# Partial differential equation pricing of contingent claims under stochastic correlation

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**Abstract**

In this paper, we study a partial differential equation (PDE) framework for option pricing where the underlying factors exhibit stochastic correlation, with an emphasis on computation. We derive a multi-dimensional time-dependent PDE for the corresponding pricing problem, and present a numerical PDE solution. We prove a stability result, and study numerical issues regarding the boundary conditions used. Moreover, we develop and analyze an asymptotic analytical approximation to the solution, leading to a novel computational asymptotic approach based on quadrature with a perturbed transition density. Numerical results are presented to verify second order convergence of the numerical PDE solution and to demonstrate its agreement with the asymptotic approximation and Monte Carlo simulations. The effect of certain problem parameters to the PDE solution, as well as to the asymptotic approximation solution, is also studied.

*Key words:* stochastic correlation, option pricing, numerical solution, asymptotic solution, partial differential equation

## 1 Introduction

In many areas of financial modeling such as pricing and risk reporting, correlation between random variables is a critical input. A sound modeling of correlation is therefore necessary for capturing the relationship between asset returns, particularly when the quantity of concern is sensitive to correlations. It has been well documented in the literature that correlation is not a constant variable, but a time-varying one (see, for example, [2, 33, 16], among many others). In particular, during periods of financial crisis, it is observed that correlations between asset returns increase (e.g. [28, 5, 32])

In the options market, there is considerable empirical evidence of a large correlation risk premium (e.g. [16, 15, 4]). In particular, in [16], it is established that correlation risk constitutes the missing link between (empirically) un-priced individual variance premium risk and priced market variance risk. Therefore, proper modelling of correlation is important in estimating correlation risk exposure.

There are several methods in the literature that seek to model the stochasticity of correlations. The dynamic conditional correlation approach (see [19, 38]) proposes a class of multivariate GARCH models that have time-varying correlations. This class of discrete-time model enjoys popularity, especially in econometric analysis. In the continuous-time literature, the Wishart process [3], sometimes considered a generalization of the Heston model [23], is often used to capture stochastic variances-covariances. This is studied in the context of derivative pricing in such works as [11, 12, 22]. Being an affine specification, the Wishart model has the advantage of analytical tractability, which is desired in many situations. However,

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in this model, variances and covariances must be specified, calibrated and evolved jointly, unlike the dynamic conditional correlation approach. This could be an inconvenient feature when a specific model of the volatility is required.

A third approach to modeling stochastic correlation is to directly specify the dynamics of the correlation variable by a stochastic process (see, for example, [37, 15, 39]). This is also the modeling approach of stochastic correlation that we adopt in this paper. Compared to Wishart processes, this approach has an advantage of possible separate calibration of parameters. However, under this class of models, analytical tractability is no longer preserved in derivative pricing, and numerical approximations become necessary.

In the domain of numerical methods, the Monte Carlo (MC) simulation is a popular choice. However, depending on the use case, this approach can have disadvantages, such as slow convergence for problems in low-dimensions, i.e. fewer than five dimensions, and the limitation that the price is obtained only at a single point in the domain, as opposed to the global character of the Partial Differential Equation (PDE) approach. In addition, unlike PDE methods, MC simulations usually suffer from difficulty in computing accurate hedging parameters. To our best knowledge, a PDE approach for pricing contingent claims under stochastic correlation has not been investigated in the literature. This forms the motivation for this work.

In this paper, we will explore the computational aspects of pricing contingent claims when the correlation variable is directly modeled by a mean-reverting stochastic process, with focus on the case of two correlated underlying risk factors. The contributions of the paper are:

- We derive a time-dependent PDE in three space dimensions of the pricing problem, and propose a numerical solution. We prove a stability result, and study the boundary conditions and their associated numerical issues, especially those arising from the correlation variable.
- Using singular perturbation theory, we develop and analyze an asymptotic solution of the PDE as the mean reversion rate of the correlation process becomes large. In cases where the price of the contingent claim and its derivatives do not have a known closed-form solution under a constant correlation model, we propose a novel asymptotic solution based on quadrature with a perturbed transition density, which, to our best knowledge, is a new computational approach to the problem.
- Through numerical results, we illustrate the accuracy of the numerical PDE and asymptotic solution. We study the effect of certain problem parameters to the PDE and asymptotic solutions.

The outline of the remainder of the paper is as follows. In Section 2, we present a pricing model with stochastic correlation and its corresponding PDE formulation. The numerical solution to the PDE is discussed in details in Section 3. In Section 4, we discuss an asymptotic solution built-upon on singular perturbation theory, and present an associated numerical algorithm that computes an approximation to the option value with stochastic correlation. Numerical experiments and results are discussed in Section 5. Section 6 concludes the paper and outlines possible future work.

## 2 Formulation

### 2.1 Model problem: contingent claims on two assets

As a model problem, we consider the pricing of a contingent claim on two (non-dividend-paying) risky assets, whose price processes, denoted by $S_1(t)$ and $S_2(t)$, under the physical measure evolve as follows:

$$\begin{aligned} dS_1(t)/S_1(t) &= \mu_{S_1} dt + \sigma_{S_1} dB_1(t), \\ dS_2(t)/S_2(t) &= \mu_{S_2} dt + \sigma_{S_2} dB_2(t), \\ dB_1(t) dB_2(t) &= \rho(t) dt. \end{aligned} \tag{2.1}$$

Here, $B_1(t)$ and $B_1(t)$ are two correlated Brownian motions, and $\mu_{S_1}, \mu_{S_2}, \sigma_{S_1}, \sigma_{S_2}$ are positive constants. The correlation variable $\rho(t)$ is unobservable, and is assumed to evolve stochastically as

$$d\rho(t) = \alpha(t, \rho(t))dt + \beta(t, \rho(t))dB_3(t), \tag{2.2}$$

where $\alpha(t, \rho(t))$ and $\beta(t, \rho(t))$ are functions that ensure a strong solution to the stochastic differential equation (SDE), and are such that $\rho(t)$ is bounded in $[-1, 1]$ with probability 1. In this specification, the Brownian motion $B_3(t)$ driving the correlation process is assumed to be independent of $B_1(t)$ and $B_2(t)$. While it is possible to include a second layer of correlation structure between the correlation process and the random shocks in asset prices, we shall restrict ourselves to the independence assumption

We remark that a second layer of correlation is considered in [15]. From a computational viewpoint, the effect of this will be extra cross terms in the pricing PDE. While this is seldom a problem in practice with a numerical PDE solver, it is not straightforward to interpret and specify second layer correlations. If one would like to specify an instantaneous correlation between $dB_3(t)$ and $dB_i(t)$, say $\psi_i$, $i = 1, 2$, then clearly $\rho(t)$ has to be bounded within an interval depending on $\psi_i$ in order that the instantaneous correlation matrix is positive semi-definite. We remark that the resulting PDE would involve two more cross derivative terms, each corresponding to the interaction between a price process and the correlation process.

We are interested in pricing a contingent claim with terminal payoff $g(S_1(T), S_2(T))$, where $T$ is the maturity of the contract. We denote by $V = V(t, S_1(t), S_2(t), \rho(t))$ the time-$t$ value of the contingent claim, $0 \le t \le T$. We assume that the value of the contingent claim is Markovian in $(S_1(t), S_2(t), \rho(t))$. We now derive the PDE that governs the price $V$.

Following a usual "no-arbitrage" argument, we consider a self-financing portfolio consisting of one long unit position in $V$, (algebraically) short $a_1(t)$ shares of $S_1$, $a_2(t)$ shares of $S_2$, and $\Delta(t)$ units of another derivative $W$ on $S_1, S_2$. We assume also the existence of a money-market account, whose value at time $t$ is denoted by $M(t)$, that pays instantaneous interest with rate $r$. For convenience, we assume $r > 0$, however, our arguments are valid with minor modifications even if $r$ is negative. Denote by $b(t)$ the (algebraically short) position in money-market account.

