Influence

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October 4, 2023

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This talk is based on the following

- [ODo21] Chapters 1, 2, 9
- [KKL88]

Bahn mi or wraps?

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Here's how we do it. I'll flip a coin and decide: Heads we get bahn mi, tails we get wraps.

There is a massive problem though. I am famously obsessed with bahn mi, so you don't trust that I will honestly report the outcome of the coin flip.

Collective Coin Flipping

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- Every flips a fair coin and writes the outcome of the flip to a shared document.
- Get bahn mi if the number of heads is even, and wraps otherwise.

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Does this fix the issue?

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Does this fix the issue? No!

Robust to Cheaters

In 1985, this question was asked by Ben-Or and Linial: Can we make a similar procedure more resilient to cheaters?

Write heads as -1 and tails as 1.

Let $f : \{-1, 1\}^n \to \{-1, 1\}$ be such that $\Pr_x[f(x) = 1] = 1/2$.

- Every flips a fair coin.
- Let $x = x_1 x_2 ... x_n$ be the outcome of the coin flips.
- Get bahn mi $f(x) = 1$, and wraps otherwise.

Questions we will answer today

1. What is functions f are the most resilient to a cheaters?
2. How big is does a coalition of cheaters need to be before they can almost always decide outcome of f?

Influence

Fourier Analysis + Hypercontractivity

Generalizations

Influence

Definition (Influence)

Let $f : \{-1, 1\}^n \to \{-1, 1\}$, for any coordinate $i \in [n]$, let

$$\mathrm{Inf}_i[f] = \Pr_{x \sim \{-1,1\}^n}[f(x) \neq f(x^{\oplus i})]$$

Also define

$$\mathrm{MaxInf}[f] = \max_{i \in [n]} \{\mathrm{Inf}_i[f]\}.$$

Plan

- Functions with low influence.
- Fourier Analysis + Hypercontractivity.
- Proof of the KKL Theorem (a lower bound on MaxInf).
- Influence of coalitions.

A contraction is a function f such that $||f(x)|| \leq ||x||$ i.e. it makes the input vector shorter. A hypercontraction is (informally) a function that makes the input much shorter.

Warmup

$$\boxed{\mathrm{Inf}_i[f] = \Pr_{x \sim \{-1,1\}^n}[f(x) \neq f(x^{\oplus i})]}$$

What are the influences for the following functions

- The constant 1 function.
- The *i*th dictator function $f(x) = x_i$.
- The Parity function.
- The OR function.
- The AND function.

Low influence function

What are some *balanced* functions where $\mathrm{MaxInf}[f]$ is small?
That is, no person should be able to pick the outcome with high probability if they decide to cheat.

Majority

What is the influence of the majority function?

Tribes

Here's another function. Split the group into many smaller groups, and order bahn mi iff there exists a group in which EVERYONE in the group wants bahn mi. This is called the Tribes function.

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Here's another function. Split the group into many smaller groups, and order bahn mi iff there exists a group in which EVERYONE in the group wants bahn mi. This is called the Tribes function.

Formally, call the groups "tribes", and let s be the number of tribes and w be the size of each tribe.
$\mathrm{Tribes}_{w,s} : \{-1,1\}^{ws} \to \{-1,1\}$ is defined by

$$\mathrm{Tribes}_{w,s}(x^{(1)}, x^{(2)}, ..., x^{(s)}) =$$

$$\mathrm{OR}_s\left(\mathrm{AND}_w(x^{(1)}), \mathrm{AND}_w(x^{(2)}), ..., \mathrm{AND}_w(x^{(s)})\right),$$

where each $x^{(i)} \in \{-1,1\}^w$.

Influence of Tribes

w is the tribe size, s is the number of tribes.

Since we want unbiased functions, we want $\Pr[\mathrm{Tribes}_{w,s} = -1] \approx 1/2$.

$$\Pr[\mathrm{Tribes}_{w,s} = -1] = 1 - (1 - 2^{-w})^s \approx 1 - \exp(-s2^{-w})$$

Setting $s = 2^w \ln(2)$ we get that this is approximately $1/2$.

