# Unifying Width-Reduced Methods for Quasi-Self-Concordant Optimization 

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#### Abstract

We provide several algorithms for constrained optimization of a large class of convex problems, including softmax, $\ell_{p}$ regression, and logistic regression. Central to our approach is the notion of width reduction, a technique which has proven immensely useful in the context of maximum flow [Christiano et al., STOC'11] and, more recently, $\ell_{p}$ regression [Adil et al., SODA'19], in terms of improving the iteration complexity from $O\left(m^{1 / 2}\right)$ to $\tilde{O}\left(m^{1 / 3}\right)$, where $m$ is the number of rows of the design matrix, and where each iteration amounts to a linear system solve. However, a considerable drawback is that these methods require both problem-specific potentials and individually tailored analyses. As our main contribution, we initiate a new direction of study by presenting the first unified approach to achieving $m^{1 / 3}$-type rates. Notably, our method goes beyond these previously considered problems to more broadly capture quasi-selfconcordant losses, a class which has recently generated much interest and includes the well-studied problem of logistic regression, among others. In order to do so, we develop a unified width reduction method for carefully handling these losses based on a more general set of potentials. Additionally, we directly achieve $m^{1 / 3}$ type rates in the constrained setting without the need for any explicit acceleration schemes, thus naturally complementing recent work based on a ball-oracle approach [Carmon et al., NeurIPS'20].


## 1 Introduction

We study a class of constrained optimization problems of the following form:

$$
\begin{equation*}
\min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}} \sum_{i} \boldsymbol{f}\left((\boldsymbol{P} \boldsymbol{x})_{i}\right) \tag{1}
\end{equation*}
$$

for convex $\boldsymbol{f}: \mathbb{R} \rightarrow \mathbb{R}$, where $\boldsymbol{A} \in \mathbb{R}^{d \times n}, \boldsymbol{b} \in \mathbb{R}^{d}, \boldsymbol{P} \in \mathbb{R}^{m \times n}$, with $d \leq n \leq m$. Specifically, we are interested in the case where $f$ satisfies a certain higher-order smoothness-like condition known as $M$-quasi-self-concordance (q.s.c.), i.e., $\left|\boldsymbol{f}^{\prime \prime \prime}(\boldsymbol{x})\right| \leq M \boldsymbol{f}^{\prime \prime}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}$. Several problems of significant interest in machine learning and numerical methods meet this condition, including logistic regression [Bac10, KSJ18], as well as softmax (often used to approximate $\ell_{\infty}$ regression) [Nes05, $\left.\mathrm{CKM}^{+} 11, ~ E V 19, ~ B u l 20\right] ~$ and (regularized) $\ell_{p}$ regression [BCLL18, AKPS19].

A very useful optimization technique, first introduced by [ $\left.\mathrm{CKM}^{+} 11\right]$ for faster approximate maximum flow and later by [CMMP13] for regression, is that of width reduction, whereby they used it to improve the iteration complexity dependence on $m$, the number of rows of the design matrix from $O\left(m^{1 / 2}\right)$ to $\tilde{O}\left(m^{1 / 3}\right)$, and where each iteration requires a linear system solve. Later work by [AKPS19] for high-accuracy $\ell_{p}$ regression, building on an $O\left(\mathrm{~m}^{1 / 2}\right)$-iteration result from [BCLL18],
again showed how width reduction could lead to improved $\tilde{O}\left(m^{1 / 3}\right)$-iteration algorithms. As a drawback, however, these approaches rely on potential methods and analyses specifically tailored to each problem.

Building on these results, we present the first unified approach to achieving $m^{1 / 3}$-type rates, at the heart of which lies a more general width reduction scheme. Notably, our method goes beyond these previously considered problems to capture quasi-self-concordant losses, thereby further including well-studied problems such as logistic regression, among others. By doing so, we directly achieve $m^{1 / 3}$-type rates in the constrained setting without relying on explicit acceleration schemes [MS13], thus complementing recent work based on a ball-oracle approach $\left\lfloor\mathrm{CJJ}^{+} 20\right]$. We additionally note that, given the ways in which our results achieve improvements similar to those of [CJJ ${ }^{+}$20], we believe our work hints at a deeper, though to our knowledge not yet fully understood, connection between the techniques of width reduction and Monteiro-Svaiter acceleration.

### 1.1 Main Results and Applications

We first present in Section 3 a width-reduced method for obtaining a crude approximation to (1) for quasi-self-concordant $\boldsymbol{f}$. At a high level, our algorithm returns an approximate solution $\tilde{\boldsymbol{x}}$ that both satisfies the linear constraints and is bounded in $\ell_{\infty}$-norm by $O(R)$, where $R$ is a bound on the norm of the optimal solution. Following from Theorem 3.3, the result below shows how, for the problem of minimizing softmax (parameterized by $\nu>0$ ), i.e., $\operatorname{smax}_{\nu}(\boldsymbol{P} \boldsymbol{x})=\nu \log \left(\sum_{i} e^{\frac{(\boldsymbol{P} x)_{i}}{\nu}}\right)$, we can bound the norm of the solution by $(1+\nu) R$.
Theorem 1.1. Let $\boldsymbol{x}^{\star}$ denote the optimum of $\min _{\boldsymbol{A x}=\boldsymbol{b}} \operatorname{smax}_{\nu}(\boldsymbol{P} \boldsymbol{x})$. Algorithm $\square$ when applied to the function $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i} e^{\frac{(\boldsymbol{P} x)_{i}}{\nu}}$ with $\epsilon=\nu$, returns $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$, and

$$
\operatorname{smax}_{\nu}(\boldsymbol{P} \widetilde{\boldsymbol{x}}) \leq(1+\widetilde{O}(\nu)) \operatorname{smax}_{\nu}\left(\boldsymbol{P}_{\boldsymbol{x}^{\star}}\right)
$$

in at most $\widetilde{O}\left(m^{1 / 3} \nu^{-5 / 3}\right)$ calls to a linear system solver.
As a consequence of Theorem 1.1 when taking $\nu=\Omega\left(\epsilon / \log ^{O(1)}(m)\right)$, we have by Theorem 5.2 a $(1+\epsilon)$ approximate solution to the problem of $\ell_{\infty}$ regression with $\tilde{O}\left(m^{1 / 3} \epsilon^{-5 / 3}\right)$ calls to a linear system solver.
Further, we show the following result which can use the approximate solution returned by Theorem 1.1 as an initial point for achieving a high-accuracy solution. We also present in Appendix A a natural extension of our results to minimizing general-self-concordant (g.s.c.) functions.
Theorem 1.2. For $M$-q.s.c. $\boldsymbol{f}, \epsilon>0$, and $\boldsymbol{x}^{(0)}$ such that $\boldsymbol{A} \boldsymbol{x}^{(0)}=\boldsymbol{b}$ and $\left\|\boldsymbol{x}^{(0)}\right\|_{\infty} \leq R$, Algorithm 2 finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\boldsymbol{f}(\widetilde{\boldsymbol{x}}) \leq \epsilon+\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)$ in $\widetilde{O}\left(M R m^{1 / 3} \log (M R) \log \left(\frac{\boldsymbol{f}\left(\boldsymbol{x}^{(0)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)}{\epsilon}\right)\right)$ calls to a linear system solver.

Resulting from the theorem above, as detailed in Section 5] are guarantees given by Theorems 5.4 and 5.5 which establish convergence rates of $\widetilde{O}\left(p^{2} \mu^{-1 /(p-2)} m^{1 / 3} R\right)$ and $\widetilde{O}\left(m^{1 / 3} R\right)$, respectively, for $\mu$-regularized $\ell_{p}$ regression and logistic regression. We emphasize that the latter is, to our knowledge, the first such use of width reduction for directly solving constrained logistic regression problems.

### 1.2 Related Works

Quasi-self-concordance and higher-order smoothness. Bac10] showed how to analyze Newton's method for quasi-self-concordant functions, with an emphasis on its application to logistic regression. Later, notions of local, or Hessian, stability which follow from quasi-self-concordance gave rise to methods with better dependence on various conditioning parameters [KSJ18, CJJ ${ }^{+}$20]. In the work by [KSJ18], the authors show how a trust-region-based Newton method [NW06] achieves linear convergence for locally stable functions without requiring, e.g., strong convexity. Meanwhile, after noting that quasi-self-concordance implies Hessian stability, [CJJ $\left.{ }^{+} 20\right]$ further improve the dependence on the distance to the optimum by leveraging Monteiro-Svaiter acceleration
[MS13], which has proven useful in the context of near-optimal methods for higher-order acceleration $\left.\operatorname{ASS19}, \mathrm{GDG}^{+} 19\right]$. However, in general these methods, which assume higher-order smoothness, require access to an oracle which minimizes a higher-order Taylor expansion model, though in some cases this may be relaxed to requiring linear system solves [Bul20].

Width reduction and $\ell_{p}$ regression. The technique of width-reduction first came to prominence in seminal work by $\left[\mathrm{CKM}^{+} 11\right]$ for achieving faster approximate maximum flow, being the first to achieve an improved $m^{1 / 3}$ dependence. At a high level, the idea behind the approach is to solve a sequence of weighted $\ell_{2}$-minimizing flow problems, whereby at each iteration one of two cases occurs: either the proposed step is added to the current solution (a "flow" step) along with the weights, or else there exist some set of coordinates that exceed a certain threshold, and so their weights are updated accordingly (a "width reduction" step). Several works have since adapted this approach to regression problems [CMMP13, AKPS19, EV19, ABKS21] and matrix scaling [AZLOW17].
In addition to their importance in machine learning, regression methods capture several fundamental problems in scientific computing and signal processing. A recent line of work initiated by [BCLL18] showed how to attain high-accuracy solutions for $\ell_{p}$ regression using $O_{p}\left(m^{1 / 2-1 / p}\right)$ linear system solves, thus going beyond what is achievable via self-concordance. Building on this work, AKPS19] showed how width reduction could be applied to this setting to achieve, as in the case of approximate maximum flow [CKM ${ }^{+} 11$, a similar improvement from $O_{p}\left(m^{1 / 2}\right)$ to $O_{p}\left(m^{1 / 3}\right)$ (for $p \rightarrow \infty)$. Further developments by [KPSW19, AS20] for graph problems showed almost-linear time solutions for $\ell_{p}$ regression for $p \approx \sqrt{\log (n)}$ which have since been a critical part of recent advances in high-accuracy maximum flow on unit-capacity graphs [LS20, KLS20].

Accelerated methods. Recent developments by $\left[\mathrm{CJJ}^{+} 20\right]$ have shown several advantages that arise in the case of unconstrained minimization for $L$-smooth $M$-quasi-self-concordant problems. By considering a certain ball oracle method (whereby each call to the oracle returns the minimizer of the function inside an $\ell_{2}$ ball of radius $r$ ), [ $\left[\mathbf{C J J}^{+} 20\right]$ implement an accelerated scheme which returns a solution to the unconstrained smooth convex minimization problem in $(R / r)^{2 / 3}$ calls to the oracle, where $R$ is the initial $\ell_{2}$-norm distance to the optimum, and they further show a matching lower bound under this oracle model. While this approach transfers its difficulty to implementing the oracle, a key insight from their work involves showing this can be done efficiently for smooth quasi-self-concordant functions when $r$ is sufficiently small, where the allowed size depends on the quasi-self-concordance parameter. One limitation to the results of $\left[\mathrm{CJJ}^{+} \mathbf{2 0}\right]$ is that they apply directly to unconstrained optimization problems and require the function to be smooth, and so we complement these results in the quasi-self-concordant setting by establishing comparable rates for directly optimizing a large class of constrained convex problems without requiring smoothness of the function.

### 1.3 Outline of the Paper

After establishing the potential functions at the heart of our width reduction techniques, we present in Section 3 our oracle for roughly approximating a solution to problem (1). We then show in Section 4 how we may attain a high-accuracy solution by using the crude approximation as a starting point. Here, the key idea is to considering a sequence of optimization problems inside $\ell_{\infty}$ balls of manageable size, similar to [CMTV17, CJJ $\left.^{+} 20\right]$. As in the case of the crude oracle, our primary advantage comes from carefully handling a pair of coupled potentials which are amenable to the large class of quasi-self-concordant problems, and in Section5]we further show how our results may be applied to several problems of interest, including logistic and $\ell_{p}$ regression.

## 2 Preliminaries

Notation: We use boldface lowercase letters to denote vectors or functions and boldface uppercase letters for matrices. Scalars are non-bold letters. Our functions are univariate, and we overload function notation to act on a vector coordinate-wise, i.e. $f(x)=\sum_{i} f\left(x_{i}\right)$. The notation $x \geq y$ for vectors refers to entry-wise inequality. Refer to the algorithm boxes for definitions of certain algorithm specific parameters that appear in lemma and theorem statements.

### 2.1 Quasi-Self-Concordance

Definition 2.1 (g.s.c. and q.s.c.). Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a thrice differentiable function with continuous third derivative, and let $\nu>0$ and $M>0$. We say that $f$ is $(M, \nu)$-general-self-concordant (g.s.c.) if

$$
\forall x, \quad\left|\boldsymbol{f}^{\prime \prime \prime}(x)\right| \leq M \boldsymbol{f}^{\prime \prime}(x)^{\frac{\nu}{2}}
$$

When $\nu=2$, we have the following condition:

$$
\forall x, \quad\left|\boldsymbol{f}^{\prime \prime \prime}(x)\right| \leq M \boldsymbol{f}^{\prime \prime}(x)
$$

and we call such functions $M$-quasi-self-concordant (q.s.c.).

### 2.2 Problem

Recall that we are solving the following problem:

$$
\min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}} \sum_{i} \boldsymbol{f}\left((\boldsymbol{P} \boldsymbol{x})_{i}\right)
$$

where $\boldsymbol{A} \in \mathbb{R}^{d \times n}, \boldsymbol{b} \in \mathbb{R}^{d}, \boldsymbol{P} \in \mathbb{R}^{m \times n}, d \leq n$, and $m \geq n$, and such that $\boldsymbol{f}$ is convex, $M$-q.s.c. and, for $\boldsymbol{w} \geq \boldsymbol{w}_{0} \geq 0, \boldsymbol{f}^{\prime \prime}(\boldsymbol{w})$ is coordinate-wise monotonic. We can ignore the case when $\boldsymbol{f}^{\prime \prime}$ is constant since that corresponds to a quadratic problem which we know how to solve directly via linear system solves.