We denote by $\Pi(t)$ the value of the portfolio. The portfolio value process can be written as

$$\Pi(t) = V(t, S_1(t), S_2(t), \rho(t)) - \Delta(t)W(t, S_1(t), S_2(t), \rho(t)) - a_1(t)S_1(t) - a_2(t)S_2(t) - b(t)M(t).$$

As the portfolio is self-financing, its infinitesimal change is

$$d\Pi(t) = dV(t, S_1(t), S_2(t), \rho(t)) - \Delta(t)dW(t, S_1(t), S_2(t), \rho(t)) - a_1(t)dS_1(t) - a_2(t)dS_2(t) - b(t)dM(t).$$

By Itô's lemma, $d\Pi(t)$ can be expanded as

$$\begin{aligned} d\Pi(t) &= \left(\left(\frac{\partial}{\partial t} + \tilde{\mathcal{L}}\right)V - \Delta(t)\left(\frac{\partial}{\partial t} + \tilde{\mathcal{L}}\right)W - a_1\mu_{S_1}S_1 - a_2\mu_{S_2}S_2 - b(t)rM(t)\right)dt \\ &+ \sigma_{S_1}S_1\left(\frac{\partial V}{\partial S_1} - \Delta(t)\frac{\partial W}{\partial S_1} - a_1(t)\right)dB_1(t) + \sigma_{S_2}S_2\left(\frac{\partial V}{\partial S_2} - \Delta(t)\frac{\partial W}{\partial S_2} - a_2(t)\right)dB_2(t) \\ &+ \beta(t, \rho(t))\left(\frac{\partial V}{\partial \rho} - \Delta(t)\frac{\partial W}{\partial \rho}\right)dB_3(t), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \tilde{\mathcal{L}}V &\doteq \frac{\sigma_{S_1}^2 {S_1}^2}{2}\frac{\partial^2 V}{\partial S_1^2} + \frac{\sigma_{S_2}^2 {S_2}^2}{2}\frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_{S_1}\sigma_{S_2}S_1S_2\frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{\beta(t,\rho)^2}{2}\frac{\partial^2 V}{\partial \rho^2} \\ &+ \mu_{S_1}S_1\frac{\partial V}{\partial S_1} + \mu_{S_2}S_2\frac{\partial V}{\partial S_2} + \alpha(t,\rho)\frac{\partial V}{\partial \rho}, \end{aligned}$$

and similarly for $\tilde{\mathcal{L}}W$. In the above and for the rest of the paper, where applicable, the dependence on $t$ of $S_1(t)$, $S_2(t)$ and $\rho(t)$ are suppressed for notational convenience.
We choose $a_1(t), a_2(t)$ and $\Delta(t)$ such that the following holds:

$$\begin{aligned}
\frac{\partial V}{\partial S_1} - \Delta(t)\frac{\partial W}{\partial S_1} - a_1(t) &= 0, \\
\frac{\partial V}{\partial S_2} - \Delta(t)\frac{\partial W}{\partial S_2} - a_2(t) &= 0, \\
\frac{\partial V}{\partial \rho} - \Delta(t)\frac{\partial W}{\partial \rho} &= 0.
\end{aligned} \tag{2.4}$$

These three equations can be solved for $a_1(t), a_2(t), \Delta(t)$. Using these choices, the terms involving real-world drifts $\mu_{S_1}$ and $\mu_{S_2}$ are cancelled. Also, from the construction of $a_1(t), a_2(t), \Delta(t)$, the terms involving $dB_i$, $i = 1, 2, 3$, also disappear from $d\Pi$ in (2.3). As a result, the portfolio is instantaneously riskless. In the presence of the risk-free money-market account paying instantaneous interest $r$, the following equation has to hold:

$$d\Pi(t) = r\Pi(t)dt = r(V - \Delta(t)W - a_1(t)S_1(t) - a_2(t)S_2(t) - b(t)M(t))dt. \tag{2.5}$$

Therefore, from (2.3)-(2.5), we have

$$\frac{\partial V}{\partial t} + \mathcal{L}V = \Delta(t)\left(\frac{\partial W}{\partial t} + \mathcal{L}W\right), \tag{2.6}$$

where

$$\begin{aligned}
\mathcal{L}V \doteq\ & \frac{\sigma_{S_1}^2 {S_1}^2}{2}\frac{\partial^2 V}{\partial S_1^2} + \frac{\sigma_{S_2}^2 {S_2}^2}{2}\frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_{S_1}\sigma_{S_2}S_1S_2\frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{\beta(t,\rho)^2}{2}\frac{\partial^2 V}{\partial \rho^2} \\
& + rS_1\frac{\partial V}{\partial S_1} + rS_2\frac{\partial V}{\partial S_2} + \alpha(t,\rho)\frac{\partial V}{\partial \rho} - rV,
\end{aligned}$$

and similarly for $\mathcal{L}W$. Assuming $\frac{\partial W}{\partial \rho} \neq 0$, we have $\Delta(t) = \frac{\frac{\partial V}{\partial \rho}}{\frac{\partial W}{\partial \rho}}$. As a result, the quantity

$$\frac{\frac{\partial V}{\partial t} + \mathcal{L}V}{\frac{\partial V}{\partial \rho}}$$

is invariant for every financial derivative $V$ that has non-zero sensitivity to $\rho$. Therefore, there exists $\Lambda(t, S_1, S_2, \rho)$ such that

$$\frac{\frac{\partial V}{\partial t} + \mathcal{L}V}{\frac{\partial V}{\partial \rho}} = \Lambda(t, S_1, S_2, \rho).$$

Consequently, we obtain the pricing PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma_{S_1}^2 {S_1}^2}{2}\frac{\partial^2 V}{\partial S_1^2} + \frac{\sigma_{S_2}^2 {S_2}^2}{2}\frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_{S_1}\sigma_{S_2}S_1S_2\frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{\beta(t,\rho)^2}{2}\frac{\partial^2 V}{\partial \rho^2} \tag{2.7}$$

$$+ rS_1\frac{\partial V}{\partial S_1} + rS_2\frac{\partial V}{\partial S_2} + (\alpha(t,\rho) - \Lambda(t, S_1, S_2, \rho))\frac{\partial V}{\partial \rho} = rV. \tag{2.8}$$

Expectedly, the quantity $\Lambda(t, S_1, S_2, \rho)$ is related to a drift adjustment from the physical measure to a risk-neutral measure. This is seen heuristically as follows. For illustration, assume that $\beta(t,\rho) \neq 0$, then $\Lambda(t, S_1, S_2, \rho)$ can be rewritten as $\Lambda(t, S_1, S_2, \rho) = \phi\beta(t,\rho)$, where $\phi \equiv \phi(t, S_1, S_2, \rho) = \frac{\Lambda(t,S_1,S_2,\rho)}{\beta(t,\rho)}$.

We have from Itô's lemma that

$$
\begin{aligned}
& d\left(e^{-rt}V(t,S_1(t),S_2(t),\rho(t))\right) \\
= \;& e^{-rt}\left(\left(\left(\frac{\partial}{\partial t}+\tilde{\mathcal{L}}\right)V-rV\right)dt+\sigma_{S_1}S_1\frac{\partial V}{\partial S_1}dB_1(t)+\sigma_{S_2}S_2\frac{\partial V}{\partial S_2}dB_2(t)+\beta(t,\rho(t))\frac{\partial V}{\partial \rho}dB_3(t)\right) \\
= \;& \left((\mu_{S_1}-r)\,S_1e^{-rt}\frac{\partial V}{\partial S_1}+(\mu_{S_2}-r)S_2e^{-rt}\frac{\partial V}{\partial S_2}+\phi\beta(t,\rho)e^{-rt}\frac{\partial V}{\partial \rho}\right)dt \\
& +\sigma_{S_1}S_1e^{-rt}\frac{\partial V}{\partial S_1}dB_1(t)+\sigma_{S_2}S_2e^{-rt}\frac{\partial V}{\partial S_2}dB_2(t)+\beta(t,\rho(t))e^{-rt}\frac{\partial V}{\partial \rho}dB_3(t) \\
= \;& \sigma_{S_1}S_1e^{-rt}\frac{\partial V}{\partial S_1}\left(dB_1(t)+\frac{\mu_{S_1}-r}{\sigma_{S_1}}dt\right)+\sigma_{S_2}S_2e^{-rt}\frac{\partial V}{\partial S_2}\left(dB_2(t)+\frac{\mu_{S_2}-r}{\sigma_{S_2}}dt\right) \\
+ \;& \beta(t,\rho(t))e^{-rt}\frac{\partial V}{\partial \rho}(dB_3(t)+\phi dt).
\end{aligned}
$$