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You have influence if everyone in your tribe votes -1, and no other tribe is already unanimous, this has probability

$$2^{-(w-1)}(1 - 2^{-w})^{s-1} \approx 2^{-w} \exp(-s2^{-w}) \approx 2^{-(w-1)}$$

Influence of Tribes

$\boxed{s = 2^w \ln(2),\ \mathrm{Inf}_i[f] \approx 2^{-(w-1)}}$

Note that $n = sw = \ln(2) w 2^w$, so $w \approx \log(n) - \log(\log(n))$. Thus,

$$\mathrm{Inf}_i[f] \approx 2^{-w} = 2^{\log(\log(n)) - \log(n)} = \log(n)/n$$

Tribes vs. Majority

$\log(n)/n$ is much smaller than $1/\sqrt{n}$, so a better choice to limit the influence of potential cheaters...

Can we do even better than Tribes?

Tribes is Optimal

Theorem (KKL (1988))

Let $f : \{-1,1\}^n \to \{-1,1\}$. *Then*

$$\mathrm{MaxInf}[f] = \mathrm{Var}[f] \cdot \Omega(\log(n)/n)$$

Note that if f is unbiased $\mathrm{Var}[f] = 1$.

Influence

Fourier Analysis + Hypercontractivity

Generalizations

Fourier Analyis over $\{-1,1\}^n$

Let $S \subseteq [n], x \in \{-1,1\}^n$, let $x^S = \prod_{i \in S} x_i$ (and $x^\emptyset = 1$).

Theorem (Fourier Expansion Theorem)

$\forall f : \{-1,1\}^n \to \mathbb{C}$, f can be uniquely expressed as a multilinear polynomial

$$f = \sum_{S \subseteq [n]} \hat{f}(S) x^S$$

I.e. $\{x^S : S \subseteq [n]\}$ is a basis for the space of functions from $\{0,1\}^n \to \mathbb{C}$.

Inner product and norms

Define the inner product

$$\langle f, g \rangle = \mathop{\mathbf{E}}_{x \sim \{-1,1\}^n} [f(x) g(x)]$$

Also define the p norm

$$||f||_p = \mathop{\mathbf{E}}_{x \sim \{-1,1\}^n} [|f(x)|^p]^{1/p}$$

.

Basic Properties

Expectation of the basis functions is 0 except for the constant 1 function.

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Proof:

$$\mathbf{E}[x^S] = \begin{cases} 1 & S = \emptyset \\ 0 & \text{else} \end{cases}$$

Proof, $\mathbf{E}[1] = 1$, obviously. $\mathbf{E}[x^S] = \prod_{i \in S} \mathbf{E}[x_i] = 0$, since x_is are independently -1, 1 with probability $1/2$.

Basic Properties

$\{x^S : S \subseteq [n]\}$ is an orthonomal basis for the space of functions.

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Proof:

$$\left\langle x^S, x^T \right\rangle = \mathbf{E}[x^S x^T] = \mathbf{E}[x^{S \Delta T}] = \begin{cases} 1 & S = T \\ 0 & \text{else} \end{cases}$$

Basic Properties

Plancherel's Theorem. Suppose $f = \sum_{S \subseteq [n]} \hat{f}(S) x^S$, and $g = \sum_{T \subseteq [n]} \hat{g}(T) x^T$, then

$$\langle f, g \rangle = \sum_{S \subseteq [n]} \hat{f}(S) \hat{g}(S)$$

Proof

$$\langle f, g \rangle = \left\langle \sum_{S \subseteq [n]} \hat{f}(S) x^S, \sum_{T \subseteq [n]} \hat{g}(T) x^T \right\rangle = \sum_{S, T \subseteq [n]} \hat{f}(S) \hat{g}(T) \left\langle x^S, x^T \right\rangle$$

Use orthonomalness.