## Assumptions on the Optimum $\boldsymbol{x}^{\star}$

We assume that $R \in \mathbb{R}_{>0}$ is such that the optimum $\boldsymbol{x}^{\star} \stackrel{\text { def }}{=} \arg \min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})$ satisfies

$$
\begin{equation*}
\left\|\boldsymbol{P} \boldsymbol{x}^{\star}\right\|_{\infty} \leq R . \tag{2}
\end{equation*}
$$

We now define the potentials that we track in the algorithm.

### 2.3 Potentials

Definition 2.2 (Dual Potential). For a weights vector $\boldsymbol{w} \in \mathbb{R}_{\geq 0}^{m}$, we define a potential

$$
\Phi(\boldsymbol{w})=\sum_{i} \Phi\left(\boldsymbol{w}_{i}\right)=\sum_{i} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)
$$

We also define the following corresponding potential, which gives rise to the linear regression problem that we will need to solve at each step of our algorithm.

Definition 2.3 (Resistances and Corresponding Potential). For a weights vector $\boldsymbol{w} \in \mathbb{R}_{\geq 0}^{m}$ and $\epsilon>0$, define resistances $\boldsymbol{r} \in \mathbb{R}_{\geq 0}^{m}$ and a corresponding potential $\Psi$ as,

$$
\begin{gathered}
\boldsymbol{r}_{i}=\frac{1}{R^{2}}\left(\boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)+\frac{\epsilon \Phi(\boldsymbol{w})}{m}\right) \\
\Psi(\boldsymbol{r})=\min _{\boldsymbol{A} \Delta=b} \sum_{i} \boldsymbol{r}_{i}(\boldsymbol{P} \Delta)_{i}^{2}
\end{gathered}
$$

We have the following relation between our two potentials $\Phi$ and $\Psi$.
Lemma 2.4. For $\epsilon>0$, resistances $\boldsymbol{r}$ (Definition[2.3), with corresponding weights $\boldsymbol{w}$, we have

$$
\Psi(\boldsymbol{r}) \leq(1+\epsilon) \Phi(\boldsymbol{w})
$$

In addition, letting $\|\boldsymbol{P}\|_{\text {min }}=\min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}}\|\boldsymbol{P} \boldsymbol{x}\|_{2}$ and $\|\boldsymbol{A}\|$ denote the operator norm of $\boldsymbol{A}$, we have

$$
\Psi(\boldsymbol{r}) \geq \frac{\epsilon \Phi(\boldsymbol{w})}{m R^{2}} \frac{\|\boldsymbol{P}\|_{\min }^{2}\|\boldsymbol{b}\|_{2}^{2}}{\|\boldsymbol{A}\|^{2}} \stackrel{\operatorname{def}}{=} \Phi(\boldsymbol{w}) L
$$

## 3 Algorithm and Analysis for a Crude Solution for Q.S.C. Functions

In this section, we give an algorithm for solving Problem (1) to a crude approximation; namely, we return a solution $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$, i.e., it satisfies the subspace constraints, and $\|\boldsymbol{P} \widetilde{\boldsymbol{x}}\|_{\infty}$ is bounded. We will later see in our applications how this translates into a constant or polynomial approximation guarantee to the function value for some functions. In the next section we will see how we can use the guarantees of the solution returned as a starting solution and boost it to an $\epsilon$ approximate solution.

Our algorithm is based on combining a multiplicative weight update (MWU) scheme with width reduction. Though such algorithms have so far only been used for $\ell_{p}$-regression, $p=1$ or $p \in[2, \infty]$, here we are able to extend the analysis to q.s.c. functions, while also providing a unified analysis for the known cases of $\ell_{p}$-regression (refer to Section 5 to see how we apply this algorithm to these instances). We note that we can extend this analysis to other general-self-concordant functions, and we have deferred these cases to the appendix.

### 3.1 Algorithm and Analysis

Our proof relies on tracking two potentials, $\Psi$ (Definition 2.3) and $\Phi$ (Definition 2.2) that depend on the weights. We first show how these potentials change with weight updates corresponding to a flow step and a width reduction step in the algorithm. We next show that if our algorithm runs for at most $K=\widetilde{O}\left(m^{1 / 3}\right)$ width reduction steps, then after $T=\widetilde{O}\left(m^{1 / 3}\right)$ flow steps we can bound $\Phi$. Further, using the relation between $\Phi$ and $\Psi$ (Lemma 2.4) and appropriately chosen parameters, we show that we cannot have more than $K$ width reduction steps. The key part of the analysis lies in the growth of $\Phi$ with respect to both flow and width steps.

```
Algorithm 1 Width-Reduced Algorithm for \(M\)-q.s.c. Functions
    procedure QSC-MWU \((\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{P}, M, R, \epsilon)\)
        \(\boldsymbol{x}^{(0,0)}=0, \boldsymbol{w}^{(0,0)}=\boldsymbol{w}_{0}\left(\Phi^{\prime}(\boldsymbol{w})\right.\) monotonic for \(\left.\boldsymbol{w} \geq \boldsymbol{w}_{0}, \Phi\left(\boldsymbol{w}_{0}\right)>0\right)\)
        \(\tau \leftarrow \widetilde{\Theta}\left(m^{1 / 3} \epsilon^{-2 / 3}\right)\)
        \(\alpha \leftarrow \widetilde{\Theta}\left(m^{-1 / 3} M^{-1} \epsilon^{2 / 3}\right)\)
        \(t=0, k=0, \quad T=\alpha^{-1} M^{-1} \epsilon^{-1}=\widetilde{\Theta}\left(m^{1 / 3} \epsilon^{-5 / 3}\right)\)
        while \(t \leq T\) do
            \(\boldsymbol{r}_{i}^{(t, k)} \leftarrow \frac{1}{R^{2}}\left(\boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}^{(t, k)}\right)+\frac{\epsilon \Phi\left(\boldsymbol{w}^{(t, k)}\right)}{m}\right) \quad \triangleright\) Resistances
            \(\widetilde{\Delta} \leftarrow \arg \min _{\boldsymbol{A} \Delta=\boldsymbol{b}} \quad \sum_{i} \boldsymbol{r}_{i}(\boldsymbol{P} \Delta)_{i}^{2} \quad \triangleright\) Oracle
            if \(\|\boldsymbol{P} \widetilde{\Delta}\|_{\infty} \leq R \tau\) then \(\quad \triangleright\) Flow Step
                    \(\boldsymbol{x}^{(t+1, k)} \leftarrow \boldsymbol{x}^{(t, k)}+\widetilde{\Delta}\)
            \(\boldsymbol{w}^{(t+1, k)} \leftarrow \boldsymbol{w}^{(t, k)}+\frac{\epsilon \alpha}{R}|\boldsymbol{P} \widetilde{\Delta}|\)
            \(t \leftarrow t+1\)
                else
                        for Indices \(i\) such that \(|\boldsymbol{P} \widetilde{\Delta}|_{i} \geq R \tau\) do \(\quad \triangleright\) Width Reduction
                        if \(f^{\prime \prime}\) is non-decreasing in \(\boldsymbol{w}\) then
                        \(\boldsymbol{w}^{(t, k+1)}\) is such that \(\boldsymbol{r}_{i}^{(t, k+1)} \leftarrow(1+\epsilon) \boldsymbol{r}_{i}^{(t, k)}\)
                else
                        \(\boldsymbol{w}^{(t, k+1)}\) is such that \(\boldsymbol{r}_{i}^{(t, k+1)} \leftarrow \frac{1}{1+\epsilon} \boldsymbol{r}_{i}^{(t, k)}\)
                \(k \leftarrow k+1\)
        return \(\boldsymbol{x}^{(T, k)} / T\)
```

[^0]
## Changes in $\Psi$ and $\Phi$

Lemma 3.1. Let $\Psi$ be as defined in 2.3 After t flow steps and $k$ width reduction steps, we have,

$$
\begin{array}{ll}
\Psi\left(\boldsymbol{r}^{(t, k)}\right) \geq \Psi\left(\boldsymbol{r}^{(0,0)}\right)\left(1+\frac{\epsilon^{2} \tau^{2}}{(1+\epsilon)^{2} m}\right)^{k} & \text { if } \boldsymbol{f}^{\prime \prime} \text { non-decreasing in } \boldsymbol{w}, \\
\Psi\left(\boldsymbol{r}^{(t, k)}\right) \leq \Psi\left(\boldsymbol{r}^{(0,0)}\right)\left(1-\frac{\epsilon^{2} \tau^{2}}{2(1+\epsilon)^{2} m}\right)^{k} & \text { if } \boldsymbol{f}^{\prime \prime} \text { non-increasing in } \boldsymbol{w} .
\end{array}
$$

Lemma 3.2. Suppose $\boldsymbol{f}$ is $M$-q.s.c. Let $\alpha$ and $\tau$ be such that $\alpha \tau \leq M^{-1}$. After $t$ flow steps and $k$ width reduction steps, our potential $\Phi$ satisfies

$$
\begin{aligned}
& \Phi\left(\boldsymbol{w}^{(t, k)}\right) \leq\left(1+\epsilon(1+\epsilon)^{2} \alpha M\right)^{t}\left(1+\epsilon(1+\epsilon) \tau^{-1}\right)^{k} \Phi\left(\boldsymbol{w}_{0}\right) \quad \text { if } \boldsymbol{f}^{\prime \prime} \text { non-decreasing in } \boldsymbol{w}, \\
& \Phi\left(\boldsymbol{w}^{(t, k)}\right) \geq\left(1-\epsilon(1+\epsilon)^{2} \alpha M\right)^{t}\left(1-\epsilon(1+\epsilon) \tau^{-1}\right)^{k} \Phi\left(\boldsymbol{w}_{0}\right) \quad \text { if } \boldsymbol{f}^{\prime \prime} \text { non-increasing in } \boldsymbol{w} .
\end{aligned}
$$

## Runtime Bound

Theorem 3.3. Let $\epsilon>0$, $\boldsymbol{f}$ be M-q.s.c. After $T \leq \frac{\alpha^{-1}}{M \epsilon}=\widetilde{\Theta}\left(m^{1 / 3} \epsilon^{-5 / 3}\right)$ flow steps and $K \leq$ $\tau=\widetilde{\Theta}\left(m^{1 / 3} \epsilon^{-2 / 3}\right)$ width reduction steps, Algorithm $\rceil$ returns $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b},\|\boldsymbol{P} \widetilde{\boldsymbol{x}}\|_{\infty} \leq$ $R M\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty}$, where $\boldsymbol{w}^{(T, K)}$ is the final weights vector that satisfies:

$$
\begin{array}{lc}
\Phi\left(\boldsymbol{w}^{(T, K)}\right) \leq \Phi\left(\boldsymbol{w}_{0}\right) e^{1+4 \epsilon} & \text { if } \boldsymbol{f}^{\prime \prime} \text { is non-decreasing in } \boldsymbol{w} \\
\Phi\left(\boldsymbol{w}^{(T, K)}\right) \geq \Phi\left(\boldsymbol{w}_{0}\right) e^{-(1+4 \epsilon)} & \text { if } \boldsymbol{f}^{\prime \prime} \text { is non-increasing in } \boldsymbol{w} .
\end{array}
$$

Proof. We show the case when $f^{\prime \prime}$ is a non-decreasing function. The other case follows similarly. We set,

$$
\tau \leftarrow \widetilde{\Theta}\left(m^{1 / 3} \epsilon^{-2 / 3}\right) \quad \alpha \leftarrow \widetilde{\Theta}\left(m^{-1 / 3} M^{-1} \epsilon^{2 / 3}\right)
$$

After $T=\frac{\alpha^{-1}}{M \epsilon}$ flow steps and $K=\tau$ width reduction steps, from Lemma 3.2 we have,

$$
\begin{aligned}
\Phi\left(\boldsymbol{w}^{(T, K)}\right) & \leq\left(1+\epsilon(1+\epsilon)^{2} \alpha M\right)^{T}\left(1+\epsilon(1+\epsilon) \tau^{-1}\right)^{K} \Phi\left(\boldsymbol{w}_{0}\right) \\
& \leq \Phi\left(\boldsymbol{w}_{0}\right) e^{\epsilon(1+\epsilon)^{2} \alpha M T+\epsilon(1+\epsilon) \tau^{-1} K} \leq \Phi\left(\boldsymbol{w}_{0}\right) e^{(1+4 \epsilon)}
\end{aligned}
$$

We now show we cannot have more width steps. Throughout the algorithm, we have $\Phi\left(\boldsymbol{w}^{(t, k)}\right) \leq$ $\Phi\left(\boldsymbol{w}_{0}\right) e^{1+4 \epsilon}$. From Lemma 2.4 we always have $\Psi\left(\boldsymbol{r}^{(0,0)}\right) \geq \Phi\left(\boldsymbol{w}_{0}\right) L$ and $\Psi\left(\boldsymbol{r}^{(T, K)}\right) \leq(1+$ $\epsilon) \Phi\left(\boldsymbol{w}^{(T, K)}\right) \leq(1+\epsilon) e^{1+4 \epsilon} \Phi\left(\boldsymbol{w}_{0}\right)$. Thus, from Lemma 3.1, we must have,

$$
(1+\epsilon) e^{1+4 \epsilon} \Phi\left(\boldsymbol{w}_{0}\right) \geq L \Phi\left(\boldsymbol{w}_{0}\right)\left(1+\frac{\epsilon^{2} \tau^{2}}{(1+\epsilon)^{2} m}\right)^{K}
$$

From the definition of $\tau$, we note that $K$ has to be less than $\tau$ for the above bound to be satisfied. Next, let $\widetilde{\Delta}^{(t)}$ denote the solution of our oracle at iteration $t$ of the flow step. From the $\boldsymbol{x}$ and $\boldsymbol{w}$ update in the algorithm,

$$
|\boldsymbol{P} \widetilde{\boldsymbol{x}}|=\left|\sum_{t} \boldsymbol{P} \widetilde{\Delta}^{(t)}\right| \epsilon \alpha M \leq \frac{\epsilon \alpha}{R} \sum_{t}\left|\boldsymbol{P} \widetilde{\Delta}^{(t)}\right| R M \leq \boldsymbol{w}^{(T, K)} R M
$$

This concludes our proof.