The Brownian motion $B_2$ can be constructed as $dB_2 = \rho(t)dB_1(t)+\sqrt{1-\rho(t)^2}dB_2'(t)$ for some $B_2'(t)$ such that $B_1(t), B_2'(t)$ and $B_3(t)$ are independent Brownian motions. The market price of risk process $\bar{\gamma}=[\gamma_i], i=1,2,3$ is given component-wise by the following:

$$
\gamma_1(t)=\frac{\mu_{S_1}-r}{\sigma_{S_1}}, \qquad \gamma_2(t)=\frac{1}{\sqrt{1-\rho(t)^2}}\left(\frac{\mu_{S_2}-r}{\sigma_{S_2}}-\rho(t)\gamma_1\right), \qquad \gamma_3(t)=\phi.
$$

In turn, the function $\Lambda$ is related to an equivalent local martingale measure, where the third market price of risk process is given by $\phi$ (subject to technical conditions that guarantee that the stochastic exponential of the market price of risk processes is a true martingale).

### 2.2 Quanto options

In this section, we present a popular type of contingent claims that decouples equity and FX risk in the terminal payoff, referred to as quanto options. The value of such derivatives depends on the instantaneous correlation between the exchange rate process and the equity price process. The pricing of such instruments in the presence of stochastic correlation is also studied in [37, 29]. In this context, we denote by $S(t)$ the underlying asset priced in the foreign currency, and by $R(t)$ the spot FX rate which is defined as the number of units of domestic currency per one unit of foreign currency. In a quanto option, the payoff depends on $S(T)$ and a fixed strike, and is paid in the domestic currency. For this kind of options, it is also important to realistically capture the correlation between $S(t)$ and $R(t)$. This is because the currency mismatch in a quanto option gives rise to a "quanto adjustment" that depends heavily on the correlation parameter between $S(t)$ and $R(t)$. Let $S(t)$ and $R(t)$ evolve as

$$
\begin{aligned}
dS(t)/S(t) &= \mu_S dt+\sigma_S dB_1(t), \\
dR(t)/R(t) &= \mu_R dt+\sigma_R dB_2(t), \\
dB_1(t)dB_2(t) &= \rho(t)dt,
\end{aligned}
\tag{2.9}
$$

where $\rho(t)$ is as specified before, and $\mu_S$, $\sigma_S$, $\mu_R$, and $\sigma_R$ are positive constants. Under the model (2.9), it can be shown that the price $V(t,S(t),\rho(t))$ of a quanto option satisfies the PDE

$$
\frac{\partial V}{\partial t}+\frac{\sigma_S^2S^2}{2}\frac{\partial^2 V}{\partial S^2}+\frac{\beta(t,\rho)^2}{2}\frac{\partial^2 V}{\partial \rho^2}+(r_f-\sigma_S\sigma_R\rho)S\frac{\partial V}{\partial S}+(\alpha(t,\rho)-\Lambda(t,S,\rho))\frac{\partial V}{\partial \rho}=r_dV, \tag{2.10}
$$

where $r_d$ and $r_f$ are positive constant domestic and foreign risk-free interest rates, respectively, and the quanto terminal condition is given by $V(T,S) = g(S)$, in which the payoff $g(S)$ is independent of the exchange rate $R$.

**REMARK 2.1.** *Considering $R$ as a risk source, Itô's lemma would have led to derivatives with respect to the $R$-variable, as shown below,*

$$\frac{\partial V}{\partial t} + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\beta(t,\rho)^2}{2}\frac{\partial^2 V}{\partial \rho^2} + \frac{\sigma_R^2 R^2}{2}\frac{\partial^2 V}{\partial R^2} + \sigma_S\sigma_R\rho SR\frac{\partial^2 V}{\partial S \partial R}$$
$$+ (r_f - \sigma_S\sigma_R\rho)S\frac{\partial V}{\partial S} + (\alpha(t,\rho) - \Lambda(t,S,\rho))\frac{\partial V}{\partial \rho} + r_R R\frac{\partial V}{\partial R} = r_d V, \tag{2.11}$$

*where $r_R = r_d - r_f$ is the risk-neutral drift of $R(t)$. The PDE is subject to the same terminal condition as that of* (2.10)*. As the quanto payoff $g(S)$ is independent of the state variable $R$, it is straightforward to see that any solution to* (2.10) *is a solution to* (2.11)*. Thus, it suffices to consider the two dimensional version* (2.10).

### 2.3 Correlation process

We now discuss the choice for the correlation process $\rho(t)$. The Jacobi process is a popular choice for modeling stochastic correlation ([15], [39], [29] and [30]). If the correlation is assumed to attain values anywhere in $(-1,1)$, then the following is a candidate choice of $\alpha$ and $\beta$:

$$\alpha(\rho) = \lambda(\eta - \rho), \qquad \beta(\rho) = \sigma_\rho\sqrt{1-\rho^2}. \tag{2.12}$$

Here, $\lambda$, $\eta$, and $\sigma_\rho$ are positive constants.

The parameter restriction $\lambda \geq \frac{\sigma_\rho^2}{1 \pm \eta}$, typically referred to as the Feller condition, is needed for the process to remain in $(-1,1)$ with probability 1 (see [39]). This is the stochastic correlation model we will focus on in this work. Unlike the Wishart process (see for example [10]), this class of models is not affine. In general, a closed-form solution to (2.7) or (2.10) for a general payoff function is not known analytically, and numerical methods are needed to approximate the solution.

In the analysis, we will focus on the model PDE (2.7). For pricing purposes and simplicity, for the rest of the paper, we assume that the specification in (2.12) is risk-neutral, i.e. $\Lambda \equiv 0$. The following theorem shows that the concept of classical solutions suffices for our purpose.

**THEOREM 2.1.** *With $\alpha(\rho)$ and $\beta(\rho)$ chosen as in (2.12), the price of a European contingent claim (with bounded payoff $g(S_1(T), S_2(T))$), given by the discounted risk-neutral expectation*

$$e^{-r(T-t)}\mathbf{E}_{t,x,y,\rho}\left[g(S_1(T), S_2(T))\right],$$

*satisfies (2.7) and is $C^{1,2,2,2}$ on $[0,T) \times (0,\infty)^2 \times (-1,1)$.*

*Proof.* See Appendix A. □

## 3 Numerical solution

Since we solve the PDE (2.7) backward in time, the change of variable $\tau = T - t$ is used. Under this change of variable, the PDE (2.7) becomes

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V \tag{3.1}$$

and is solved forward in $\tau$. The pricing of the option price is defined in an unbounded domain

$$\{(\tau, S_1, S_2, \rho) \in (0,T] \times (0,\infty)^2 \times (-1,1)\}, \tag{3.2}$$

subject to the initial condition $g(\cdot,\cdot)$.

While an implementation of the finite difference scheme may seem straightforward, caution must be taken to ensure proper discretization due to the unique features of this problem. Most of the techniques of this section can be modified to other choices of $\alpha(t,\rho)$ and $\beta(t,\rho)$.

## 3.1 Localization

Localization estimates are well studied in the literature. See, for example, [27] (multi-dimensional Black-Scholes equation), [8] (exponential Lévy models and jump diffusion processes), and [7] (two-asset jump diffusion models), among many others. Relevant to our work, in [8] and also in [25], for various models (with constant correlations, though) and different assumption on payoffs, it has been proved that the price of a European option is approximated exponentially well by that of a corresponding barrier option (in log of the barrier). We now extend this result to the context of stochastic correlation.

For the statement of the result, we will switch to log scaling. We denote $X_t = \log(S_1(t))$ and $Y_t = \log(S_2(t))$, and generic variables $x = \log(S_1), y = \log(S_2)$. Let also $-R^{\log}$ and $R^{\log}$, where $R^{\log} > 0$, respectively denote generic lower and upper barriers for the processes $X_t$ and $Y_t$.