Basic Properties

This special case of the previous slides is called Parseval's Theorem.

$$||f||_2^2 = \langle f, f \rangle = \sum_{S \subseteq [n]} \hat{f}(S)^2$$

Spectral Sample

If $f : \{-1, 1\}^n \to \{-1, 1\}$, $||f||_2^2 = \mathbf{E}[f^2] = 1$. Thus, by Parseval's Theorem:

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = 1$$

Define the spectral sample $\mathcal{S}$ to be a distribution on subsets of $[n]$ that takes value S with probability $\hat{f}(S)^2$.

Decomposition

Write f as a multilinear polynomial using the Fourier Expansion Theorem. We can write f as $x_i d(x) + e(x)$ where d and e are polynomials that don't depend on x_i.

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Write f as a multilinear polynomial using the Fourier Expansion Theorem. We can write f as $x_i d(x) + e(x)$ where d and e are polynomials that don't depend on x_i. Note that

$$f(x^{i\to 1}) = d(x) + e(x), f(x^{i\to -1}) = -d(x) + e(x),$$

Rearranging for $d(x)$ and $e(x)$, we have that

$$d(x) = \frac{f(x^{i\to 1}) - f(x^{i\to -1})}{2}, e(x) = \frac{f(x^{i\to 1}) + f(x^{i\to -1})}{2}$$

Decomposition

For each i, let D_i, E_i be the operators mapping f to the first and second part of this decomposition respectively. I.e. $D_i f : \{-1,1\}^n \to \{-1,1\}$ is the function such that

$$D_i f(x) = \frac{f(x^{i\to 1}) - f(x^{i\to -1})}{2},$$

and $E_i f : \{-1,1\}^n \to \{-1,1\}$ is such that

$$E_i f(x) = \frac{f(x^{i\to 1}) + f(x^{i\to -1})}{2},$$

and $f = x_i D_i f + E_i f$

The Discrete Derivative

$$\boxed{D_i f(x) = \frac{f(x^{i\to 1}) - f(x^{i\to -1})}{2}}$$

D_i is called the discrete derivative. If $f : \{-1,1\}^n \to \{-1,1\}$, then

$$D_i f(x) = \begin{cases} 0 & \text{if } f(x) = f(x^{\oplus i}) \\ \pm 1 & \text{if } f(x) \neq f(x^{\oplus i}) \end{cases}.$$

Thus, $\mathrm{Inf}_i[f] = \mathbf{E}[D_i f^2] = ||D_i f||_2^2$

Fourier Transform of $D_i f$

$$\boxed{f(x) = x_i D_i f(x) + E_i f(x)}$$

From the definition of $D_i f$, we find that

$$D_i f = \sum_{S \subseteq [n], i \in S} \hat{f}(S) x^{S \setminus \{i\}}$$

Fourier Transform of $\mathrm{Inf}_i[f]$

$\boxed{D_i f = \sum_{S \subseteq [n], i \in S} \hat{f}(S) x^{S \setminus \{i\}}}$

Using the fourier expansion of $D_i f$, and Parseval's we get

$$\mathrm{Inf}_i[f] = ||D_i f||^2 = \sum_{S \subseteq [n], i \in S} \hat{f}(S)^2.$$

Summing over i, we get

$$I[f] = \sum_{S \subseteq [n]} |S| \hat{f}(S)^2 = \mathop{\mathbf{E}}_{S \sim \mathcal{S}} [|S|].$$

Variance

Suppose f is unbiased (i.e. $\mathbf{E}[f] = 0$), then
$\mathrm{Var}[f] = \mathbf{E}[f^2] - \mathbf{E}[f]^2 = 1$, since f^2 is the constant 1 function.

$$\mathrm{Var}[f] = \langle f, f \rangle - \langle f, 1 \rangle^2 = \left(\sum_{S \subseteq [n]} \hat{f}(S)^2 \right) - \hat{f}(\emptyset)^2 = \sum_{S \subseteq [n], S \neq \emptyset} \hat{f}(S)^2$$

Comparing

Summarizing the previous two slides,

$$I[f] = \sum_{S \subseteq [n], S \neq \emptyset} |S| \hat{f}(S)^2,$$

and

$$\mathrm{Var}[f] = \sum_{S \subseteq [n], S \neq \emptyset} \hat{f}(S)^2.$$

Additionally, if f is unbiased, then $\mathrm{Var}[f] = 1$.