## 4 Boosting to a High-Accuracy Solution for Q.S.C. Functions

In this section, we give a width-reduced multiplicative weights update algorithm that, given a starting solution $\boldsymbol{x}^{(0)}$ satisfying $\left\|\boldsymbol{x}^{(0)}\right\|_{\infty} \leq R$ and $\boldsymbol{A} \boldsymbol{x}^{(0)}=\boldsymbol{b}$, finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\boldsymbol{f}(\widetilde{\boldsymbol{x}}) \leq$ $(1+\epsilon) \boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)$ for any q.s.c. function $\boldsymbol{f}$. We would mention here that for the algorithms in this section, it is key that we have a starting solution that satisfies our subspace constraints and has $\ell_{\infty^{-}}$ norm bounded by $R$. Thus, the algorithms here may be of independent interest if such a starting solution is available. We can otherwise use Algorithm 1 with $\epsilon=1$ to obtain such a solution.

For any $\boldsymbol{x}$, we define a residual problem, and we show how it is sufficient to solve the residual problem approximately $\log \left(\epsilon^{-1}\right)$ times to obtain our high-accuracy solution. Similar approaches have been applied to specific functions such as softmax [AZLOW17] and $\ell_{p}$-regression [AKPS19]. We unify these approaches and give a version that works for any q.s.c. function.

We further note that our residual problem is to optimize a simple quadratic objective inside an $\ell_{\infty}$ box. The difficulty lies in solving such $\ell_{\infty}$ box constraints fast. We use a binary search followed by a width-reduced multiplicative weights routine analogous to $\left[\mathrm{CKM}^{+} 11\right]$ to solve our residual problem.
Definition 4.1 (Residual Problem). We define the residual objective at any $\boldsymbol{x}$ satisfying $\|\boldsymbol{P} \boldsymbol{x}\|_{\infty} \leq$ $R$ as

$$
\operatorname{res}(\Delta)=\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \boldsymbol{P} \Delta-e^{-1}(\boldsymbol{P} \Delta)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta
$$

and the residual problem as

$$
\begin{align*}
& \quad \max _{\Delta} \operatorname{res}(\Delta) \\
& \text { s.t. } \quad \boldsymbol{A} \Delta=0, \quad \text { and } \quad\|\boldsymbol{P} \Delta-\boldsymbol{z}\|_{\infty} \leq \frac{1}{2 M} \tag{3}
\end{align*}
$$

Here, $\boldsymbol{z}$ is a vector that depends on $\boldsymbol{x}$, and is defined as

$$
\boldsymbol{z}_{i}= \begin{cases}\left.\left(-\frac{1}{2 M}+R+(\boldsymbol{P} \boldsymbol{x})_{i}\right) \in\left[-\frac{1}{2 M}, 0\right)\right], & \text { if }(\boldsymbol{P} \boldsymbol{x})_{i}-\frac{1}{2 M}<-R \\ \left(-R+(\boldsymbol{P} \boldsymbol{x})_{i}+\frac{1}{2 M}\right) \in\left(0, \frac{1}{2 M}\right], & \text { if }(\boldsymbol{P} \boldsymbol{x})_{i}+\frac{1}{2 M}>R \\ 0, & \text { otherwise }\end{cases}
$$

We note that any solution $\Delta$ satisfying the above box constraint satisfies $\|\boldsymbol{P} \Delta\|_{\infty} \leq M^{-1}$ and $\left\|\boldsymbol{P} \boldsymbol{x}-e^{-2} \boldsymbol{P} \Delta\right\|_{\infty} \leq R$.
Lemma 4.2. [Iterative Refinement] Let $\boldsymbol{f}$ be $M$-q.s.c. and $\widetilde{\Delta}^{(t)}$ a $\kappa$-approximate solution to the residual problem at $\boldsymbol{x}^{(t)}$ (Problem (3). Starting from $\boldsymbol{x}^{(0)}$ such that $\boldsymbol{A} \boldsymbol{x}^{(0)}=\boldsymbol{b},\left\|\boldsymbol{x}^{(0)}\right\|_{\infty} \leq R$, and iterating as $\boldsymbol{x}^{(t+1)}=\boldsymbol{x}^{(t)}-e^{-2} \widetilde{\Delta}^{(t)}$, after at most $O\left(\kappa M R \log \left(\frac{\left.f\left(\boldsymbol{x}^{(0)}\right)-\boldsymbol{f (} \boldsymbol{x}^{\star}\right)}{\epsilon}\right)\right)$ iterations we get $\boldsymbol{x}$ such that $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}$ and $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)+\epsilon$.

### 4.1 Approximately Solving the Residual Problem

## Binary Search

Lemma 4.3. Let $\nu$ be such that $\boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right) \in(\nu / 2, \nu]$ and $\Delta^{\star}$ denote the optimum of the residual problem at $\boldsymbol{x}^{(t)}$. Then, $\operatorname{res}\left(\Delta^{\star}\right) \in\left(\frac{\nu}{8 M R}, e^{2} \nu\right]$.

From the above lemma we may do a binary search in the range $\left(\frac{\nu}{8 M R}, e^{2} \nu\right]$. Let us start with the assumption that the residual problem has a solution between $(\zeta / 2, \zeta]$.
Lemma 4.4. Let $\zeta$ be such that $\operatorname{res}\left(\Delta^{\star}\right) \in(\zeta / 2, \zeta]$ and $\Delta^{\star}$ the optimum of the residual problem. Then, $\left(\boldsymbol{P} \Delta^{\star}\right)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta^{\star} \leq e \cdot \zeta$.

## Using Width Reduction

We will show that Algorithm 3 returns $\Delta$ such that $\|\boldsymbol{P} \Delta-\boldsymbol{z}\|_{\infty} \leq \frac{1}{2 M}$ and $\operatorname{res}(\Delta) \geq \frac{1}{400} \zeta$.

```
Algorithm 2 Boosting to \(\epsilon\)-approximation
    procedure \(\mathrm{QSC}-\mathrm{MIN}\left(\left(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{P}, \boldsymbol{x}_{0}, M, \epsilon\right)\right.\) such that \(\left.\boldsymbol{A} \boldsymbol{x}_{0}=\boldsymbol{b},\left\|\boldsymbol{x}_{0}-\boldsymbol{x}^{\star}\right\|_{\infty} \leq 2 R\right)\)
        \(\boldsymbol{x}^{(0)}=\boldsymbol{x}_{0}, \tau \leftarrow m^{1 / 3}, \alpha \leftarrow m^{-1 / 3}\)
        for \(i \leq O\left(M R \log \epsilon^{-1}\right)\) do
            for \(\nu \in(\epsilon, \boldsymbol{f}(\boldsymbol{x})]\) do \(\quad \triangleright\) Decrease \(\nu\) by 2 in each iteration
                    for \(\zeta \in\left(\frac{\nu}{8 M R}, e^{2} \nu\right]\) do \(\quad \triangleright\) Decrease \(\zeta\) by 2 in each iteration
                    \(\boldsymbol{y}_{\zeta, \nu} \leftarrow M W U\left(\boldsymbol{A}, \boldsymbol{P}, \boldsymbol{x}^{(i)}, M, \zeta\right)\)
        \(\boldsymbol{x}^{(i+1)} \leftarrow \boldsymbol{x}^{(i)}-e^{-2} \arg \min _{\boldsymbol{y}_{\zeta, \nu}} \boldsymbol{f}\left(\boldsymbol{x}-e^{-2} \boldsymbol{y}_{\zeta, \nu}\right)\)
```

```
Algorithm 3
    procedure \(\operatorname{MWU}(\boldsymbol{A}, \boldsymbol{P}, \boldsymbol{x}, M, \zeta)\)
        \(\boldsymbol{y}^{(0)}=0, \boldsymbol{w}^{(0)}=\frac{\zeta}{m}\)
        \(t=0\)
        \(\boldsymbol{A}^{\prime}=\left[\boldsymbol{A}^{\top}, \boldsymbol{P}^{\top} \nabla \boldsymbol{f}(\boldsymbol{x})\right]^{\top}, \boldsymbol{b}=\left[0, \frac{\zeta}{2}\right]\)
        while \(\|\boldsymbol{w}\|_{1} \leq 10 \zeta\) do
            \(\widetilde{\Delta} \leftarrow \arg \min _{\boldsymbol{A}^{\prime} \Delta=\boldsymbol{b}^{\prime}} \sum_{j} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{x}_{j}\right)(\boldsymbol{P} \Delta)_{j}^{2}+4 M^{2} \sum_{j}\left(\boldsymbol{w}_{j}^{(t)}+\frac{\left\|\boldsymbol{w}^{(t)}\right\|_{1}}{m}\right)(\boldsymbol{P} \Delta-\boldsymbol{z})_{j}^{2}\)
            if \(2 M\|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}\|_{\infty} \leq \tau\) then \(\quad \triangleright\) Flow Step
                    \(\boldsymbol{y}^{(t+1)} \leftarrow \boldsymbol{y}^{(t)}+\widetilde{\Delta}\)
                    \(\boldsymbol{w}^{(t+1)} \leftarrow \boldsymbol{w}^{(t)}\left(1+\frac{1}{2} \alpha M|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}|\right)\)
                else
                    for Indices \(i\) such that \(2 M|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}|_{i} \geq \tau\) do
                    \(\boldsymbol{w}_{i}^{(t+1)} \leftarrow 2 \boldsymbol{w}_{i}^{(t)} \quad \triangleright\) Width Step
            \(t \leftarrow t+1\)
        return \(\frac{y^{(t)}}{100 t}\)
```

Lemma 4.5. Let $\zeta$ be such that res $\left(\Delta^{\star}\right) \in(\zeta / 2, \zeta]$. Algorithm 3 returns $\boldsymbol{y}$ such that $\boldsymbol{A} \boldsymbol{y}=0$, $\|\boldsymbol{P} \boldsymbol{y}-\boldsymbol{z}\|_{\infty} \leq \frac{1}{2 M}$ and $\operatorname{res}(\boldsymbol{y}) \geq \frac{1}{400} \operatorname{res}\left(\Delta^{\star}\right)$ in $O\left(m^{1 / 3}\right)$ calls to a linear system solver.

We now state the main result of the section which follows directly from Lemmas 4.2, 4.3 and 4.5,
Theorem 4.6. For $\epsilon>0$, M-q.s.c. function $\boldsymbol{f}$ and, $\boldsymbol{x}^{(0)}$ such that $\boldsymbol{A} \boldsymbol{x}^{(0)}=\boldsymbol{b}$, $\left\|\boldsymbol{x}^{(0)}\right\|_{\infty} \leq R$, Algorithm 2 finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\boldsymbol{f}(\widetilde{\boldsymbol{x}})-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right) \leq \epsilon$ in $\widetilde{O}\left(M R m^{1 / 3} \log (M R) \log \left(\frac{f\left(\boldsymbol{x}^{(0)}\right)-f\left(x^{\star}\right)}{\epsilon}\right)\right)$ calls to a linear system solver.

## 5 Applications

We now show how our methods may be applied to various quasi-self-concordant functions.

### 5.1 Sum of Exponentials, Softmax and $\ell_{\infty}$-regression

We recall the softmax function $\operatorname{smax}_{\nu}(\boldsymbol{P} \boldsymbol{x})=\nu \log \left(\sum_{i} e^{\frac{(\boldsymbol{P} \boldsymbol{x})_{i}}{\nu}}\right)$, which we may note is $1 / \nu$-q.s.c.
We start by assuming that at the optimum, for $R \geq \Omega\left((\log m)^{-1}\right), \operatorname{smax}_{\nu}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right) \leq R$.
We apply Algorithm 1 to $\sum_{i} e^{\frac{(\boldsymbol{P} x)_{i}}{\nu}}$, which is also $1 / \nu$-q.s.c. We can use the following weight update step for the width reduction step: $\boldsymbol{w}_{i}^{(t, k+1)} \leftarrow \boldsymbol{w}_{i}^{(t, k)}+\nu \log (1+\epsilon)$.

Theorem 5.1. Let $\boldsymbol{x}^{\star}$ denote the optimum of $\min _{\boldsymbol{A x}=\boldsymbol{b}} \operatorname{smax}_{\nu}(\boldsymbol{P} \boldsymbol{x})$. Algorithm 1 when applied to the function $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i} e^{\frac{(\boldsymbol{P} x)_{i}}{\nu}}$, returns $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$, and

$$
\operatorname{smax}_{t}(\boldsymbol{P} \widetilde{\boldsymbol{x}}) \leq(1+\widetilde{O}(\nu)) \operatorname{smax}_{\nu}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)
$$

in at most $\widetilde{O}\left(m^{1 / 3} \nu^{-5 / 3}\right)$ calls to a linear system solver.
Proof. We know that $\operatorname{smax}_{\nu}(\boldsymbol{P} \widetilde{\boldsymbol{x}}) \leq\|\boldsymbol{P} \widetilde{\boldsymbol{x}}\|_{\infty}+\nu \log m$. From Lemma 3.3, we have that $\widetilde{\boldsymbol{x}}$ is obtained in at most $\widetilde{O}\left(m^{1 / 3} \nu^{-5 / 3}\right)$ calls to a linear system solver satisfying $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$. Further, we also have, $\|\boldsymbol{P} \widetilde{\boldsymbol{x}}\|_{\infty} \leq M R\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty}=R \frac{\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty}}{\nu}$. We will now bound $\frac{\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty}}{\nu}$. We note that $\Phi\left(\boldsymbol{w}^{(T, K)}\right) \leq \Phi\left(\boldsymbol{w}_{0}\right) e^{1+4 \nu}$. For $\boldsymbol{w}_{0}=0$,

$$
\Phi\left(\boldsymbol{w}^{(T, K)}\right)=\frac{1}{\nu^{2}} \sum_{i} e^{\frac{w_{i}^{(T, K)}}{\nu}}=\Phi\left(\boldsymbol{w}_{0}\right) \sum_{i} e^{\frac{w_{i}^{(T, K)}}{\nu}} \leq \Phi\left(\boldsymbol{w}_{0}\right) e^{1+4 \nu}
$$

Therefore, we must have $\boldsymbol{w}^{(T, K)} \leq(1+4 \nu) \nu$. Our bound is

$$
\operatorname{smax}_{\nu}(\boldsymbol{P} \widetilde{\boldsymbol{x}}) \leq(1+4 \nu) R+\nu \log m \leq(1+\widetilde{O}(\nu)) R
$$

for $R \geq \Omega(1 / \log m)$. We can now do a binary search on $R$ as follows to obtain

$$
\operatorname{smax}_{\nu}(\boldsymbol{P} \widetilde{\boldsymbol{x}}) \leq(1+\widetilde{O}(\nu)) \operatorname{smax}_{\nu}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)
$$

Binary search on $R$ : Let $R_{0}$ denote the value $\left\|\boldsymbol{P} \boldsymbol{x}^{\star}\right\|_{\infty}=R_{0}$. Now, for any $R \geq R_{0}$, we attain an $\widetilde{\boldsymbol{x}}$ which has an objective value at most $R(1+4 \nu)$. For any $R<R_{0}$, as long as $R$ is such that the plane $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}$ has at least one point with infinity norm at most $R$, we will get a feasible solution to our problem. However, the objective value guarantee of $R(1+4 \nu)$ may not hold. Since the optimum is $R_{0}$, the solution returned in such cases must give an objective value larger than $R_{0}$. We can thus do a binary search on $R$ and reach $O(\nu)$ close to the value $R_{0}$. This will require running our algorithm $O\left(\log \left(R_{0} \nu^{-1}\right)\right)$ times. In the end we can return the $\boldsymbol{x}$ which gives the smallest objective values among all these runs.