**Proposition 3.1.** *Let $u_{\log}(\tau,x,y,\rho) = e^{-r\tau}\mathbf{E}_{x,y,\rho}\left(g\left(e^{X_\tau}, e^{Y_\tau}\right)\right)$ be the option price with the bounded payoff function $g$ in log scaling ($||g||_\infty < \infty$). Define $M_\tau^x = \sup_{\hat\tau\in[0,\tau]}|X_{\hat\tau}|$, $M_\tau^y = \sup_{\hat\tau\in[0,\tau]}|Y_{\hat\tau}|$ and $M_\tau^{x,y} = \max(M_\tau^x, M_\tau^y)$. Furthermore, denote by $\hat\theta$ is the first exit time of $(X_{\hat\tau}, Y_{\hat\tau})$, $\hat\tau \in [0,\tau]$ from the region $[-R^{\log}, R^{\log}] \times [-R^{\log}, R^{\log}]$. Let*

$$u^1_{R^{\log}}(\tau,x,y,\rho) = e^{-r\tau}\mathbf{E}\left[g\left(e^{X_\tau}, e^{Y_\tau}\right)\mathbf{1}_{\{M_\tau^{x,y} < R^{\log}\}}\right],$$

$$u^2_{R^{\log}}(\tau,x,y,\rho) = e^{-r\tau}\mathbf{E}\left[g\left(e^{X_\tau}, e^{Y_\tau}\right)\mathbf{1}_{\{M_\tau^{x,y} < R^{\log}\}} + g\left(e^{X_{\hat\theta}}, e^{Y_{\hat\theta}}\right)\mathbf{1}_{\{M_\tau^{x,y} \geq R^{\log}\}}\right].$$

*Then, for $\gamma > 0$, there exists constant $C(\gamma, \sigma_{S_1}, \sigma_{S_2}, r, \tau)$ independent of $R^{\log}$ such that, for $i = 1, 2$,*

$$|u_{\log}(\tau,x,y,\rho) - u^i_{R^{\log}}(\tau,x,y,\rho)| \leq C(\gamma, \sigma_{S_1}, \sigma_{S_2}, r, \tau)||g||_\infty (e^{-\gamma(R^{\log}-|x|)} + e^{-\gamma(R^{\log}-|y|)}),$$

*pointwise in $(0,T] \times [-R^{\log}, R^{\log}] \times [-R^{\log}, R^{\log}] \times (-1,1)$.*

*Proof.* By construction

$$|u_{\log}(\tau,x,y,\rho) - u^1_{R^{\log}}(\tau,x,y,\rho)| \leq ||g||_\infty \mathbf{Q}\left(\{M_\tau^{x,y} \geq R^{\log}\}\right),$$

where $\mathbf{Q}$ is the pricing measure. Similarly, we have

$$|u_{\log}(\tau,x,y,\rho) - u^2_{R^{\log}}(\tau,x,y,\rho)| \leq 2||g||_\infty \mathbf{Q}\left(\{M_\tau^{x,y} \geq R^{\log}\}\right).$$

We can write $X_{\hat\tau} = x + U_{\hat\tau}$ and $Y_{\hat\tau} = y + \tilde U_{\hat\tau}$, $\hat\tau \in [0,\tau]$, where $U_{\hat\tau}$, and $\tilde U_{\hat\tau}$ start from 0 and have drifts $r - \frac{\sigma_{S_1}^2}{2}$ and $r - \frac{\sigma_{S_2}^2}{2}$, respectively. We have $M_\tau^x = \sup_{\hat\tau\in[0,\tau]}|x + U_{\hat\tau}|$. Theorem 25.18 of [36] implies that for any $\gamma > 0$, $C_1(\gamma, \sigma_{S_1}, r, \tau) = \mathbf{E}\left[e^{\gamma \sup_{\hat\tau\in[0,\tau]}|U_{\hat\tau}|}\right] < \infty$. Therefore, by the exponential Chebyshev's inequality, for every $R_1^{\log} > 0$,

$$\mathbf{Q}\left(\left\{\sup_{\hat\tau\in[0,\tau]}|U_{\hat\tau}| \geq R_1^{\log}\right\}\right) \leq C_1 e^{-\gamma R_1^{\log}}.$$

As a result,

$$\mathbf{Q}\left(\left\{M_\tau^x \geq R^{\log}\right\}\right) \leq C_1(\gamma, \sigma_{S_1}, r, \tau) e^{-\gamma\left(R^{\log}-|x|\right)}.$$

A similar bound can be obtained for $\mathbf{Q}\left(\{M_\tau^y \geq R^{\log}\}\right)$, i.e.

$$\mathbf{Q}\left(\left\{M_\tau^y \geq R^{\log}\right\}\right) \leq C_2(\gamma, \sigma_{S_2}, r, \tau) e^{-\gamma\left(R^{\log}-|y|\right)}.$$

The result follows by noting that

$$\mathbf{Q}\left(\{M_\tau^{x,y} \geq R^{\log}\}\right) \leq \mathbf{Q}\left(\{M_\tau^x \geq R^{\log}\}\right) + \mathbf{Q}\left(\{M_\tau^y \geq R^{\log}\}\right).$$

□

Proposition 3.1 shows that the price of a continuously monitored barrier option approximates that of a European option arbitrarily well by extending the log barrier $R^{\log}$. As only a bounded domain is required to compute the price of a barrier option, in the present case of European options, truncation of the domain is effective when the truncation boundary is far enough from the points of interest.

**REMARK 3.1.** *Proposition 3.1 is formulated in log price scaling. Technically, the equation in log space corresponding to (3.1) is*

$$\frac{\partial u_{\log}}{\partial \tau} = \mathcal{L}_{\log} u_{\log}, \tag{3.3}$$

*defined on* $(0, T] \times \mathbf{R}^2 \times (-1, 1)$ *with terminal condition* $u_{\log}(\tau = 0, x, y, \rho) = g(e^x, e^y)$, *where*

$$\begin{aligned} \mathcal{L}_{\log} \quad \doteq \quad & \frac{\sigma_{S_1}^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\sigma_{S_2}^2}{2}\frac{\partial^2}{\partial y^2} + \rho\sigma_{S_1}\sigma_{S_2}\frac{\partial^2}{\partial x \partial y} + \frac{\beta(\rho)^2}{2}\frac{\partial^2}{\partial \rho^2} \\ & + \left(r - \frac{\sigma_{S_1}^2}{2}\right)\frac{\partial}{\partial x} + \left(r - \frac{\sigma_{S_2}^2}{2}\right)\frac{\partial}{\partial y} + \alpha(\rho)\frac{\partial}{\partial \rho} - r. \end{aligned}$$

*However, if analytic solution is known along the* $S_i = 0$ *boundaries,* $i = 1, 2$, *such as in the case of spread/basket/exchange options, then the price of the option can be similarly approximated by the price of the corresponding barrier option. Specifically, this can be achieved by solving the original Black-Scholes equation (3.1) in a truncated domain, with approximation error of the order* $O\left(\frac{1}{(R_{price})^\gamma}\right)$ *pointwise in the domain, where* $R_{price}$ *is the far boundary in price scaling. In the case of non-negative risk-neutral drift of the asset prices, this can also be seen from Doob's martingale inequality.*

**REMARK 3.2.** *The boundedness condition imposed on the payoff function* $g$ *may seem restrictive. In particular, the analysis in Proposition 3.1 is applicable to only put payoffs, and not to call ones. However, it is possible to extend the results of Proposition 3.1 to more general, unbounded payoffs, such as those given in [25]. If* $g$ *is of polynomial growth, say bounded asymptotically by a multivariate polynomial of order* $q$, *then using again Theorem 25.18 of [36], it suffices to provide a similar bound for* $e^{-r\tau}\mathbf{E}_{x,y,\rho}\left(e^{q\max(X_\tau, Y_\tau)}\mathbf{1}_{\{\max(X_\tau,Y_\tau)>R^{\log}\}}\right)$. *The bound can be obtained using tail decay estimates for the normal distribution. Similar arguments can be found in [35]. Alternatively, when put-call parity formulae are available, the analysis in Proposition 3.1 can be extended to European options with call-type payoffs. In our numerical experiments, in cases with unbounded payoffs, such as a European call option, we do not observe a problem, and notice good agreement of the numerical PDE price with that obtained from Monte Carlo simulations.*

