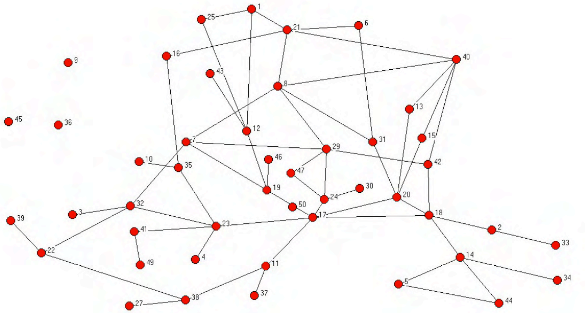
$(n = 50,\ p = 0.01)$ threshold is 0.0004

Property: the network has a cycle.

$$t(n) = \frac{1}{n}$$



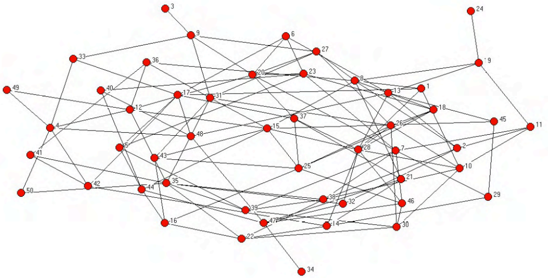
$(n = 50,\ p = 0.03)$ threshold is 0.02



$(n = 50,\ p = 0.05)$ threshold is 0.02

Property: network is connected.

$$t(n) = \frac{\log(n)}{n}$$



$(n = 50,\ p = 0.1)$ threshold is ~ 0.078

Connectedness (a special phase transition)

Theorem [Erdös-Renyi]

A threshold function for the connectedness of the Poisson random network is $t(n) = \log(n)/n$.

Proof Sketch:

- If $p(n)/t(n) \to 0$ then there will be isolated nodes with probability 1.
- If $p(n)/t(n) \to \infty$ then there will not be any component of size less than $n/2$.
- KEY IDEA: threshold for the isolated node is the same as threshold for small components.

proof sketch continued:

- We first show that for $E[d]$, expected degree, $\log(n)$ is the threshold above which we expect each node to have some links.
- Once nodes have many links, the chance of disconnected components vanishes.
- Let $E[d] = (1-n)p(n) = f(n) + log(n)$ for some function f.
- Probability that some node is isolated:

$$(1-p(n))^{n-1} = (1-(f(n)+\log(n))/(n-1))^{n-1}$$

proof sketch continued:

- Probability that a given node is completely isolated:

$$\underbrace{\underbrace{(1-p(n))^{n-1} \sim (1-p(n))^{n}}_{\text{as } p(n)\to 0} \sim \exp^{-np(n)}}_{e^{x}=lim_n(1+x/n)^n} \text{ as } p(n) \to 0.$$

  and similarly:

$$(1-p(n))^{n-1} \sim e^{-(f(n)+\log(n))} = e^{-f(n)}/n$$

proof sketch continued:

Thus the expected number of isolated nodes is $e^{f(n)}$

- $\underbrace{f(n)}_{E[d]-\log(n)} \to \infty$: expected number of isolated nodes approaches 0.
- $\underbrace{f(n)}_{E[d]-\log(n)} \to -\infty$: expected number of isolated nodes becomes infinite.