Theorem 5.2. Let $\boldsymbol{x}^{\star}$ denote the optimum of the $\ell_{\infty}$-regression problem, $\min _{\boldsymbol{A} x=b}\|\boldsymbol{P} \boldsymbol{x}\|_{\infty}$. Algorithm $\square$ when applied to the function $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i}\left(e^{\frac{(P x)_{i}}{\nu}}+e^{\frac{-(P x)_{i}}{\nu}}\right)$ for $\nu=\Omega\left(\frac{\epsilon}{\log m}\right)$, returns $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and

$$
\|\boldsymbol{P} \widetilde{\boldsymbol{x}}\|_{\infty} \leq(1+\epsilon)\left\|\boldsymbol{P} \boldsymbol{x}^{\star}\right\|_{\infty}
$$

in at most $\widetilde{O}\left(m^{1 / 3} \epsilon^{-5 / 3}\right)$ calls to a linear system solve.
Theorem 5.3. For $\delta>0$, let $\overline{\boldsymbol{x}}$ be the solution returned by Algorithm $\square$ (with $\epsilon=1$ ) applied to $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i} e^{\frac{(\boldsymbol{P} \boldsymbol{x})_{i}}{\nu}}$. Now, Algorithm $\left[\right.$ with starting solution $\boldsymbol{x}^{(0)}=\overline{\boldsymbol{x}}$, applied to $\boldsymbol{f}$ finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\sum_{i} e^{\frac{(P \widetilde{x})_{i}}{\nu}} \leq(1+\delta) \sum_{i} e^{\frac{\left(\boldsymbol{P x}^{\star}\right)_{i}}{\nu}}$ in at most $O\left(m^{1 / 3} R^{2} \nu^{-2} \log \left(\frac{m}{\delta}\right)\right)$ calls to $a$ linear system solver.

## $5.2 p$-Norm Regression

We will solve, $\min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\|\boldsymbol{P} \boldsymbol{x}\|_{p}^{p}+\mu\|\boldsymbol{P} \boldsymbol{x}\|_{2}^{2}$, for $p \geq 3$ which is $p \mu^{-1 /(p-2)}$-q.s.c. w.r.t. its argument. We first apply Algorithm 1 to this function and use the returned solution as the starting point of Algorithm 2 We can use the following weight update step for the width reduction step: $\boldsymbol{w}_{i}^{(t, k+1)} \leftarrow(1+\epsilon)^{1 /(p-2)} \boldsymbol{w}_{i}^{(t, k)}$.
Theorem 5.4. For $\delta>0$ and $p \geq 3$, let $\overline{\boldsymbol{x}}$ be the solution returned by Algorithm $\mathbb{\square}$ (with $\epsilon=1)$ applied to $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\|\boldsymbol{P} \boldsymbol{x}\|_{p}^{p}+\mu\|\boldsymbol{P} \boldsymbol{x}\|_{2}^{2}$. Now, Algorithm 2 with starting solution $\boldsymbol{x}^{(0)}=\overline{\boldsymbol{x}}$, applied to $\boldsymbol{f}$ finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\boldsymbol{f}(\boldsymbol{P} \widetilde{\boldsymbol{x}}) \leq \boldsymbol{f}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)+\delta$ in at most $O\left(p^{2} \mu^{-1 /(p-2)} m^{1 / 3} R \log \left(\frac{p m R}{\mu \delta}\right)\right)$ calls to a linear system solver.

### 5.3 Logistic Regression

We consider the function $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i} \log \left(1+e^{(\boldsymbol{P} \boldsymbol{x})_{i}}\right)$ which is 1-q.s.c. w.r.t its argument. We will use Algorithm 1 with the following weight update for the width reduction step which reduces the resistance by a factor of $(1+\epsilon): \boldsymbol{w}_{i}^{(t, k+1)} \leftarrow \boldsymbol{w}_{i}^{(t, k)}+0.9 \epsilon$
Theorem 5.5. For $\delta>0$, let $\overline{\boldsymbol{x}}$ be the solution returned by Algorithm $\square$ (with $\epsilon=1$ ) applied to $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i} \log \left(1+e^{(\boldsymbol{P} \boldsymbol{x})_{i}}\right)$. Now, Algorithm 2 with starting solution $\boldsymbol{x}^{(0)}=\overline{\boldsymbol{x}}$, applied to $\boldsymbol{f}$ finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\sum_{i} \log \left(1+e^{(\boldsymbol{P} \widetilde{\boldsymbol{x}})_{i}}\right) \leq \sum_{i} \log \left(1+e^{\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)_{i}}\right)+\delta$ in at most $O\left(m^{1 / 3} R \log \left(\frac{m R}{\delta}\right)\right)$ calls to a linear system solver.

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## A Algorithm for General-Self-Concordant Functions

In this section we will show how to use our algorithms for the following classes of general-selfconcordant functions.

1. $6>\nu \geq 2$ : $f$ is $(N, \nu)$-g.s.c. and $L$-smooth.
2. $\nu<2$ : $f$ is $(N, \nu)$-g.s.c., $L$-smooth and $\mu$-strongly convex.

We will use the following result to reduce these problems to ( $M, 2$ )-g.s.c. problems and use our algorithms.
Lemma A. 1 (Prop 4. [STD19]). Let $\boldsymbol{f}$ be $(M, \nu)$-g.s.c. with $\nu>0$. Then:
(a) If $\nu \in(0,3]$ and $f$ is also strongly convex with strong convexity parameter $\mu>0$ in $\ell_{2}$-norm, then $\boldsymbol{f}$ is also $\left(\frac{M}{\sqrt{\mu}^{3-\nu}}, 3\right)$-g.s.c.
(b) If $\nu \geq 2$ and $\nabla \boldsymbol{f}$ is Lipschitz continuous with finite Lipschitz constant $L$ in $\ell_{2}$-norm, then $f$ is also $\left(M L^{\frac{\nu}{2}-1}, 2\right)$-g.s.c.

We thus have the following result.
Theorem A.2. For $\delta>0, \boldsymbol{f}(N, \nu)$-g.s.c. $6>\nu \geq 2$ and $L$-smooth, let $\overline{\boldsymbol{x}}$ be the solution returned by Algorithm $\mathbb{\square}$ (with $\epsilon=1$ ) applied to $\boldsymbol{f}(\boldsymbol{x})$. Now, Algorithm 2 with starting solution $\boldsymbol{x}^{(0)}=\overline{\boldsymbol{x}}$, applied to $\boldsymbol{f}$ finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\sum_{i} \boldsymbol{f}\left(\boldsymbol{P} \widetilde{\boldsymbol{x}}_{i}\right) \leq \sum_{i} \boldsymbol{f}\left(\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)+\delta$ in at most

$$
O\left(m^{1 / 3} N L^{\frac{\nu-2}{2}} R \log \left(\frac{\left.\boldsymbol{f}\left(\boldsymbol{x}^{(0)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)\right)}{\delta}\right)\right)
$$

calls to a linear system solver.
Proof. From LemmaA. 1 f is $\left(N L^{(\nu-2) / 2}, 2\right)$-g.s.c. We now use Lemma 3.3 with $M=N L^{(\nu-2) / 2}$ followed by Theorem4.6.

Theorem A.3. For $\delta>0, \boldsymbol{f}(N, \nu)$-g.s.c. $2>\nu \geq 0$ and $L$-smooth $\mu$-strongly convex, let $\overline{\boldsymbol{x}}$ be the solution returned by Algorithm $\square$ (with $\epsilon=1$ ) applied to $\boldsymbol{f}(\boldsymbol{x})$. Now, Algorithm 2 with starting solution $\boldsymbol{x}^{(0)}=\overline{\boldsymbol{x}}$, applied to $\boldsymbol{f}$ finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\sum_{i} \boldsymbol{f}\left(\boldsymbol{P} \widetilde{\boldsymbol{x}}_{i}\right) \leq \sum_{i} \boldsymbol{f}\left(\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)+\delta$ in at most

$$
O\left(m^{1 / 3} N \mu^{-\frac{3-\nu}{2}} L^{1 / 2} R \log \left(\frac{\left.\boldsymbol{f}\left(\boldsymbol{x}^{(0)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)\right)}{\delta}\right)\right)
$$

calls to a linear system solver.
Proof. From Lemma A. $1 f$ is $\left(N \mu^{-\frac{3-\nu}{2}} L^{1 / 2}, 2\right)$-g.s.c. We now use Lemma 3.3with $M=$ $N \mu^{-\frac{3-\nu}{2}} L^{1 / 2}$ followed by Theorem4.6

## B Missing Proofs

## B. 1 Proofs from Section 2

Definition B.1. [Hessian Stability] For distance $r \in \mathbb{R}_{\geq 0}$ and function $\boldsymbol{d}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ acting on $r$, a function $\boldsymbol{f}$ is $(r, \boldsymbol{d}(r))$-hessian stable w.r.t. a norm $\|\cdot\|$ if for all $\boldsymbol{x}, \boldsymbol{y}$ such that $\|\boldsymbol{x}-\boldsymbol{y}\| \leq r$,

$$
\frac{1}{\boldsymbol{d}(r)} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \preceq \nabla^{2} \boldsymbol{f}(\boldsymbol{y}) \preceq \boldsymbol{d}(r) \nabla^{2} \boldsymbol{f}(\boldsymbol{x})
$$

Lemma B. 2 (Lemma $11\left[\mathrm{CJJ}^{+} 20 \mid\right.$ ). If $\boldsymbol{f}$ is a univariate $M$-quasi-self-concordant (q.s.c.) function, then $\boldsymbol{f}(\boldsymbol{x})=\sum_{i} \boldsymbol{f}\left(\boldsymbol{x}_{i}\right)$ is $\left(r, e^{M r}\right)$ hessian stable in the $\ell_{\infty}$-norm.

Lemma 2.4. For $\epsilon>0$, resistances $\boldsymbol{r}$ (Definition 2.3]), with corresponding weights $\boldsymbol{w}$, we have

$$
\Psi(\boldsymbol{r}) \leq(1+\epsilon) \Phi(\boldsymbol{w})
$$

In addition, letting $\|\boldsymbol{P}\|_{\min }=\min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}}\|\boldsymbol{P} \boldsymbol{x}\|_{2}$ and $\|\boldsymbol{A}\|$ denote the operator norm of $\boldsymbol{A}$, we have

$$
\Psi(\boldsymbol{r}) \geq \frac{\epsilon \Phi(\boldsymbol{w})}{m R^{2}} \frac{\|\boldsymbol{P}\|_{\min }^{2}\|\boldsymbol{b}\|_{2}^{2}}{\|\boldsymbol{A}\|^{2}} \stackrel{\operatorname{def}}{=} \Phi(\boldsymbol{w}) L
$$

Proof. Let $\widetilde{\Delta}$ be the minimizer of $\Psi(\boldsymbol{r})$ and $\boldsymbol{x}^{\star}$ be the optimum of (11).

$$
\begin{aligned}
\Psi(\boldsymbol{r}) & =\sum_{i} \boldsymbol{r}_{i}(\boldsymbol{P} \widetilde{\Delta})_{i}^{2} \leq \sum_{i} \boldsymbol{r}_{i}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)_{i}^{2} \\
& =\sum_{i}\left(\boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)+\frac{\epsilon \Phi(\boldsymbol{w})}{m}\right) \frac{\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)_{i}^{2}}{R^{2}} \\
& \leq \sum_{i} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)+\frac{\epsilon \Phi(\boldsymbol{w})}{m} \cdot m, \\
& =\Phi(\boldsymbol{w})(1+\epsilon)
\end{aligned}
$$

We next look at a lower bound for $\Psi$. We note that, any solution to the oracle must satisfy $\boldsymbol{A} \widetilde{\Delta}=\boldsymbol{b}$. This implies, $\|\boldsymbol{A}\|\|\widetilde{\Delta}\|_{2} \geq\|\boldsymbol{b}\|_{2}$, where $\|\cdot\|$ denotes the operator norm. Now,

$$
\Psi(\boldsymbol{r}) \geq \frac{\epsilon \Phi(\boldsymbol{w})}{m R^{2}}\|\boldsymbol{P} \widetilde{\Delta}\|_{2}^{2} \geq \frac{\epsilon \Phi(\boldsymbol{w})}{m R^{2}}\|\boldsymbol{P}\|_{\min }^{2}\|\widetilde{\Delta}\|_{2}^{2} \geq \frac{\epsilon \Phi(\boldsymbol{w})}{m R^{2}} \frac{\|\boldsymbol{P}\|_{\min }^{2}\|\boldsymbol{b}\|_{2}^{2}}{\|\boldsymbol{A}\|^{2}}
$$

## Lemma B.3.

$$
\sum_{i} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)|\boldsymbol{P} \widetilde{\Delta}|_{i} \leq(1+\epsilon) R \Phi(\boldsymbol{w})
$$

Proof.

$$
\begin{array}{rlrl}
\sum_{i} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)|\boldsymbol{P} \widetilde{\Delta}|_{i} & \leq \sqrt{\sum_{i} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right) \sum_{i} f^{\prime \prime}\left(\boldsymbol{w}_{i}\right)|\boldsymbol{P} \widetilde{\Delta}|_{i}^{2}} & & \text { Cauchy Schwarz } \\
& \leq \sqrt{\Phi(\boldsymbol{w})} \sqrt{R^{2} \Psi(\boldsymbol{r})} & & \\
& \leq R \sqrt{\Phi(\boldsymbol{w})} \sqrt{(1+\epsilon) \Phi(\boldsymbol{w})} & & \text { From Lemman2.4 } \\
& =R(1+\epsilon) \Phi(\boldsymbol{w}) &
\end{array}
$$