### 3.2 Boundary conditions

To solve the PDE (3.1) numerically by FD methods, we need to truncate the unbounded domain (3.2) into a finite-sized computational one

$$\{(\tau, S_1, S_2, \rho) \in (0,T] \times [0, S_1^{\max}] \times [0, S_2^{\max}] \times (-1,1)\} \equiv (0,T] \times \Omega, \tag{3.4}$$

where $S_1^{\max}$ and $S_2^{\max}$ are sufficiently large (see Proposition 3.1). Denote $\partial_{S_1}\Omega = \{(S_1, S_2, \rho) \in \{0, S_1^{\max}\} \times [0, S_2^{\max}] \times [-1,1]\}$ and similarly for $\partial_{S_2}\Omega$, and $\partial_\rho \Omega$. In the same fashion, we can define a localized spatial domain in the log scaling $\Omega^{\log} = \{(x,y,\rho) \in [-R_1^{\log}, R_1^{\log}] \times [-R_2^{\log}, R_2^{\log}] \times [-1,1]\}$ and its boundaries.

From Proposition 3.1, we know the localization errors on the boundaries $\partial_{S_1}\Omega$, and $\partial_{S_2}\Omega$ can be made negligible if $S_1^{\max}$ and $S_2^{\max}$ are chosen sufficiently large. In our experiments, we choose a Dirichlet condition for these boundaries. However, there is a difficulty with choosing the boundary conditions on $\partial_\rho\Omega$, as, for an arbitrary option payoff, they are not known as $\rho \to \pm 1$. Note that, from a computational perspective with a finite difference method, it is necessary to specify the behaviour at $\partial_\rho\Omega$. The focus of the rest of this subsection is how to obtain appropriate $\rho$-boundary conditions.

We note that, under the choice (2.12) for $\alpha(\cdot)$ and $\beta(\cdot,\cdot)$, the correlation process $\rho$ is of the Cox-Ingersoll-Ross (CIR) type [9]. This type of processes are commonly used in modelling of interest rate and volatility. In particular, it is reported in [24] that there can be multiple solutions to the CIR bond pricing PDE under certain parameter choices. In [18], it is pointed out that only the stochastic representation of the solution given by the discounted risk neutral expectation is $C^1$ *up to including the boundary*.

Furthermore, in [17] (resp. [18]), the authors have studied the problems of boundary behaviors of the discounted risk neutral expectation under stochastic volatility (resp. one factor term structure) models, where at the zero boundary of the variance variable (resp. the short rate variable), a boundary condition is unclear for the specification of the PDE. In their works, they proved that under regularity assumptions such as bounded smooth payoff, linear growth of coefficients, square of volatility of the variance (resp. the short rate) being continuously differentiable with a Hölder continuous derivative, and the vanishing of the volatility of variance (resp. the short rate) as variance (resp. the short rate) goes to zero, the discounted risk neutral expectation is $C^1$ everywhere in the solution domain up to including the boundary of 0 in volatility (resp. the short rate). Moreover using interior Schauder estimates, it was shown that the solution satisfies a reduced PDE at the boundary, corresponding to the pricing equation with vanishing second derivative term with respect to the variance (resp. the short rate).

From Theorem 2.1, the solution we seek is $C^2$ in the interior of space and $C^1$ in time. We do not intend to carry out a similar analysis to [17] or [18] here, but instead, we assume that the $C^1$-ness extends to the $\rho$-boundaries, noting that the coefficients of the correlation process in our case satisfy the assumptions in [17, 18] . Hence, a similar boundary condition holds for our European option pricing problem:

$$\begin{aligned}\lim_{\rho\to\pm1} \quad & \Bigg( -\frac{\partial V}{\partial \tau} + \frac{\sigma_{S_1}^2 {S_1}^2}{2}\frac{\partial^2 V}{\partial S_1^2} + \frac{\sigma_{S_2}^2 {S_2}^2}{2}\frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_{S_1}\sigma_{S_2} S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \\ & + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_1} + \lambda(\eta - \rho)\frac{\partial V}{\partial \rho} - rV \Bigg) = 0.\end{aligned} \tag{3.5}$$

Note that, in log-price variables, this becomes

$$\begin{aligned}\lim_{\rho\to\pm1} \quad & \Bigg( -\frac{\partial u_{\log}}{\partial \tau} + \frac{\sigma_{S_1}^2}{2}\frac{\partial^2 u_{\log}}{\partial x^2} + \frac{\sigma_{S_2}^2}{2}\frac{\partial^2 u_{\log}}{\partial y^2} + \rho\sigma_{S_1}\sigma_{S_2} \frac{\partial^2 u_{\log}}{\partial S_1 \partial S_2} \\ + \quad & \left(r - \frac{\sigma_{S_1}^2}{2}\right)\frac{\partial u_{\log}}{\partial x} + \left(r - \frac{\sigma_{S_2}^2}{2}\right)\frac{\partial u_{\log}}{\partial y} + \lambda(\eta - \rho)\frac{\partial u_{\log}}{\partial \rho} - ru_{\log} \Bigg) = 0.\end{aligned} \tag{3.6}$$

The choice of finite difference discretization of (3.5) (or (3.6)) is to be further elaborated in Section 3.3, with consideration given to numerical stability.

From the viewpoint of the Fichera theory, the Fichera function is given by $\mathcal{F}(\rho) = \lambda(\eta - \rho) + \sigma_\rho^2 \rho$ and the boundary points relevant to our problem are $\rho_{\max} = 1$ and $\rho_{\min} = -1$. The outflow conditions $\mathcal{F}(\rho_{\max}) = \mathcal{F}(+1) \leq 0$ and $\mathcal{F}(\rho_{\min}) = \mathcal{F}(-1) \geq 0$ can be written, respectively, as

$$\lambda(\eta - 1) + \sigma_\rho^2 \leq 0, \text{ and } \lambda(\eta + 1) - \sigma_\rho^2 \geq 0.$$

These are identical to the Feller conditions $\lambda \geq \frac{\sigma_\rho^2}{1 \pm \eta}$ mentioned in Section 2.3. If such Feller conditions are satisfied, then mathematically, no boundary condition should be supplied at $\rho = \pm 1$. However, for computational purposes, the boundary behaviour needs to be specified, and, therefore, the boundary condition (3.5) or (3.6) coming from the PDE itself is needed for the finite difference implementation.

In summary, the localized problem in the original price scaling is

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V \tag{3.7}$$

on $(0, T] \times (0, S_1^{\max}) \times (0, S_2^{\max}) \times (-1, 1)$ subject to the terminal and boundary conditions

$$\begin{aligned} V(\tau = 0, S_1, S_2, \rho) &= g(S_1, S_2) \\ V(\tau, S_1, S_2, \rho) &= V_{Dir}(\tau, S_1, S_2) \text{ on } \partial_{S_1}\Omega \cup \partial_{S_2}\Omega, \end{aligned}$$

where $V_{Dir}(\tau, S_1, S_2)$ is a Dirichlet condition of choice. And finally, $V$ satisfies (3.5) along the spatial $\rho$-boundary $\partial\Omega \setminus (\partial_{S_1}\Omega \cup \partial_{S_2}\Omega)$. Similar boundary conditions hold for the formulation in log-price space (equation (3.3)).