From these we can see that $\mathrm{Var}[f] \leq I[f]$, so we get an immediate lower bound on $\mathrm{MaxInf}[f]$ of $\mathrm{Var}[f]/n$. In this case, that's $1/n$ since f is unbiased.

Comparing

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and

$$\mathrm{Var}[f] = \sum_{S \subseteq [n], S \neq \emptyset} \hat{f}(S)^2.$$

Additionally, if f is unbiased, then $\mathrm{Var}[f] = 1$.

We want a better lower bound. Here's some intuition for the proof: Either $I[f] = \Omega(\log(n))$, in which case $\mathrm{MaxInf}[f] = \Omega(\log(n)/n)$ by averaging. OR, $I[f] = o(\log(n))$. Since the first sum is small but the second sum needs to sum up to 1, $\hat{f}(S)$ should be concentrated on small sets S, the proof will show that this is impossible when $\mathrm{MaxInf}[f]$ is small.

Fourier Weight Concentrated on Small Sets

The game is to distribute the $\hat{f}(S)$ such that the sum of the squares is 1, but the sum weighted by the size of $|S|$ is small. We might define a score like this, where w_S is large for small S and small for large S.

$$\sum_{S \subseteq [n]} w_S \cdot \hat{f}(S)^2$$

Noise Operator

Let $\rho \in [0,1]$. For fixed $x \in \{-1,1\}^n$, write $y \sim N_\rho(x)$ to denote a random string y drawn as follows. For each $i \in [n]$, independently,

$$y_i = \begin{cases} x_i & \text{with probability } \rho \\ \text{uniformly random} & \text{with probability } 1-\rho \end{cases}$$

Then define the **noise operator** T_ρ such that

$$T_\rho f(x) = \mathop{\mathbf{E}}_{y \sim N_\rho(x)}[f(y)]$$

FT of the Noise Operator

$$\boxed{T_\rho f = \mathbf{E}_{y\sim N_\rho(x)}[f(y)]}$$

Note that T_ρ is linear since expectation is linear. I.e.
$T_\rho(f+\alpha g) = T_\rho f + \alpha T_\rho g$

Thus, $T_\rho f = \sum_{S\subseteq[n]} \hat{f}(S) T_\rho x^S$, and

$$\begin{aligned} T_\rho x^S(x) &= \mathop{\mathbf{E}}_{y\sim N_\rho(x)}[y^S] \\ &= \prod_{i\in S} E_{y\sim N_\rho(x)}[y_i] \\ &= \prod_{i\in S}(\rho x_i) \\ &= \rho^{|S|}x^S \end{aligned}$$

Fourier Transform of the Noise Operator

Thus,

$$T_\rho f = \sum_{S \subseteq [n]} \rho^{|S|} \hat{f}(S) x^S$$

$$\boxed{T_\rho f = \sum_{S\subseteq[n]} \rho^{|S|}\hat{f}(S)x^S}$$

$$I[f] = \sum_{S\subseteq[n], S\neq\emptyset} |S|\hat{f}(S)^2,$$

$$\mathrm{Var}[f] = \sum_{S\subseteq[n], S\neq\emptyset} \hat{f}(S)^2 = 1$$

$$||T_{\sqrt{\rho}}f||_2^2 = \sum_{S\subseteq[n]} \rho^{|S|}\hat{f}(S)^2$$

Contraction

An operator T is called a contraction if $||Tf|| \leq ||f||$.