## B. 2 Proofs from Section 3

## Change in $\Psi$

Lemma 3.1. Let $\Psi$ be as defined in 2.3 After $t$ flow steps and $k$ width reduction steps, we have,

$$
\begin{array}{ll}
\Psi\left(\boldsymbol{r}^{(t, k)}\right) \geq \Psi\left(\boldsymbol{r}^{(0,0)}\right)\left(1+\frac{\epsilon^{2} \tau^{2}}{(1+\epsilon)^{2} m}\right)^{k} & \text { if } \boldsymbol{f}^{\prime \prime} \text { non-decreasing in } \boldsymbol{w}, \\
\Psi\left(\boldsymbol{r}^{(t, k)}\right) \leq \Psi\left(\boldsymbol{r}^{(0,0)}\right)\left(1-\frac{\epsilon^{2} \tau^{2}}{2(1+\epsilon)^{2} m}\right)^{k} & \text { if } \boldsymbol{f}^{\prime \prime} \text { non-increasing in } \boldsymbol{w} .
\end{array}
$$

Proof. We show this by induction. It is clear that this holds for $t=k=0$. We know from Lemma C.2. for $\boldsymbol{r}^{\prime} \geq \boldsymbol{r}$,

$$
\Psi\left(\boldsymbol{r}^{\prime}\right) \geq \Psi(\boldsymbol{r})+\sum_{i}\left(1-\frac{\boldsymbol{r}_{i}}{\boldsymbol{r}_{i}^{\prime}}\right) \boldsymbol{r}_{i}(\boldsymbol{P} \widetilde{\Delta})_{i}^{2}
$$

Since the weights are only increasing, this corresponds to the case $f^{\prime \prime}$ is an increasing function. Similarly, when $f^{\prime \prime}$ is a non-increasing function, we have the following bound: for $\boldsymbol{r}^{\prime} \leq \boldsymbol{r}$ from LemmaC. 1

$$
\Psi\left(\boldsymbol{r}^{\prime}\right) \leq \Psi(\boldsymbol{r})-\frac{1}{2} \sum_{i}\left(1-\frac{\boldsymbol{r}_{i}^{\prime}}{\boldsymbol{r}_{i}}\right) \boldsymbol{r}_{i}(\boldsymbol{P} \widetilde{\Delta})_{i}^{2}
$$

We first consider a flow step. We note that our weights $\boldsymbol{w}$ are increasing, and if $\boldsymbol{f}^{\prime \prime}$ is increasing then $\boldsymbol{r}^{(t+1)} \geq \boldsymbol{r}^{(t)}$. Similarly if $\boldsymbol{f}^{\prime \prime}$ is decreasing, $\boldsymbol{r}^{(t+1, k)} \leq \boldsymbol{r}^{(t, k)}$. We can use the above relations to now get $\Psi\left(\boldsymbol{r}^{(t+1, k)}\right) \geq \Psi\left(\boldsymbol{r}^{(t, k)}\right)$ for the first case and $\Psi\left(\boldsymbol{r}^{(t+1, k)}\right) \leq \Psi\left(\boldsymbol{r}^{(t, k)}\right)$ for the second. We next consider a width reduction step. Let $i$ be one edge that has $\left|\boldsymbol{P} \widetilde{\Delta}_{i}\right| \geq R \tau$. We have,

$$
\boldsymbol{r}_{i}^{(t, k)}(\boldsymbol{P} \widetilde{\Delta})_{i}^{2} \geq \frac{\epsilon \Phi\left(\boldsymbol{w}^{(t, k)}\right)}{R^{2} m}|\boldsymbol{P} \widetilde{\Delta}|_{i}^{2} \geq \frac{\epsilon \Phi\left(\boldsymbol{w}^{(t, k)}\right)}{R^{2} m} R^{2} \tau^{2} \geq \frac{\epsilon \tau^{2}}{(1+\epsilon) m} \Psi\left(\boldsymbol{r}^{(t, k)}\right)
$$

where the last inequality follows from Lemma 2.4 Now, since we are changing our resistances by a factor of $(1+\epsilon)$, we get the following bounds for the two cases,

$$
\begin{aligned}
& \Psi\left(\boldsymbol{r}^{(t, k+1)}\right) \geq \Psi\left(\boldsymbol{r}^{(t, k)}\right)+\left(1-\frac{\boldsymbol{r}_{i}}{(1+\epsilon) \boldsymbol{r}_{i}}\right) \frac{\epsilon \tau^{2}}{(1+\epsilon) m} \Psi\left(\boldsymbol{r}^{(t, k)}\right)=\Psi\left(\boldsymbol{r}^{(t, k)}\right)\left(1+\frac{\epsilon^{2} \tau^{2}}{(1+\epsilon)^{2} m}\right) \\
& \Psi\left(\boldsymbol{r}^{(t, k+1)}\right) \leq \Psi\left(\boldsymbol{r}^{(t, k)}\right)-\frac{1}{2}\left(1-\frac{\boldsymbol{r}_{i} /(1+\epsilon)}{\boldsymbol{r}_{i}}\right) \frac{\epsilon \tau^{2}}{(1+\epsilon) m} \Psi\left(\boldsymbol{r}^{(t, k)}\right)=\Psi\left(\boldsymbol{r}^{(t, k)}\right)\left(1-\frac{\epsilon^{2} \tau^{2}}{2(1+\epsilon)^{2} m}\right)
\end{aligned}
$$

With these two relations we conclude our proof.

## Change in $\Phi$

Lemma 3.2. Suppose $\boldsymbol{f}$ is $M$-q.s.c. Let $\alpha$ and $\tau$ be such that $\alpha \tau \leq M^{-1}$. After $t$ flow steps and $k$ width reduction steps, our potential $\Phi$ satisfies

$$
\begin{aligned}
& \Phi\left(\boldsymbol{w}^{(t, k)}\right) \leq\left(1+\epsilon(1+\epsilon)^{2} \alpha M\right)^{t}\left(1+\epsilon(1+\epsilon) \tau^{-1}\right)^{k} \Phi\left(\boldsymbol{w}_{0}\right) \quad \text { if } \boldsymbol{f}^{\prime \prime} \text { non-decreasing in } \boldsymbol{w} \\
& \Phi\left(\boldsymbol{w}^{(t, k)}\right) \geq\left(1-\epsilon(1+\epsilon)^{2} \alpha M\right)^{t}\left(1-\epsilon(1+\epsilon) \tau^{-1}\right)^{k} \Phi\left(\boldsymbol{w}_{0}\right) \quad \text { if } \boldsymbol{f}^{\prime \prime} \text { non-increasing in } \boldsymbol{w} .
\end{aligned}
$$

Proof. We first show the case when $f^{\prime \prime}$ is increasing. The same calculation will work for the other case too by just considering the sign of $\Phi^{\prime}$.
We will use induction. It is easy to see the claim holds for the initial iteration, $t=k=0$. We next assume that it holds for some $\boldsymbol{w}^{(t, k)}$. If the next step is a flow step, we update to $\boldsymbol{w}^{(t+1, k)} \leq$ $\boldsymbol{w}^{(t, k)}+\epsilon \alpha \tau$. Since $\alpha \tau \leq M^{-1}$, we have that $\Phi$ is $\left(M^{-1}, e^{\epsilon}\right)$ hessian stable around this update. We will use $\boldsymbol{w}$ to denote $\boldsymbol{w}^{(t, k)}$ for simplicity. We thus have,

$$
\begin{aligned}
\Phi\left(\boldsymbol{w}^{(t+1)}\right) & =\Phi\left(\boldsymbol{w}+\frac{\epsilon \alpha}{R}|\boldsymbol{P} \widetilde{\Delta}|\right) \\
& =\Phi(\boldsymbol{w})+\frac{\epsilon \alpha}{R} \nabla \Phi(\boldsymbol{y})^{\top}|\boldsymbol{P} \widetilde{\Delta}|
\end{aligned}
$$

(For some $\boldsymbol{y}$ between $\boldsymbol{w}$ and $\boldsymbol{w}+\alpha|\boldsymbol{P} \Delta|$ )

$$
\begin{aligned}
& =\Phi(\boldsymbol{w})+\frac{\epsilon \alpha}{R} \sum_{i} \boldsymbol{f}^{\prime \prime \prime}\left(\boldsymbol{y}_{i}\right)|\boldsymbol{P} \widetilde{\Delta}|_{i} \\
& \leq \Phi(\boldsymbol{w})+\frac{\epsilon \alpha}{R} M \sum_{i} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{y}_{i}\right)|\boldsymbol{P} \widetilde{\Delta}|_{i}
\end{aligned}
$$

(Since $f$ is $M$-q.s.c.)

$$
\leq \Phi(\boldsymbol{w})+\frac{\epsilon \alpha}{R} M e^{\epsilon} \sum_{i} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)|\boldsymbol{P} \widetilde{\Delta}|_{i}
$$

(Since $f$ is hessian stable in this range)

$$
\leq \Phi(\boldsymbol{w})+\epsilon(1+\epsilon)^{2} \alpha M \Phi(\boldsymbol{w})
$$

(From LemmaB.3)

We thus get the following bound,

$$
\Phi\left(\boldsymbol{w}^{(t+1, k)}\right) \leq \Phi\left(\boldsymbol{w}^{(t, k)}\right)\left(1+\epsilon(1+\epsilon)^{2} \alpha M\right)
$$

Now, suppose the next step is a width reduction step.

$$
\begin{aligned}
\Phi\left(\boldsymbol{w}^{(t, k+1)}\right) & =\sum_{i \notin \mathcal{I}} \Phi\left(\boldsymbol{w}_{i}\right)+\sum_{i \in \mathcal{I}} \Phi\left(\boldsymbol{w}_{i}^{(t+1)}\right) \\
& =\sum_{i \notin \mathcal{I}} \Phi\left(\boldsymbol{w}_{i}\right)+\sum_{i \in \mathcal{I}} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}^{(t+1)}\right) \\
& \leq \sum_{i \notin \mathcal{I}} \Phi\left(\boldsymbol{w}_{i}\right)+(1+\epsilon) \sum_{i \in \mathcal{I}} f^{\prime \prime}\left(\boldsymbol{w}_{i}\right) \\
& \leq \Phi(\boldsymbol{w})+\frac{\epsilon}{R \tau} \sum_{i \in \mathcal{I}} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)|\boldsymbol{P} \widetilde{\Delta}|_{i} \\
& \leq \Phi(\boldsymbol{w})+\frac{\epsilon}{R \tau} \sum_{i} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{w}_{i}\right)|\boldsymbol{P} \widetilde{\Delta}|_{i} \\
& \leq \Phi(\boldsymbol{w})+\frac{\epsilon(1+\epsilon)}{\tau} \Phi(\boldsymbol{w})
\end{aligned}
$$

From LemmaB. 3
We thus get the following bound,

$$
\Phi\left(\boldsymbol{w}^{(t, k+1)}\right) \leq \Phi\left(\boldsymbol{w}^{(t, k)}\right)\left(1+\epsilon(1+\epsilon) \tau^{-1}\right)
$$

## B. 3 Proofs from Section 4

## Iterative Refinement

Lemma B.4. Let $\boldsymbol{f}$ be a $(r, d(r))$-hessian stable function in $\ell_{\infty}$-norm, and $\widetilde{\boldsymbol{x}}=\boldsymbol{x}+\Delta$ such that $\|\Delta\|_{\infty} \leq r$. We then have,

$$
\frac{1}{d(r)} \Delta^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \Delta \leq \boldsymbol{f}(\widetilde{\boldsymbol{x}})-\boldsymbol{f}(\boldsymbol{x})-\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \Delta \leq d(r) \Delta^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \Delta
$$

Proof. We have for some $\boldsymbol{z}$ along the line joining $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$,

$$
\boldsymbol{f}(\widetilde{\boldsymbol{x}})=\boldsymbol{f}(\boldsymbol{x})+\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \Delta+\Delta^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{z}) \Delta
$$

Since $\|\boldsymbol{z}-\boldsymbol{x}\|_{\infty} \leq\|\widetilde{\boldsymbol{x}}-\boldsymbol{x}\|_{\infty} \leq r$, from hessian stability, we have,

$$
\frac{1}{d(r)} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \preceq \nabla^{2} \boldsymbol{f}(\boldsymbol{z}) \preceq d(r) \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) .
$$

Using this relation in the above, we get our lemma.
Lemma B.5. Let $\Delta$ be any feasible solution to the residual problem at $\boldsymbol{x}$. We then have,

$$
\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{x}-\Delta) \leq \operatorname{res}(\Delta), \quad \boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}\left(\boldsymbol{x}-e^{-2} \Delta\right) \geq e^{-2} \cdot \operatorname{res}(\Delta)
$$

Proof. Since our function is $M$-q.s.c., from Lemmas B. 4 and B.2, for all $\Delta$ such that $\|\boldsymbol{P} \Delta\|_{\infty} \leq$ $M^{-1}$,

$$
e^{-1}(\boldsymbol{P} \Delta)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta \leq \boldsymbol{f}(\boldsymbol{x}-\Delta)-\boldsymbol{f}(\boldsymbol{x})+\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \boldsymbol{P} \Delta \leq e(\boldsymbol{P} \Delta)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta
$$

The first bound directly follows from the left inequality. For the second bound, we first note that $e^{-2}\|\boldsymbol{P} \Delta\| \leq M^{-1}$. We can now use the right inequality.

$$
\begin{aligned}
\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}\left(\boldsymbol{x}-e^{-2} \Delta\right) & \geq e^{-2} \nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \boldsymbol{P} \Delta-e^{-3}(\boldsymbol{P} \Delta)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta \\
& =e^{-2}\left(\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \boldsymbol{P} \Delta-e^{-1}(\boldsymbol{P} \Delta)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta\right) \\
& =e^{-2} \operatorname{res}(\Delta) .
\end{aligned}
$$

Lemma B.6. Assume $f$ is $M-q . s . c . ~ L e t ~ \boldsymbol{x}^{\star}$ denote the minimizer of Problem (1) and $\Delta^{\star}$ the optimizer of Problem (3) at $\boldsymbol{x}^{(t)}$. We then have,

$$
\operatorname{res}\left(\Delta^{\star}\right) \geq \frac{1}{4 M R}\left(\boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)\right)
$$