### 3.3 Discretization

To obtain a provably monotone discretization, we switch to log scaling (3.3). Let $h_1, h_2$ and $h_3$ be step-sizes of a uniform spatial discretization of $\Omega^{\log}$, and $i \in I_1 = \{0, 1, \ldots, n_1\}$, $j \in I_2 = \{0, 1, \ldots, n_2\}$, $k \in I_3 = \{0, 1, \ldots, n_3\}$. Let

$$\Omega^{\Delta,\log} = \{(x_i, y_j, \rho_k) = (-R_1^{\log} + ih_1, -R_2^{\log} + jh_2, -1 + kh_3), i \in I_1, j \in I_2, k \in I_3\}.$$

Recall that we solve the equation (3.3) which is in the log space. We use the finite difference method to obtain a discrete representation $\mathcal{L}_{\log}^{\Delta}$ of $\mathcal{L}_{\log}$ on the discretized grid $\Omega^{\Delta,\log}$. The time dimension is discretized using the $\theta$-timestepping. Let $u_{\log}^{(l)}$ be the vectorized numerical solution at the $l$-th timestep. At the $(l+1)$-th timestep, the timestepping reads as follows:

$$\frac{u_{\log}^{(l+1)} - u_{\log}^{(l)}}{\Delta\tau} = \theta \mathcal{L}_{\log}^{\Delta} u_{\log}^{(l+1)} + (1-\theta)\mathcal{L}_{\log}^{\Delta} u_{\log}^{(l)}. \tag{3.8}$$

When $\theta = 0.5$, the scheme is known as Crank-Nicolson (CN), and the choice $\theta = 1$ is known as fully implicit timestepping.

Let $A$ be the discretization matrix arising from $\mathcal{L}_{\log}^{\Delta}$ including the boundary conditions. For every $i' \in I_1$, every $j' \in I_2$, and every $k' \in I_3$, we denote $A_{i,j,k}^{i',j',k'}$ to be the matrix element where the index $(i, j, k)$ corresponds to the column index, while $(i', j', k')$ corresponds to the row index in a vectorized ordering of $(i, j, k)$. We require that the discretization matrix satisfies the following:

$$A_{i,j,k}^{i,j,k} \leq 0 \text{ for all } (i, j, k), \tag{3.9}$$

$$A_{i,j,k}^{i',j',k'} \geq 0 \text{ for } (i', j', k') \neq (i, j, k) \text{ where equality is component-wise, and} \tag{3.10}$$

$$\sum_{i',j',k'} A_{i,j,k}^{i',j',k'} \leq 0 \text{ for all } (i, j, k). \tag{3.11}$$

To ensure these properties are satisfied, first and cross derivatives have to be carefully discretized. For first derivatives, one could choose between forward, backward and central differences in such a way that the signs of the matrices are correctly obtained. For cross derivatives, a 7-point stencil can be used according to the sign of the correlation variable. Details are provided in Appendix B.

While in this work we are primarily concerned with classical solutions, maintaining these conditions has the advantage that the discretization matrix arising from the fully implicit timestepping is monotone, a key requirement for convergence towards viscosity solutions. This is particularly relevant for options with early exercise features. We plan to investigate this in a future work. Furthermore, for such a discretization, implicit timestepping methods can be easily proved to have bounded $l^\infty$ norm. In the following, we present a stability result.

**THEOREM 3.1.** *Assume fully implicit timestepping is used ($\theta = 1$), $\mathcal{L}^\Delta$ is discretized as in Appendix B, and that the $(x, y)$ grid satisfies $\frac{h_1}{h_2} = \frac{\sigma_{S_1}}{\sigma_{S_2}}$. Then at each timestep l, we have*

$$||u_{\log}^{(l+1)}||_\infty \leq \max(||u_{\log}^{(l)}||_\infty, ||u_{Dir}^{(l+1)}||_\infty),$$

*where $u_{Dir}^{(l+1)}$ is the Dirichlet condition imposed on $\partial_x \Omega^{\log} \cup \partial_y \Omega^{\log}$ during this timestep. The same conclusion holds for Crank-Nicolson timestepping ($\theta = \frac{1}{2}$) with a timestep restriction that scales with $O(\min(h_1^2, h_2^2))$.*

*Proof.* The result follows from Appendix B and steps in [7]. □

Solving a time-dependent PDE in three space dimensions, while feasible, could be expensive on traditional computing architecture. The computational time can be improved by using computing techniques such as implementing an ADI scheme on graphic processing units (see e.g. [13, 14]).

While we have presented Theorem 3.1 in log-price scaling, in practice we have not observed a numerical stability problem when a finite difference scheme is implemented on the original PDE (2.7) or (2.10). Note that convection is strong in the $\rho$ direction away from the mean $\eta$, and upwind differencing should be utilized where necessary.

# 4 Asymptotic solution and approximation algorithm

In practical cases, solving a full time-dependent PDE in three space dimensions could be time-consuming and undesirable. It is possible to trade accuracy for computational efficiency. Approximation formulas are desirable in that rapid computation is possible, which makes calibration or pricing much more efficient.

Following [21], we assume that the mean reversion speed $\lambda$ in (2.12) is fast, i.e. $\lambda = 1/\epsilon$, where $\epsilon \to 0$. We scale $\sigma_\rho$ such that the variance of the correlation process' invariant distribution is finite and fixed. Therefore, the volatility of correlation has the corresponding scale $\sigma_\rho = \frac{\tilde{\sigma}_\rho}{\sqrt{\epsilon}}$.
Define the differential operators

$$\begin{aligned}
\mathcal{A}_0 &= (\eta - \rho)\frac{\partial}{\partial \rho} + \frac{\tilde{\sigma_\rho}^2(1-\rho^2)}{2}\frac{\partial^2}{\partial \rho^2}, \quad \text{and} \\
\mathcal{A}_1 &= \frac{\partial}{\partial t} + \frac{\sigma_{S_1}^2 {S_1}^2}{2}\frac{\partial^2}{\partial S_1^2} + \frac{\sigma_{S_2}^2 {S_2}^2}{2}\frac{\partial^2}{\partial S_2^2} + \rho\sigma_{S_1}\sigma_{S_2}S_1S_2\frac{\partial^2}{\partial S_1 \partial S_2} \\
&\quad + rS_1\frac{\partial}{\partial S_1} + rS_2\frac{\partial}{\partial S_2} - r\mathcal{I},
\end{aligned}$$

where $\mathcal{I}$ is the identity operator. Note that $\frac{\partial}{\partial t} + \mathcal{L} = \mathcal{A}_1 + \frac{1}{\epsilon}\mathcal{A}_0$.
The pricing equation (3.1) can be written as

$$\mathcal{A}_1 V + \frac{1}{\epsilon}\mathcal{A}_0 V = 0.$$

Let $V^\epsilon$ be a power series expansion of $V$ in $\epsilon$

$$V^\epsilon = V^{(0)} + \epsilon V^{(1)} + \epsilon^2 V^{(2)} + \dots.$$

We will determine $V^{(0)}$ and $V^{(1)}$. We impose the terminal condition $V^{(0)}(T, S_1, S_2, \rho) = g(S_1, S_2)$. Upon substitution the following equations are obtained from setting the lower order terms to zero:

$$O\left(\frac{1}{\epsilon}\right) \;:\; \mathcal{A}_0 V^{(0)} = 0 \tag{4.1}$$

$$O(1) \;:\; \mathcal{A}_1 V^{(0)} + \mathcal{A}_0 V^{(1)} = 0 \tag{4.2}$$

$$O(\epsilon) \;:\; \mathcal{A}_1 V^{(1)} + \mathcal{A}_0 V^{(2)} = 0 \tag{4.3}$$

Equation (4.1) implies that we could choose $V^{(0)} = V^{(0)}(t, S_1, S_2)$, i.e. independent of $\rho$. For $\lambda > \frac{2\sigma_\rho^2}{1\pm\eta}$ (equivalently $\tilde{\sigma_\rho}^2 < \frac{1\pm\eta}{2}$), Equation (4.2) implies a centering condition

$$\langle \mathcal{A}_1 V^{(0)} \rangle = 0,$$

where $\langle \cdot \rangle$ denotes expectation with respect to the invariant distribution of $\rho$. This is because (4.2) leads to

$$\langle \mathcal{A}_1 V^{(0)} \rangle = -\int \mathcal{A}_0 V^{(1)} \Phi(\rho) d\rho = \int V^{(1)} \mathcal{A}_0^* \Phi(\rho) d\rho = 0,$$

where $\mathcal{A}_0^*$ is the adjoint of $\mathcal{A}_0$, taking also into account that $\Phi$ satisfies the stationary form of the Fokker-Planck equation $\mathcal{A}_0^*\Phi = 0$. The vanishing of the boundary terms requires that $\Phi(\pm 1) = \Phi'(\pm 1) = 0$. This can be satisfied by imposing $\lambda > \frac{2\sigma_\rho^2}{1\pm\eta}$. The explicit form of $\Phi$ is given in Appendix C. Note that condition $\lambda > \frac{2\sigma_\rho^2}{1\pm\eta}$ is more restrictive than Feller's condition by a factor of 2.