Norms

For any $1 \leq p \leq q$,

$$||f||_p \leq ||f||_q$$

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Proof: Apply Jensen's Inequality

$$\mathbf{E}[|f|^p]^{q/p} \leq \mathbf{E}[(|f|^p)^{q/p}] \implies \mathbf{E}[|f|^p]^{1/p} \leq \mathbf{E}[|f|^q]^{1/q}$$

For example, if $f : \{-1, 1\} \to \{0, 1\}$ was a function with expectation $1/2$, $||f||_2 = 1/\sqrt{2} \approx 0.71$, $||f||_4 = 1/\sqrt[4]{2} \approx 0.84$.

Note: Usually, if using the sum norm instead of the expectation norm, the inequality is the other way around.

(2, 4)-Hypercontractivity Theorem

Theorem

$$||T_{1/\sqrt{3}}f||_4 \leq ||f||_2$$

(2, 4)-Hypercontractivity Theorem

Theorem

$$||T_{1/\sqrt{3}}f||_4 \leq ||f||_2$$

Proof: By induction on n, (use the decomposition $f = x_n d + e$).
See [ODo21], p253 for the details.

(4/3, 2)-HC Theorem

(2, 4)-HC Theorem:

$$||T_{1/\sqrt{3}}f||_4 \leq ||f||_2$$

Theorem

$$||T_{1/\sqrt{3}}f||_2 \leq ||f||_{4/3}$$

Proof: Let $T = T_{1/\sqrt{3}}$

$$||Tf||_2^2 = \langle Tf, Tf \rangle = \langle f, TTf \rangle \leq ||f||_{4/3} ||TTf||_4 \leq ||f||_{4/3} ||Tf||_2$$

Where the first inequality follows from Hölder's Inequality, and the second follows from the $(2, 4)$-Hypercontractivity Theorem.

Proof of the KKL Theorem

The proof considers the sum $\sum_{i\in[n]} ||T_{1/\sqrt{3}} D_i f||_2^2$.
Upper bound. By the $(4/3, 2)$-Hypercontractivity Theorem, we have

$$||T_{1/\sqrt{3}} D_i f||_2 \leq ||D_i f||_{4/3} = \mathbf{E}[|D_i f|^{4/3}]^{3/4} = \mathrm{Inf}_i[f]^{3/4}$$

Thus,

$$\begin{aligned}
\sum_{i\in[n]} ||T_{1/\sqrt{3}} D_i f||_2^2 &\leq \sum_{i\in[n]} \mathrm{Inf}_i[f]^{3/2} \\
&= \sum_{i\in[n]} \mathrm{Inf}_i[f]\sqrt{\mathrm{Inf}_i[f]} \\
&\leq \sqrt{\mathrm{MaxInf}[f]}\, I[f]
\end{aligned}$$

Proof of the KKL Theorem

Lower bound.

$$D_i f = \sum_{S\subseteq[n], i\in S} \hat{f}(S) x^{S\setminus\{i\}}$$
$$T_\rho f = \sum_{S\subseteq[n]} \rho^{|S|}\hat{f}(S) x^S$$
$$||f||_2^2 = \sum_{S\subseteq[n]} \hat{f}(S)^2 \text{ (Parseval's Theorem)}$$

$$\begin{aligned}
\sum_{i\in[n]} ||T_{1/\sqrt{3}} D_i f||_2^2 &= \sum_{i=1}^{n} \sum_{S\subseteq[n], i\in S} \hat{f}(S)^2/3^{|S|-1} \\
&= \sum_{|S|\geq 1} |S|\hat{f}(S)^2/3^{|S|-1} \\
&\geq \sum_{|S|\geq 1} \hat{f}(S)^2/3^{|S|} \\
&= \mathop{\mathbf{E}}_{S\sim\mathcal{S}}[3^{-|S|}] \\
&\geq 3^{-\mathbf{E}_{S\sim\mathcal{S}}[|S|]} \\
&= 3^{-I[f]}
\end{aligned}$$

Proof of the KKL Theorem

Combining the two inequalities, we get that $3^{-I[f]} \leq I[f]\sqrt{\mathrm{MaxInf}[f]}$, so

$$\left(\frac{3^{-I[f]}}{I[f]}\right)^2 \leq \mathrm{MaxInf}[f].$$

Proof of the KKL Theorem

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Case 1. $I[f] \geq 0.1 \log_3(n)$. By averaging, $\mathrm{MaxInf}[f] \geq \Omega(\log(n)/n)$.