Proof. Let $\boldsymbol{x}^{(t)}$ be such that $\boldsymbol{A} \boldsymbol{x}^{(t)}=\boldsymbol{b}$ and $\boldsymbol{x}^{\star}$ is the optimum of (11). Note that we have $\left\|\boldsymbol{P} \boldsymbol{x}^{(t)}\right\|_{\infty} \leq R$ and therefore, $\left\|\boldsymbol{P} \boldsymbol{x}^{(t)}-\boldsymbol{P} \boldsymbol{x}^{\star}\right\|_{\infty} \leq 2 R$. Let $r=\frac{1}{2 M}$ and $\boldsymbol{x}=\left(1-\frac{r}{2 R}\right) \boldsymbol{x}^{(t)}+$ $\frac{r}{2 R} \boldsymbol{x}^{\star}$. Let $\widetilde{\Delta}=\boldsymbol{x}^{(t)}-\boldsymbol{x}=\frac{r}{2 R}\left(\boldsymbol{x}^{(t)}-\boldsymbol{x}^{\star}\right)$. We have,

$$
\|\boldsymbol{P} \widetilde{\Delta}\|_{\infty}=\left\|\boldsymbol{P} \boldsymbol{x}^{(t)}-\boldsymbol{P} \boldsymbol{x}\right\|_{\infty}=\frac{r}{2 R}\left\|\boldsymbol{P} \boldsymbol{x}^{(t)}-\boldsymbol{P} \boldsymbol{x}^{\star}\right\|_{\infty} \leq r,
$$

and

$$
\boldsymbol{A} \widetilde{\Delta}=\boldsymbol{A}\left(\boldsymbol{x}^{(t)}-\boldsymbol{x}\right)=\frac{r}{2 R}\left(-\boldsymbol{A} \boldsymbol{x}^{\star}+\boldsymbol{A} \boldsymbol{x}^{(t)}\right)=0
$$

We next show that $\|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}\|_{\infty} \leq \frac{1}{2 M}$.

$$
\|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}\|_{\infty}=\left\|\frac{r}{2 R} \boldsymbol{P} \boldsymbol{x}^{(t)}-\frac{r}{2 R} \boldsymbol{P} \boldsymbol{x}^{\star}-\boldsymbol{z}\right\|_{\infty}
$$

We will do a case by case analysis. Consider some coordinate $i$.

1. $\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}-\frac{1}{2 M}<-R$ : From the definition of $\boldsymbol{z}_{i}$, we note that $\boldsymbol{z}_{i}=R-\frac{1}{2 M}+\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}$ and $-R<\boldsymbol{P} \boldsymbol{x}_{i}^{(t)} \leq-R+\frac{1}{2 M}$. Suppose $\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}=-R+a$ for some $0 \leq a<\frac{1}{2 M}$. We have,

$$
\begin{aligned}
|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}|_{i} & =\left|\frac{r}{2 R}\left(\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}-\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)-\boldsymbol{z}_{i}\right| \\
& =\left|\frac{r}{2 R}\left(-R+a-\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)-a+\frac{1}{2 M}\right| \\
& =\left|\frac{r}{2 R}\left(-R-\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)-a\left(1-\frac{r}{2 R}\right)+\frac{1}{2 M}\right| \\
& \leq \frac{1}{2 M} .
\end{aligned}
$$

The last inequality follows since $-2 R \leq-R-\boldsymbol{P} \boldsymbol{x}_{i}^{\star} \leq 0$.
2. $\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}+\frac{1}{2 M}>R$ : From the definition of $\boldsymbol{z}_{i}$, we note that $\boldsymbol{z}_{i}=-R+\frac{1}{2 M}+\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}$ and $R-\frac{1}{2 M}<\boldsymbol{P} \boldsymbol{x}_{i}^{(t)} \leq R$. Suppose $\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}=R-a$ for some $0 \leq a<\frac{1}{2 M}$. We have,

$$
\begin{aligned}
|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}|_{i} & =\left|\frac{r}{2 R}\left(\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}-\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)-\boldsymbol{z}_{i}\right| \\
& =\left|\frac{r}{2 R}\left(R-a-\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)+a-\frac{1}{2 M}\right| \\
& =\left|\frac{r}{2 R}\left(R-\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)+a\left(1-\frac{r}{2 R}\right)-\frac{1}{2 M}\right| \\
& \leq \frac{1}{2 M} .
\end{aligned}
$$

The last inequality follows since $0 \leq R-\boldsymbol{P} \boldsymbol{x}_{i}^{\star} \leq 2 R$.
3. $-R+\frac{1}{2 M} \leq \boldsymbol{P} \boldsymbol{x}_{i}^{(t)} \leq-\frac{1}{2 M} R$ : In this case $\boldsymbol{z}_{i}=0$.

$$
|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}|_{i}=\left|\frac{r}{2 R}\left(\boldsymbol{P} \boldsymbol{x}_{i}^{(t)}-\boldsymbol{P} \boldsymbol{x}_{i}^{\star}\right)\right| \leq r=\frac{1}{2 M} .
$$

We thus conclude, that $\boldsymbol{x}-\boldsymbol{x}^{(t)}$ is a feasible solution for the residual problem and from convexity,

$$
\frac{r}{2 R}\left(\boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)\right) \leq \boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}(\boldsymbol{x}) .
$$

Let $\Delta^{\star}$ denote the optimum of the residual problem at $\boldsymbol{x}^{(t)}$ (3). From Lemma B.5,

$$
\frac{r}{2 R}\left(\boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)\right) \leq \boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}(\boldsymbol{x}) \leq \operatorname{res}\left(\boldsymbol{x}^{(t)}-\boldsymbol{x}\right) \leq \operatorname{res}\left(\Delta^{\star}\right)
$$

Lemma 4.2. [Iterative Refinement] Let $\boldsymbol{f}$ be $M$-q.s.c. and $\widetilde{\Delta}^{(t)}$ a $\kappa$-approximate solution to the residual problem at $\boldsymbol{x}^{(t)}$ (Problem (3). Starting from $\boldsymbol{x}^{(0)}$ such that $\boldsymbol{A} \boldsymbol{x}^{(0)}=\boldsymbol{b},\left\|\boldsymbol{x}^{(0)}\right\|_{\infty} \leq R$, and iterating as $\boldsymbol{x}^{(t+1)}=\boldsymbol{x}^{(t)}-e^{-2} \widetilde{\Delta}^{(t)}$, after at most $O\left(\kappa M R \log \left(\frac{f\left(\boldsymbol{x}^{(0)}\right)-f\left(\boldsymbol{x}^{\star}\right)}{\epsilon}\right)\right)$ iterations we get $\boldsymbol{x}$ such that $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}$ and $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)+\epsilon$.

Proof. From LemmaB.6,

$$
\operatorname{res}\left(\widetilde{\Delta}^{(t)}\right) \geq \frac{1}{\kappa} \operatorname{res}\left(\Delta^{\star}\right) \geq \frac{1}{4 \kappa M R}\left(\boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)\right)
$$

Now, from LemmaB. 5

$$
\boldsymbol{f}\left(\boldsymbol{x}^{(t+1)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right) \leq \boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)-e^{-2} \operatorname{res}\left(\widetilde{\Delta}^{(t)}\right) \leq\left(1-\frac{e^{-2}}{4 \kappa M R}\right)\left(\boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)\right)
$$

Inductively applying the above equation,

$$
\boldsymbol{f}\left(\boldsymbol{x}^{(T)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right) \leq\left(1-\frac{e^{-2}}{4 \kappa M R}\right)^{T}\left(\boldsymbol{f}\left(\boldsymbol{x}^{(0)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right)\right)
$$

## Binary Search

Lemma 4.3. Let $\nu$ be such that $\boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right) \in(\nu / 2, \nu]$ and $\Delta^{\star}$ denote the optimum of the residual problem at $\boldsymbol{x}^{(t)}$. Then, res $\left(\Delta^{\star}\right) \in\left(\frac{\nu}{8 M R}, e^{2} \nu\right]$.

Proof. The lower bound follows formB.6. For the upper bound, fromB.5,

$$
\nu \geq \boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}^{\star}\right) \geq \boldsymbol{f}\left(\boldsymbol{x}^{(t)}\right)-\boldsymbol{f}\left(\boldsymbol{x}-e^{-2} \Delta^{\star}\right) \geq e^{-2} \operatorname{res}\left(\Delta^{\star}\right)
$$

Lemma 4.4. Let $\zeta$ be such that $\operatorname{res}\left(\Delta^{\star}\right) \in(\zeta / 2, \zeta]$ and $\Delta^{\star}$ the optimum of the residual problem. Then, $\left(\boldsymbol{P} \Delta^{\star}\right)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta^{\star} \leq e \cdot \zeta$.

Proof. Consider scaling $\Delta^{\star}$ by $O(1)>\lambda>0$. We must have,

$$
\left[\frac{d}{d \lambda} \operatorname{res}\left(\lambda \Delta^{\star}\right)\right]_{\lambda=1}=0
$$

This implies,

$$
\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \boldsymbol{P} \Delta^{\star}-2 e^{-1}\left(\boldsymbol{P} \Delta^{\star}\right)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta^{\star}=0
$$

or

$$
e^{-1}\left(\boldsymbol{P} \Delta^{\star}\right)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta^{\star}=\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \boldsymbol{P} \Delta^{\star}-e^{-1}\left(\boldsymbol{P} \Delta^{\star}\right)^{\top} \nabla^{2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{P} \Delta^{\star}=\operatorname{res}\left(\Delta^{\star}\right) \leq \zeta
$$

## Width Reduction

Lemma 4.5. Let $\zeta$ be such that res $\left(\Delta^{\star}\right) \in(\zeta / 2, \zeta]$. Algorithm 3 returns $\boldsymbol{y}$ such that $\boldsymbol{A} \boldsymbol{y}=0$, $\|\boldsymbol{P} \boldsymbol{y}-\boldsymbol{z}\|_{\infty} \leq \frac{1}{2 M}$ and $\operatorname{res}(\boldsymbol{y}) \geq \frac{1}{400} \operatorname{res}\left(\Delta^{\star}\right)$ in $O\left(m^{1 / 3}\right)$ calls to a linear system solver.

Proof. This algorithm is basically an implementation of the width-reduced MWU algorithm from $\left[\mathrm{CKM}^{+} 11\right]$. We will give a proof for completeness. For the purpose of this proof, we denote,

$$
\begin{gathered}
\Psi(\boldsymbol{r})=\min _{\boldsymbol{A} \Delta=0, \nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \boldsymbol{P} \Delta=\zeta / 2} \sum_{j}\left(\boldsymbol{f}^{\prime \prime}\left(\boldsymbol{x}_{j}\right)(\boldsymbol{P} \Delta)_{j}^{2}+\sum_{j} 4 M^{2}\left(\boldsymbol{w}_{j}+\frac{\|\boldsymbol{w}\|_{1}}{m}\right)\right)(\boldsymbol{P} \Delta-\boldsymbol{z})_{j}^{2} \\
\Phi(\boldsymbol{w})=\|\boldsymbol{w}\|_{1} .
\end{gathered}
$$

Let $\widetilde{\Delta}$ be the solution returned by $\Psi$. We first note that, for $\Delta^{\star}$ the optimum of the residual problem,

$$
\begin{aligned}
\Psi(\boldsymbol{r}) & \leq \sum_{j}\left(\boldsymbol{f}^{\prime \prime}\left(\boldsymbol{x}_{j}\right)\left(\boldsymbol{P} \Delta^{\star}\right)_{j}^{2}+\sum_{j} 4 M^{2}\left(\boldsymbol{w}_{j}+\frac{\|\boldsymbol{w}\|_{1}}{m}\right)\right)\left(\boldsymbol{P} \Delta^{\star}-\boldsymbol{z}\right)_{j}^{2} \\
& \leq e \cdot \zeta+\sum_{j} 4 M^{2}\left(\boldsymbol{w}_{j}+\frac{\|\boldsymbol{w}\|_{1}}{m}\right)\left(\boldsymbol{P} \Delta^{\star}-\boldsymbol{z}\right)_{j}^{2}, \text { From Lemma4.4 } \\
& \leq e \cdot \zeta+\|\boldsymbol{w}\|_{1}+\Phi(\boldsymbol{w}), \text { Since }\left\|\boldsymbol{P} \Delta^{\star}-\boldsymbol{z}\right\|_{\infty} \leq \frac{1}{2 M} \\
& \leq(e+2) \Phi(\boldsymbol{w})
\end{aligned}
$$

We note that,
$\sum_{j} \boldsymbol{w}_{j}(4 M)(\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z})_{j} \leq \sqrt{\sum_{j} \boldsymbol{w}_{j} \sum_{j} \boldsymbol{w}_{j}(4 M)^{2}(\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z})_{j}^{2}} \leq \sqrt{\Phi(\boldsymbol{w}) \Psi(\boldsymbol{r})} \leq \sqrt{e+2} \Phi(\boldsymbol{w})$.
For a flow step, from the above calculation, note that,
$\Phi\left(\boldsymbol{w}^{(t+1)}\right)=\sum_{j} \boldsymbol{w}_{j}+\frac{\alpha}{2} \sum_{j} \boldsymbol{w}_{j} M(\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z})_{j} \leq \Phi\left(\boldsymbol{w}^{(t)}\right)+\frac{\sqrt{e+2}}{8} \alpha \Phi\left(\boldsymbol{w}^{(t)}\right)=\Phi\left(\boldsymbol{w}^{(t)}\right)(1+\alpha)$.
For a width reduction step let $\mathcal{I}$ denote the indices which have the weights doubled,

$$
\begin{aligned}
\Phi\left(\boldsymbol{w}^{(t+1)}\right) & =\sum_{j \notin \mathcal{I}} \boldsymbol{w}_{j}^{(t)}+2 \sum_{j \in \mathcal{I}} \boldsymbol{w}_{j}^{(t)} \leq \Phi\left(\boldsymbol{w}^{(t)}\right)+\frac{2}{\tau} \sum_{j \in \mathcal{I}} \boldsymbol{w}_{j}^{(t)}(2 M)|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}|_{j} \\
& \leq \Phi\left(\boldsymbol{w}^{(t)}\right)+\frac{\sqrt{e+2}}{\tau} \Phi(\boldsymbol{w}) \leq \Phi\left(\boldsymbol{w}^{(t)}\left(1+3 \tau^{-1}\right)\right.
\end{aligned}
$$