As $V^{(0)}$ is independent of $\rho$, the centering condition implies that

$$\mathcal{A}_{BS}(\bar{\rho}) V^{(0)} = 0,$$

where $\bar{\rho} = \eta$ is the mean of $\rho$ with respect to the invariant distribution (see Appendix C), and that $\mathcal{A}_{BS}(\bar{\rho})$ is the same as $\mathcal{A}_1$, except that $\rho$ is changed to the constant $\bar{\rho}$. Therefore, the zeroth order approximation $V^{(0)}$ is given by the solution to the two-dimensional Black-Scholes equation with constant correlation equal to $\bar{\rho} = \eta$.

We proceed to find $V^{(1)}$. From (4.2) we have

$$\mathcal{A}_0 V^{(1)} = -\mathcal{A}_1 V^{(0)} = -\sigma_{S_1}\sigma_{S_2} S_1 S_2 (\rho - \bar{\rho}) \frac{\partial^2 V^{(0)}}{\partial S_1 \partial S_2},$$

where the equality $\langle \mathcal{A}_{BS}(\bar{\rho}) V^{(0)} \rangle = 0$ is used to eliminate the non-cross derivative terms. Consequently, $V^{(1)} = -(\phi(\rho))\sigma_{S_1}\sigma_{S_2} S_1 S_2 \frac{\partial^2 V^{(0)}}{\partial S_1 \partial S_2} + C(t, S_1, S_2)$, where $\phi(y)$ is a solution to the equation $\mathcal{A}_0 \phi = \rho - \bar{\rho}$ and $C(t, S_1, S_2)$, arising from the integration of this ODE, is independent of $\rho$. We take, as a particular solution, $\phi(\rho) = \bar{\rho} - \rho = \eta - \rho$.

Next, we determine $C(t,S_1,S_2)$. Define the differential operator $D_{1,1} = S_1S_2\frac{\partial^2}{\partial S_1\partial S_2}$. One sees immediately that $D_{1,1}$ commutes with $\mathcal{A}_1$. The solvability of the Poisson equation (4.3) requires

$$\langle \mathcal{A}_1 V^{(1)}\rangle = 0. \tag{4.4}$$

As a result,

$$\begin{aligned}
\langle \mathcal{A}_1 V^{(1)}\rangle &= 0\\
\langle \mathcal{A}_1 C(t,S_1,S_2)\rangle &= \langle \mathcal{A}_1((\eta-\rho)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)})\rangle\\
\langle \mathcal{A}_1\rangle C(t,S_1,S_2) &= \langle(\mathcal{A}_1-\langle\mathcal{A}_1\rangle)((\eta-\rho)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)})\rangle + \langle\langle\mathcal{A}_1\rangle((\eta-\rho)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)})\rangle\\
&= \langle(\rho-\eta)(\eta-\rho)\rangle\sigma_{S_1}^2\sigma_{S_2}^2D_{1,1}^2V^{(0)}\\
&= -E\sigma_{S_1}^2\sigma_{S_2}^2D_{1,1}^2V^{(0)},
\end{aligned}$$

where $E=\frac{(1-\eta^2)\tilde{\sigma}_\rho^2}{2+\tilde{\sigma}_\rho^2}$ (see Appendix C). We specify $C(T,S_1,S_2)=0$. By commutativity, one can verify that $C(t,S_1,S_2)=(T-t)E\sigma_{S_1}^2\sigma_{S_2}^2D_{1,1}^2V^{(0)}$ is a solution, because $\langle\mathcal{A}_1\rangle=\mathcal{A}_{BS}(\eta)$, and $\mathcal{A}_{BS}(\eta)V^{(0)}=0$. Therefore, we obtain the approximation $V^{\epsilon,1}$ to $V$ given by

$$V^{\epsilon,1} = V^{(0)} + \epsilon(\rho-\eta)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)} + \epsilon(T-t)\frac{(1-\eta^2)\tilde{\sigma}_\rho^2\sigma_{S_1}^2\sigma_{S_2}^2}{2+\tilde{\sigma}_\rho^2}D_{1,1}^2V^{(0)}, \tag{4.5}$$

where, as defined earlier, $V^{(0)}$ is the solution to the two-dimensional Black-Scholes equation with constant correlation $\eta$. The error of the approximation (4.5) is given in the following theorem.

**THEOREM 4.1.** *Assume the payoff function $g$ is smooth and that $g$ and its derivatives have at most polynomial growth as their arguments approach $\pm\infty$. Assume also $\lambda > \frac{2\sigma_\rho^2}{1\pm\eta}$ (equivalently $\tilde{\sigma}_\rho{}^2 < \frac{1\pm\eta}{2}$). Then for $t<T$,*

$$\left|V(t,S_1,S_2,\rho) - V^{\epsilon,1}(t,S_1,S_2,\rho)\right| = O(\epsilon^2).$$

*Proof.* See Appendix D. □

## 4.1 Asymptotic solution based on transition density

The approximation (4.5) requires knowing the price of the derivative and its derivatives under a constant correlation model. This is known, however, only for a few derivatives, such as exchange options (via the Margrabe's formula [31]). For many other popular derivatives, such as spread options, a closed form solution is not currently available, hence the approximation (4.5) is limited in these cases. To deal with this difficulty, we propose a heuristic solution based on (4.5). The idea is simple. Since the set of European option prices fully determines the transition density, we apply heuristically (4.5) to compute the *transition density* instead.

Specifically, we denote by $f(T,S_1(T),S_2(T),\rho(T)|t,S_1,S_2,\rho)$, the joint transition density function of the terminal prices $S_1(T)$ and $S_2(T)$ and the correlation value $\rho(T)$, given the asset prices $S_1$, $S_2$ and correlation value $\rho$ at an earlier time $t$. The time-0 option price can be computed by

$$\begin{aligned}
\int_0^\infty\int_0^\infty\int_{-1}^1 \quad & e^{-rT}g\left(S_1(T),S_2(T)\right)f\left(T,S_1(T),S_2(T),\rho(T)|0,S_1,S_2,\rho\right)d\rho(T)dS_1(T)dS_2(T)\\
&= \int_0^\infty\int_0^\infty e^{-rT}g\left(S_1(T),S_2(T)\right)p_m\left(T,S_1(T),S_2(T)|0,S_1,S_2,\rho\right)dS_1(T)dS_2(T),
\end{aligned}$$

where $p_m(\cdot|\cdot)$ is the associated marginal transition density. The idea is to approximate $p_m(\cdot|\cdot)$ by $p_m^{\epsilon,1}(\cdot|\cdot)$, a perturbed version of $p_m(\cdot|\cdot)$, obtained from formally applying (4.5) to $p_m(\cdot|\cdot)$. Explicitly, this means

$$p_m^{\epsilon,1} = p_m^{(0)} + \epsilon(\rho - \eta)\sigma_{S_1}\sigma_{S_2} D_{1,1} p_m^{(0)} + \epsilon(T-t)\frac{(1-\eta^2)\tilde{\sigma}_\rho^2 \sigma_{S_1}^2 \sigma_{S_2}^2}{2 + \tilde{\sigma}_\rho^2} D_{1,1}^2 p_m^{(0)}. \tag{4.6}$$

We denote by

$$p(T, S_1(T), S_2(T)|t, S_1, S_2)$$

the joint transition density in the case of the constant correlation $\bar{\rho} = \eta$. Its explicit form is known and is given in Appendix E. It is easy to see that the zeroth order approximation of $p_m(\cdot|\cdot)$ is given by $p(\cdot|\cdot)$. The rest of the right-side on (4.5) depends on derivatives of $p(\cdot|\cdot)$. These involve some algebraic work, given also in Appendix E. Finally, we propose approximating the time-0 price of a European option by the double integral

$$\int_0^\infty \int_0^\infty e^{-rT} g\left(S_1(T), S_2(T)\right) p_m^{\epsilon,1}\left(T, S_1(T), S_2(T)|0, S_1, S_2, \rho\right) dS_1(T) dS_2(T), \tag{4.7}$$

which can be computed using quadrature methods. We present the details of this approximation in Appendix E. We demonstrate the effectiveness of this approach in Section 5.
An example algorithm that uses (4.7) for pricing is shown in Algorithm 1.