Proof of the KKL Theorem

Combining the two inequalities, we get that $3^{-I[f]} \leq I[f]\sqrt{\mathrm{MaxInf}[f]}$, so

$$\left(\frac{3^{-I[f]}}{I[f]}\right)^2 \leq \mathrm{MaxInf}[f].$$

Case 1. $I[f] \geq 0.1 \log_3(n)$. By averaging, $\mathrm{MaxInf}[f] \geq \Omega(\log(n)/n)$.

Case 2. $I[f] < 0.1 \log_3(n)$. Then

$$\mathrm{MaxInf}[f] \geq (n^{-0.1}/0.1 \log_3(n))^2 = \Omega(n^{-0.21}) = \Omega(\log(n)/n)$$

Influence

Fourier Analysis + Hypercontractivity

Generalizations

Generalizations

- Generalizing to coalitions.
- Generalizing the domain to X^n.

Extension to Coalitions

The KKL theorem says that someone has influence at least $\Omega(\log(n)/n)$. How about the influence of a coalition?

KKL for Coalitions

Theorem

For all unbiased $f\ \{-1,1\}^n \to \{-1,1\}$*, there exists a set* J*, with* $|J| = O(n/\log(n))$*, such that* $\mathrm{Inf}_J[f] \geq 0.99$*.*

Proof

Suppose that f is monotone (monotone functions minimize influence anyway so this is fine). First we'll show that there is a set of coordinates that you can bribe to make the value of f 1 with high probability.

You iteratively bribe the coordinate with the most influence. Let $f_0 = f$, define $f_t = f_{t-1}^{j \to 1}$ where j is the coordinate of largest influence in f_{t-1}. We have $\mathbf{E}[f_t] \geq \mathbf{E}[f_{t-1}] + \mathrm{MaxInf}[f_{t-1}]$.

Run this for t iterations. If $\mathbf{E}[f_t] > 0.99$, then great. Otherwise, $\mathrm{Var}[f] = \Omega(1)$, so by KKL the $\mathrm{MaxInf}[f_i]$ for each $i < t$ is $\Omega(\log(n)/n)$. Thus, $\mathbf{E}[f_t] \geq \Omega(\log(n)/n)t$, setting $t = O(n/\log(n))$ suffices to make this 0.99. Let J_1 be the set of coordinates you bribed along the way.

Proof

Do the same thing but bribe them to vote the other way instead, get another set J_{-1} such that having them all bribed to vote -1, the probability that the outcome is -1 is -0.99. Then $J_1 \cup J_{-1}$ has large influence.

KKL for Coalitions

More generally,

Theorem

For all $\epsilon > 0$*, and unbiased* $f\ \{-1,1\}^n \to \{-1,1\}$*, there exists a set* J*, with* $|J| \leq O(\log(1/\epsilon)n/\log(n))$*, such that* $\mathrm{Inf}_J[f] \geq 1-\epsilon$*.*

Function Resilient to Coalitions

The Tribes Function minimizes MaxInf but is not resilient to large coalitions - you just need to bribe a single tribe (of size rougly $\log(n) - \log(\log(n)))$ to get large influence.

The best known construction is due to [AL93], which has $o(1)$ influences for all coalitions of size $o(n/\log^2(n))$. It's kind of like a randomized tribes.

Open Questions

- Combinatorial proof?
- Coalitions lower bound for $[0, 1]$.
- Is the [AL93] construction optimal? Also, can we derandomize it?
- Generalizing the codomain? E.g. what if we were choosing between bahn mi, wraps, AND burgers.

References

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