We can bound the number of width reduction steps by $O\left(m / \tau^{2}\right)$ similar to Lemma 3.1. We now show that our final solution has $\left\|\frac{1}{T} \boldsymbol{P} \boldsymbol{y}-\boldsymbol{z}\right\|_{\infty} \leq \frac{1}{2 M}$. After $T$ iterations, let $j$ denote the index with max value in vector $\boldsymbol{w}$. For $\alpha \tau \leq 1,\left(1+\frac{\alpha}{2} M|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}|_{j}\right) \geq \exp \left(\frac{3}{4} \alpha M|\boldsymbol{P} \widetilde{\Delta}-\boldsymbol{z}|_{j}\right)$.

$$
\begin{aligned}
10 \zeta & \geq \Phi\left(\boldsymbol{w}^{T}\right) \geq \boldsymbol{w}_{j}^{(T)} \geq \frac{\zeta}{m} \Pi_{t=1}^{T}\left(1+\frac{\alpha}{2} M\left|\boldsymbol{P} \widetilde{\Delta}^{(t)}-\boldsymbol{z}\right|_{j}\right) \\
& \geq \frac{\zeta}{m} \exp \left(\frac{3}{8} \alpha(2 M) \sum_{t}\left|\boldsymbol{P} \widetilde{\Delta}^{(t)}-\boldsymbol{z}\right|_{j}\right)=\frac{\zeta}{m} \exp \left(\frac{3}{8} \alpha(2 M)(\boldsymbol{P} \boldsymbol{y}-T \boldsymbol{z})_{j}\right) .
\end{aligned}
$$

We thus have for all coordinates $j$ and $T \geq \alpha^{-1} O(\log m)$,

$$
\frac{|\boldsymbol{P} \boldsymbol{y}-T \boldsymbol{z}|_{j}}{T} \leq \frac{O\left(M^{-1} \log m\right)}{\alpha T} \leq \frac{1}{2 M}
$$

It remains to show that $\boldsymbol{y} /(100 T)$ has the required value for the residual. First note that,

$$
\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \frac{\boldsymbol{y}}{100 T}=\frac{1}{100 T} \sum_{t} \nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \boldsymbol{P} \widetilde{\Delta}^{(t)}=\frac{\zeta}{2 \cdot 100}
$$

We next look at the quadratic term.

$$
\begin{aligned}
& \frac{1}{(100)^{2} T^{2}} \sum_{j} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{x}_{j}\right) \boldsymbol{y}_{j}^{2}=\frac{1}{T^{2}(100)^{2}} \sum_{j} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{x}_{j}\right)\left(\sum_{t}\left|\boldsymbol{P} \widetilde{\Delta}^{(t)}\right|_{j}\right)^{2} \\
& \leq \frac{1}{T^{2}(100)^{2}} \sum_{j} T \sum_{t} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{x}_{j}\right)\left|\boldsymbol{P} \widetilde{\Delta}^{(t)}\right|_{j}^{2}=\frac{1}{T(100)^{2}} \sum_{t} \Psi\left(\boldsymbol{r}^{(t)}\right) \\
& \quad \leq \frac{1}{T(100)^{2}} T(e+2) \Phi\left(\boldsymbol{w}^{(T)}\right) \leq \frac{10(e+2)}{(100)^{2}} \zeta
\end{aligned}
$$

Choose $c$ such that we have,

$$
e^{-1} \frac{1}{(100)^{2}} \sum_{j} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{x}_{j}\right) \boldsymbol{y}_{j}^{2} \leq \frac{\zeta}{4 \cdot 100}
$$

We thus have,

$$
\operatorname{res}\left(\frac{\boldsymbol{y}}{100 T}\right)=\nabla \boldsymbol{f}(\boldsymbol{x})^{\top} \frac{\boldsymbol{y}}{100 T}-e^{-1} \frac{1}{(100)^{2} T^{2}} \sum_{j} \boldsymbol{f}^{\prime \prime}\left(\boldsymbol{x}_{j}\right) \boldsymbol{y}_{j}^{2} \geq \frac{\zeta}{4 \cdot 100} \geq \frac{1}{400} \operatorname{res}\left(\Delta^{\star}\right)
$$

## B. 4 Proofs from Section 5

## Sum of exponential, soft-max and $\ell_{\infty}$ regression

Theorem 5.2. Let $\boldsymbol{x}^{\star}$ denote the optimum of the $\ell_{\infty}$-regression problem, $\min _{\boldsymbol{A x}=\boldsymbol{b}}\|\boldsymbol{P} \boldsymbol{x}\|_{\infty}$. Algorithm $\square$ when applied to the function $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i}\left(e^{\frac{(\boldsymbol{P} \boldsymbol{x})_{i}}{\nu}}+e^{\frac{-(\boldsymbol{P})_{i}}{\nu}}\right)$ for $\nu=\Omega\left(\frac{\epsilon}{\log m}\right)$, returns $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and

$$
\|\boldsymbol{P} \widetilde{\boldsymbol{x}}\|_{\infty} \leq(1+\epsilon)\left\|\boldsymbol{P} \boldsymbol{x}^{\star}\right\|_{\infty}
$$

in at most $\widetilde{O}\left(m^{1 / 3} \epsilon^{-5 / 3}\right)$ calls to a linear system solve.
Proof. Let $\boldsymbol{Q}=\left[\begin{array}{c}P \\ -P\end{array}\right]$. We note that $\boldsymbol{f}(\boldsymbol{x})=\sum_{i} e^{\frac{(Q x)_{i}}{\nu}}$. Let $\overline{\boldsymbol{x}}$ denote the optimum of $\boldsymbol{f}$, which is also the optimum of $\operatorname{smax}_{\nu}(\boldsymbol{Q} \boldsymbol{x})$. We have the following relation,

$$
\forall \boldsymbol{x},\|\boldsymbol{P} \boldsymbol{x}\|_{\infty} \leq \operatorname{smax}_{\nu}(\boldsymbol{Q} \boldsymbol{x}) \leq\|\boldsymbol{P} \boldsymbol{x}\|_{\infty}+\nu \log m .
$$

Let $R=\left\|\boldsymbol{P} \boldsymbol{x}^{\star}\right\|_{\infty}$ (we can find this up to $\epsilon$ error using binary search), then the above relation implies $\operatorname{smax}_{\nu}(\boldsymbol{Q} \overline{\boldsymbol{x}}) \leq R(1+\epsilon)$. From Theorem5.1.

$$
\|\boldsymbol{P} \widetilde{\boldsymbol{x}}\|_{\infty} \leq \operatorname{smax}_{\nu}(\boldsymbol{Q} \widetilde{\boldsymbol{x}}) \leq R(1+\epsilon)=\left\|\boldsymbol{P} \boldsymbol{x}^{\star}\right\|_{\infty}(1+\epsilon) .
$$

Theorem 5.3. For $\delta>0$, let $\overline{\boldsymbol{x}}$ be the solution returned by Algorithm $\square$ (with $\epsilon=1$ ) applied to $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i} e^{\frac{(\boldsymbol{P} \boldsymbol{x})_{i}}{\nu}}$. Now, Algorithm 2 with starting solution $\boldsymbol{x}^{(0)}=\overline{\boldsymbol{x}}$, applied to $\boldsymbol{f}$ finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\sum_{i} e^{\frac{(\overrightarrow{\boldsymbol{P}})_{i}}{\nu}} \leq(1+\delta) \sum_{i} e^{\frac{\left(P x^{\star}\right)_{i}}{\nu}}$ in at most $O\left(m^{1 / 3} R^{2} \nu^{-2} \log \left(\frac{m}{\delta}\right)\right)$ calls to a linear system solver.

Proof. From Lemma [3.3, Algorithm 1 returns $\overline{\boldsymbol{x}}$ in $O\left(m^{1 / 3}\right)$ iterations such that $\boldsymbol{A} \overline{\boldsymbol{x}}=\boldsymbol{b}$ and $\|\boldsymbol{P} \overline{\boldsymbol{x}}\|_{\infty} \leq M R\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty}$. Since $\frac{1}{\nu^{2}} \sum_{i} e^{\frac{w_{i}^{(T, K)}}{\nu}}=\Phi\left(\boldsymbol{w}^{(T, K)}\right) \leq \Phi\left(\boldsymbol{w}_{0}\right) e^{5}$, we have $\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty} \leq 5 \nu$. This gives, $\|\boldsymbol{P} \overline{\boldsymbol{x}}\|_{\infty} \leq 5 R$. We next bound the function value.

$$
\boldsymbol{f}(\boldsymbol{P} \overline{\boldsymbol{x}})=\sum_{i} e^{\frac{P x_{i}}{\nu}} \leq \sum_{i} e^{\frac{w_{i}^{(T, K)} M R}{\nu}}
$$

If $M R \leq 1$, then $\boldsymbol{f}(\boldsymbol{P} \overline{\boldsymbol{x}}) \leq \nu^{2} \Phi\left(\boldsymbol{w}^{(T, K)}\right) \leq m$. Otherwise,

$$
\boldsymbol{f}(\boldsymbol{P} \overline{\boldsymbol{x}}) \leq \sum_{i}\left(e^{\frac{w_{i}^{(T, K)}}{\nu}}\right)^{M R} \leq\left(\sum_{i} e^{w_{i}^{(T, K)}}\right)^{M R} \leq\left(\nu^{2} \Phi\left(\boldsymbol{w}^{(T, K)}\right)\right)^{M R} \leq O\left(m^{M R}\right)
$$

Now, we use Algorithm 2 Using the above calculated bounds in Theorem4.6, we get our result.

## $\ell_{p}$-Regression

Theorem 5.4. For $\delta>0$ and $p \geq 3$, let $\bar{x}$ be the solution returned by Algorithm 1 (with $\epsilon=1)$ applied to $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\|\boldsymbol{P} \overline{\boldsymbol{x}}\|_{p}^{p}+\mu\|\boldsymbol{P} \boldsymbol{x}\|_{2}^{2}$. Now, Algorithm 2 with starting solution $\boldsymbol{x}^{(0)}=\overline{\boldsymbol{x}}$, applied to $\boldsymbol{f}$ finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\boldsymbol{f}(\boldsymbol{P} \widetilde{\boldsymbol{x}}) \leq \boldsymbol{f}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)+\delta$ in at most $O\left(p^{2} \mu^{-1 /(p-2)} m^{1 / 3} R \log \left(\frac{p m R}{\mu \delta}\right)\right)$ calls to a linear system solver.

Proof. From Lemma 3.3, we get $\overline{\boldsymbol{x}}$ such that $\|\overline{\boldsymbol{x}}\|_{\infty} \leq R M\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty}$. We now want to bound $\boldsymbol{f}(\overline{\boldsymbol{x}})$.

$$
\boldsymbol{f}(\overline{\boldsymbol{x}})=(R M)^{p}\left\|\boldsymbol{w}^{(T, K)}\right\|_{p}^{p}+\mu(R M)^{2}\left\|\boldsymbol{w}^{(T, K)}\right\|_{2}^{2}
$$

We next note that for $\boldsymbol{w}^{(T, K)} \geq \boldsymbol{w}_{0}=1$,

$$
\Phi\left(\boldsymbol{w}^{(T, K)}\right)=p(p-1)\left\|\boldsymbol{w}^{(T, K)}\right\|_{p-2}^{p-2}+2 \mu \leq \Phi\left(\boldsymbol{w}_{0}\right) e^{O(1)}
$$

This implies that $\boldsymbol{w}^{(T, K)} \leq O(1) \boldsymbol{w}_{0}$ and $\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty} \leq O(1)$. Therefore,

$$
\boldsymbol{f}(\overline{\boldsymbol{x}}) \leq\left((O(1) R M)^{p} m .\right.
$$

Now, using this bound on $f(\bar{x})$ and $\bar{x}$ as a starting solution for Algorithm 2 we get our result by applying Theorem4.6

## B.4.1 Logistic Regression

Theorem 5.5. For $\delta>0$, let $\overline{\boldsymbol{x}}$ be the solution returned by Algorithm $\square($ with $\epsilon=1)$ applied to $\boldsymbol{f}(\boldsymbol{P} \boldsymbol{x})=\sum_{i} \log \left(1+e^{(\boldsymbol{P} \boldsymbol{x})_{i}}\right)$. Now, Algorithm 2 with starting solution $\boldsymbol{x}^{(0)}=\overline{\boldsymbol{x}}$, applied to $\boldsymbol{f}$ finds $\widetilde{\boldsymbol{x}}$ such that $\boldsymbol{A} \widetilde{\boldsymbol{x}}=\boldsymbol{b}$ and $\sum_{i} \log \left(1+e^{(\boldsymbol{P} \widetilde{\boldsymbol{x}})_{i}}\right) \leq \sum_{i} \log \left(1+e^{\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)_{i}}\right)+\delta$ in at most $O\left(m^{1 / 3} R \log \left(\frac{m R}{\delta}\right)\right)$ calls to a linear system solver.

Proof. From Lemma 3.3, we get $\overline{\boldsymbol{x}}$ such that $\|\overline{\boldsymbol{x}}\|_{\infty} \leq R M\left\|\boldsymbol{w}^{(T, K)}\right\|_{\infty}$. We now want to bound $\boldsymbol{f}(\overline{\boldsymbol{x}})$.

$$
\boldsymbol{f}(\overline{\boldsymbol{x}})=\sum_{i} \log \left(1+e^{R M \boldsymbol{w}_{i}^{(T, K)}}\right) \leq 2 R M \sum_{i} \boldsymbol{w}_{i}^{(T, K)}
$$

We next note that for $\boldsymbol{w}^{(T, K)} \geq \boldsymbol{w}_{0}$,

$$
\Phi\left(\boldsymbol{w}^{(T, K)}\right)=\sum_{i} \frac{e^{\boldsymbol{w}_{i}^{(T, K)}}}{\left(1+e^{\boldsymbol{w}_{i}^{(T, K)}}\right)^{2}} \geq \Phi\left(\boldsymbol{w}_{0}\right) e^{-O(1)}
$$

This implies that $\boldsymbol{w}^{(T, K)} \leq O(1) \boldsymbol{w}_{0}$. Therefore,

$$
\boldsymbol{f}(\overline{\boldsymbol{x}}) \leq O(R m)
$$

Now, using this bound on $f(\bar{x})$ and $\bar{x}$ as a starting solution for Algorithm 2, we get our result by applying Theorem4.6

## C Energy Lemma

Lemma C.1. Let $\widetilde{\Delta}=\arg \min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{c}} \boldsymbol{x}^{\top} \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}$. Then one has for any $\boldsymbol{r}$ and $\boldsymbol{r}^{\prime}$ such that $\boldsymbol{r}^{\prime} \leq \boldsymbol{r}$,

$$
\Psi\left(\boldsymbol{r}^{\prime}\right) \leq \Psi(\boldsymbol{r})-\frac{1}{2} \sum_{i}\left(1-\frac{\boldsymbol{r}_{i}^{\prime}}{\boldsymbol{r}_{i}}\right) \boldsymbol{r}_{i}(\boldsymbol{P} \widetilde{\Delta})_{i}
$$

Proof.

$$
\Psi(\boldsymbol{r})=\min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{c}} \boldsymbol{x}^{\top} \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}
$$

Constructing the Lagrangian and noting that strong duality holds,

$$
\begin{array}{rll}
\Psi(\boldsymbol{r}) & =\min _{\boldsymbol{x}} \max _{\boldsymbol{y}} & \boldsymbol{x}^{\top} \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}+2 \boldsymbol{y}^{\top}(\boldsymbol{c}-\boldsymbol{A} \boldsymbol{x}) \\
& =\max _{\boldsymbol{y}} \min _{\boldsymbol{x}} & \boldsymbol{x}^{\top} \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}+2 \boldsymbol{y}^{\top}(\boldsymbol{c}-\boldsymbol{A} \boldsymbol{x}) .
\end{array}
$$

Optimality conditions with respect to $x$ give us,

$$
2 \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}^{\star}=2 \boldsymbol{A}^{\top} \boldsymbol{y}
$$

Substituting this in $\Psi$ gives us,

$$
\Psi(\boldsymbol{r})=\max _{y} 2 \boldsymbol{y}^{\top} \boldsymbol{c}-\boldsymbol{y}^{\top} \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \boldsymbol{y}
$$

Optimality conditions with respect to $y$ now give us,

$$
2 \boldsymbol{c}=2 \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \boldsymbol{y}^{\star}
$$

which upon re-substitution gives,

$$
\Psi(\boldsymbol{r})=\boldsymbol{c}^{\top}\left(\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\right)^{-1} \boldsymbol{c}
$$

We also note that

$$
\begin{equation*}
\boldsymbol{x}^{\star}=\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\left(\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\right)^{-1} \boldsymbol{c} \tag{5}
\end{equation*}
$$

We now want to see what happens when we change $r$. Let $\boldsymbol{R}$ denote the diagonal matrix with entries $\boldsymbol{r}$ and let $\boldsymbol{R}^{\prime}=\boldsymbol{R}-\boldsymbol{S}$, where $\boldsymbol{S}$ is the diagonal matrix with the changes in the resistances. We will use the following version of the Sherman-Morrison-Woodbury formula multiple times,

$$
(\boldsymbol{X}+\boldsymbol{U} \boldsymbol{C} \boldsymbol{V})^{-1}=\boldsymbol{X}^{-1}-\boldsymbol{X}^{-1} \boldsymbol{U}\left(\boldsymbol{C}^{-1}+\boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{U}\right)^{-1} \boldsymbol{V} \boldsymbol{X}^{-1}
$$

We begin by applying the above formula for $\boldsymbol{X}=\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}, \boldsymbol{C}=-\boldsymbol{I}, \boldsymbol{U}=\boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}$ and $\boldsymbol{V}=$ $\boldsymbol{S}^{1 / 2} \boldsymbol{P}$. We thus get,

$$
\begin{align*}
&\left(\boldsymbol{P}^{\top} \boldsymbol{R}^{\prime} \boldsymbol{P}\right)^{-1}=\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1}+\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2} \\
&\left(\boldsymbol{I}-\boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \tag{6}
\end{align*}
$$

We next observe that,

$$
\boldsymbol{I}-\boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2} \preceq \boldsymbol{I}
$$

which gives us,

$$
\begin{equation*}
\left(\boldsymbol{P}^{\top} \boldsymbol{R}^{\prime} \boldsymbol{P}\right)^{-1} \succeq\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1}+\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \tag{7}
\end{equation*}
$$

This further implies,

$$
\begin{equation*}
\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R}^{\prime} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \succeq \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}+\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \tag{8}
\end{equation*}
$$

We apply the Sherman-Morrison formula again for, $\boldsymbol{X}=\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}, \boldsymbol{C}=\boldsymbol{I}, \boldsymbol{U}=$ $\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}$ and $\boldsymbol{V}=\boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}$. Let us look at the term $\boldsymbol{C}^{-1}+$ $\boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{U}$.

$$
\begin{aligned}
\boldsymbol{C}^{-1}+\boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{U} & =\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\left(\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\right)^{-1} \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2} \\
& \preceq \boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2} \\
& \preceq \boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}
\end{aligned}
$$

Using this, we get,

$$
\left(\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R}^{\prime} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\right)^{-1} \preceq \boldsymbol{X}^{-1}-\boldsymbol{X}^{-1} \boldsymbol{U}\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{V} \boldsymbol{X}^{-1}
$$

which on multiplying by $\boldsymbol{c}^{\top}$ and $\boldsymbol{c}$ gives,

$$
\Psi\left(\boldsymbol{r}^{\prime}\right) \leq \Psi(\boldsymbol{r})-\boldsymbol{c}^{\top} \boldsymbol{X}^{-1} \boldsymbol{U}\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{c}
$$

We note from Equation (5) that $\boldsymbol{x}^{\star}=\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \boldsymbol{X}^{-1} \boldsymbol{c}$. We thus have,

$$
\begin{aligned}
\Psi\left(\boldsymbol{r}^{\prime}\right) & \leq \Psi(\boldsymbol{r})-\left(\boldsymbol{x}^{\star}\right)^{\top} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{S}^{1 / 2} \boldsymbol{P} \boldsymbol{x}^{\star} \\
& =\Psi(\boldsymbol{r})-\sum_{e}\left(\boldsymbol{r}_{e}-\boldsymbol{r}_{e}^{\prime}\right)\left(1+\frac{\boldsymbol{r}_{e}-\boldsymbol{r}_{e}^{\prime}}{\boldsymbol{r}_{e}}\right)^{-1}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)_{e} \\
& =\Psi(\boldsymbol{r})-\sum_{e}\left(\frac{\boldsymbol{r}_{e}-\boldsymbol{r}_{e}^{\prime}}{2 \boldsymbol{r}_{e}-\boldsymbol{r}_{e}^{\prime}}\right) \boldsymbol{r}_{e}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)_{e} \\
& \leq \Psi(\boldsymbol{r})-\frac{1}{2} \sum_{e}\left(\frac{\boldsymbol{r}_{e}-\boldsymbol{r}_{e}^{\prime}}{\boldsymbol{r}_{e}}\right) \boldsymbol{r}_{e}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)_{e}
\end{aligned}
$$

Where the last line follows from the fact $2 \boldsymbol{r}_{e}-\boldsymbol{r}_{e}^{\prime} \leq 2 \boldsymbol{r}_{e}$.
The next lemma is Lemma C. 4 in ABKS21] which is included here for completeness.
Lemma C.2. Let $\widetilde{\Delta}=\arg \min _{\boldsymbol{A} \boldsymbol{x}=c} \boldsymbol{x}^{\top} \boldsymbol{P}^{\top} R \boldsymbol{P} \boldsymbol{x}$. Then one has for any $\boldsymbol{r}^{\prime}$ and $\boldsymbol{r}$ such that $\boldsymbol{r}^{\prime} \geq \boldsymbol{r}$,

$$
\Psi\left(\boldsymbol{r}^{\prime}\right) \geq \Psi(\boldsymbol{r})+\sum_{e}\left(1-\frac{\boldsymbol{r}_{e}}{\boldsymbol{r}_{e}^{\prime}}\right) \boldsymbol{r}_{e}(\boldsymbol{P} \widetilde{\Delta})_{e}^{2}
$$

Proof.

$$
\Psi(\boldsymbol{r})=\min _{\boldsymbol{A} \boldsymbol{x}=\boldsymbol{c}} \boldsymbol{x}^{\top} \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}
$$

Constructing the Lagrangian and noting that strong duality holds,

$$
\begin{array}{rll}
\Psi(\boldsymbol{r}) & =\min _{\boldsymbol{x}} \max _{y} & \boldsymbol{x}^{\top} \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}+2 \boldsymbol{y}^{\top}(\boldsymbol{c}-\boldsymbol{A} \boldsymbol{x}) \\
& =\max _{y} \min _{x} & \boldsymbol{x}^{\top} \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}+2 \boldsymbol{y}^{\top}(\boldsymbol{c}-\boldsymbol{A} \boldsymbol{x})
\end{array}
$$

Optimality conditions with respect to $\boldsymbol{x}$ give us,

$$
2 \boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P} \boldsymbol{x}^{\star}=2 \boldsymbol{A}^{\top} \boldsymbol{y}
$$

Substituting this in $\Psi$ gives us,

$$
\Psi(\boldsymbol{r})=\max _{y} \quad 2 \boldsymbol{y}^{\top} \boldsymbol{c}-\boldsymbol{y}^{\top} \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \boldsymbol{y}
$$

Optimality conditions with respect to $\boldsymbol{y}$ now give us,

$$
2 \boldsymbol{c}=2 \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \boldsymbol{y}^{\star}
$$

which upon re-substitution gives,

$$
\Psi(\boldsymbol{r})=\boldsymbol{c}^{\top}\left(\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\right)^{-1} \boldsymbol{c}
$$

We also note that

$$
\begin{equation*}
\boldsymbol{x}^{\star}=\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\left(\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\right)^{-1} \boldsymbol{c} \tag{9}
\end{equation*}
$$

We now want to see what happens when we change $r$. Let $\boldsymbol{R}$ denote the diagonal matrix with entries $\boldsymbol{r}$ and let $\boldsymbol{R}^{\prime}=\boldsymbol{R}+\boldsymbol{S}$, where $\boldsymbol{S}$ is the diagonal matrix with the changes in the resistances. We will use the following version of the Sherman-Morrison-Woodbury formula multiple times,

$$
(\boldsymbol{X}+\boldsymbol{U} \boldsymbol{C} \boldsymbol{V})^{-1}=\boldsymbol{X}^{-1}-\boldsymbol{X}^{-1} \boldsymbol{U}\left(\boldsymbol{C}^{-1}+\boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{U}\right)^{-1} \boldsymbol{V} \boldsymbol{X}^{-1}
$$

We begin by applying the above formula for $\boldsymbol{X}=\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}, \boldsymbol{C}=\boldsymbol{I}, \boldsymbol{U}=\boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}$ and $\boldsymbol{V}=$ $\boldsymbol{S}^{1 / 2} \boldsymbol{P}$. We thus get,

$$
\begin{align*}
&\left(\boldsymbol{P}^{\top} \boldsymbol{R}^{\prime} \boldsymbol{P}\right)^{-1}=\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1}-\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2} \\
&\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R P}\right)^{-1} \tag{10}
\end{align*}
$$

We next claim that

$$
\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2} \preceq \boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}
$$

which gives us,

$$
\begin{align*}
\left(\boldsymbol{P}^{\top} \boldsymbol{R}^{\prime} \boldsymbol{P}\right)^{-1} \preceq & \left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1}- \\
& \left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \tag{11}
\end{align*}
$$

This further implies,

$$
\begin{align*}
& \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R}^{\prime} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \preceq \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}- \\
& \boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \tag{12}
\end{align*}
$$

We apply the Sherman-Morrison formula again for, $\boldsymbol{X}=\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}, \boldsymbol{C}=-(\boldsymbol{I}+$ $\left.\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1}, \boldsymbol{U}=\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}$ and $\boldsymbol{V}=\boldsymbol{S}^{1 / 2} \boldsymbol{P}\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}$. Let us look at the term $\boldsymbol{C}^{-1}+\boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{U}$.

$$
-\left(\boldsymbol{C}^{-1}+\boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{U}\right)^{-1}=\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}-\boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{U}\right)^{-1} \succeq\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1}
$$

Using this, we get,

$$
\left(\boldsymbol{A}\left(\boldsymbol{P}^{\top} \boldsymbol{R}^{\prime} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top}\right)^{-1} \succeq \boldsymbol{X}^{-1}+\boldsymbol{X}^{-1} \boldsymbol{U}\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{V} \boldsymbol{X}^{-1}
$$

which on multiplying by $\boldsymbol{c}^{\top}$ and $\boldsymbol{c}$ gives,

$$
\Psi\left(\boldsymbol{r}^{\prime}\right) \geq \Psi(\boldsymbol{r})+\boldsymbol{c}^{\top} \boldsymbol{X}^{-1} \boldsymbol{U}\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{V} \boldsymbol{X}^{-1} \boldsymbol{c}
$$

We note from Equation (9) that $\boldsymbol{x}^{\star}=\left(\boldsymbol{P}^{\top} \boldsymbol{R} \boldsymbol{P}\right)^{-1} \boldsymbol{A}^{\top} \boldsymbol{X}^{-1} \boldsymbol{c}$. We thus have,

$$
\begin{aligned}
\Psi\left(\boldsymbol{r}^{\prime}\right) & \geq \Psi(\boldsymbol{r})+\left(\boldsymbol{x}^{\star}\right)^{\top} \boldsymbol{P}^{\top} \boldsymbol{S}^{1 / 2}\left(\boldsymbol{I}+\boldsymbol{S}^{1 / 2} \boldsymbol{R}^{-1} \boldsymbol{S}^{1 / 2}\right)^{-1} \boldsymbol{S}^{1 / 2} \boldsymbol{P} \boldsymbol{x}^{\star} \\
& =\Psi(\boldsymbol{r})+\sum_{e}\left(\frac{\boldsymbol{r}_{e}^{\prime}-\boldsymbol{r}_{e}}{\boldsymbol{r}_{e}^{\prime}}\right) \boldsymbol{r}_{e}\left(\boldsymbol{P} \boldsymbol{x}^{\star}\right)_{e}
\end{aligned}
$$


[^0]:    ${ }^{1}$ We will see later how such weight/resistance changes can be realized for some special cases.