**Algorithm 1** A sample pricing algorithm with the perturbed density at time $t = 0$

1: Retrieve market data: prices $S_1, S_2$, current correlation $\rho$, risk-free rate $r$, mean reversion level of correlation $\eta$ applicable forward-looking volatilities $\sigma_{S_1}, \sigma_{S_2}, \sigma_\rho$.
2: Retrieve European option data: payoff $g(\cdot,\cdot)$, maturity $T$.
3: Obtain the constant correlation (set to mean reversion level $\eta$) marginal density $p(T, S_1(T), S_2(T)|0, S_1, S_2)$, and their derivatives $D_{1,1}(p)$ and $D_{1,1}^2(p)$. These formulae are explicitly given in Appendix E.
4: Calculate $p_m^{\epsilon,1}\left(T, S_1(T), S_2(T)|0, S_1, S_2, \rho\right)$ from (4.6).
5: Choose a grid $(0, S_1^{\max}) \times (0, S_2^{\max})$ and discretize.
6: For the particular discretization chosen, calculate the price from (4.7) with truncated $S_i$-boundaries using quadrature.

We note that the complexity of the algorithm does not depend on the maturity of the option. As a result, given model parameters, long-term options are priced with the same computational complexity, which is an advantageous feature compared to timestepping-based methods, such as simulation or PDE approaches. This feature is especially desirable in computational scenarios such as calibration, in which model parameters are implied from a given set of option prices by solving an inverse problem.

We conclude this section by noting that, although we restrict the analysis to non-dividend-paying assets, it is relatively straightforward to generalize the asymptotic solution in the case of non-zero continuous dividend rate by adjusting the risk-neutral drifts of $S_1$ and $S_2$.

## 5 Numerical experiments

In this section, we present numerical results from the implementation of our method on the following options:

(a) spread and basket options on two assets with stochastic correlation using the three-dimensional PDEs (3.1) and (3.3);

(b) quanto options with stochastic correlation using the two-dimensional PDE (2.10); and

(c) max options on two assets with stochastic correlation using a two-dimensional PDE obtained after a similarity reduction.

For all experiments, the $S$-boundary conditions are of Dirichlet type where the value on the boundary is simply the discounted payoff for the current values of the state variables. The $\rho$-boundary conditions are as those described in Section 3.2. See Equations (3.5) and (3.6) for the boundary conditions used in the price space and log-price space formulations, respectively. Similar conditions are used for the two-dimensional PDEs.

Unless otherwise stated, we use standard second order differences for all spatial derivatives, including the cross-derivatives. The only exception is when the convection term $\alpha(t,\rho)\frac{\partial V}{\partial \rho}$ is large, in which case we use forward or backward first order differences, depending on the sign of $\alpha(\cdot,\cdot)$. Whenever solving the log-price space formulation PDE (3.3), the discretization is carried out as described in Section 3.3 and Appendix B, so that a monotone discretization scheme is obtained. In all cases, the timestepping is Crank-Nicolson-Rannacher, i.e. we use the fully implicit timestepping for the first few timesteps, then switch to Crank-Nicolson for the remaining timesteps.

While it is easier to carry out stability analysis in log price space, in practice there could be computational disadvantages in solving the log transformed equation. A uniform grid in log price space becomes a non-uniform grid in price space. A fine discretization around the region of interest in the price space could require a much finer discretization of the log price grids, and hence higher computational cost. Moreover, for non-smooth payoffs, it could be hard to align the points of discontinuity (of derivatives) with nodes on the log-price grid. For this reason, in our numerical experiments with the log-price space PDE formulation (3.3), an averaging procedure has been applied to smooth out the payoff function (see [34]).

In the numerical experiments, we also report a quantity

$$\zeta_q \equiv \log\left(\left|\frac{\text{change}_{q-1}}{\text{change}_q}\right|\right) / \log(2),$$

where $\text{change}_q$ is the absolute difference of solution value from the $(q-1)$-th grid refinement to the $q$-th grid refinement. This is an estimate of numerical order of convergence, as each successive refinement involves halving the time and space step-sizes.

## 5.1 Options on two assets

We report numerical results of the proposed method on two types of options on two assets, namely spread and basket options. The market parameters are listed in Table 5.1.

### *5.1.1 Spread options*

As an illustration, we will price a spread option, more specifically a call option on spread. Mathematically, the terminal payoff of such an option is given by

$$g(S_1, S_2) = \max(S_1 - S_2 - K, 0).$$

In Table 5.2, we list numerical results for the pricing of a 1-year spread option with $K = 10$ for different values of spot prices and current levels of correlation $(S_1(0), S_2(0), \rho(0))$. For this test, we solve the log-price space formulation PDE (3.3). In the numerical experiments, instead of $[-R^{\log}, R^{\log}] \times [-R^{\log}, R^{\log}] \times [-1, 1]$ for a sufficiently large $R^{\log} > 0$ as in Proposition 3.1, we have localized the grid

| | |
|---|---|
| Volatility of first asset $\sigma_{S_1}$ | 30 % |
| Volatility of second asset $\sigma_{S_2}$ | 30 % |
| Mean reversion level of correlation $\eta$ | 0.0 |
| Mean reversion speed of correlation $\lambda$ | 3.0 |
| Volatility of correlation $\sigma_\rho$ | 50 % |
| Risk-free rate $r$ | 5 % |

Table 5.1: Market parameters for Section 5.1

to $[1,5] \times [1,5] \times [-1,1]$ in the log grid for more efficient use of computing power. As indicated, the numerical method exhibits second-order convergence.

| $n_1, n_2$ | $n_3$ | $\Delta t$ | (50,50,-0.2) | (40,50,-0.2) | (50,40,-0.2) | (50,50,0.2) | (40,50,0.2) | (50,40,0.2) |
|---|---|---|---|---|---|---|---|---|
| 20 | 10 | 0.1 | 6.3630 | 2.6413 | 9.7262 | 5.9775 | 2.3802 | 9.3587 |
| 40 | 20 | 0.05 | 5.1514 | 1.8937 | 8.4685 | 4.7068 | 1.6277 | 8.0308 |
| 80 | 40 | 0.025 | 4.8444 | 1.6994 | 8.1535 | 4.3815 | 1.4314 | 7.6944 |
| $\zeta_3$ | | | 1.98 | 1.94 | 2.00 | 1.97 | 1.94 | 1.98 |

Table 5.2: Value of the 1-year spread option with $K = 10$ at different values of $(S_1(0), S_2(0), \rho(0))$, in three successive grid refinements, using the log-price space formulation PDE (3.3). The domain is $\Omega^{\Delta,\log} = [1,5] \times [1,5] \times [-1,1]$. Nodes are placed uniformly, with $n_1$ (resp. $n_2, n_3$) being the number of subintervals in the $\log(S_1)$ (resp. $\log(S_2), \rho$) direction.

In Table 5.3, we present similar results as in Table 5.2, but from solving the price space formulation PDE (3.1). The numerical results indicate that solving directly (3.1) does not seem to pose a problem in terms of stability, although a positive discretization is no longer guaranteed.

| $n_1, n_2$ | $n_3$ | $\Delta t$ | (50,50,-0.2) | (40,50,-0.2) | (50,40,-0.2) | (50,50,0.2) | (40,50,0.2) | (50,40,0.2) |
|---|---|---|---|---|---|---|---|---|
| 20 | 10 | 0.1 | 4.5456 | 1.5370 | 7.7828 | 4.0878 | 1.2727 | 7.3315 |
| 40 | 20 | 0.05 | 4.7035 | 1.6080 | 7.9973 | 4.2367 | 1.3385 | 7.5352 |
| 80 | 40 | 0.025 | 4.7418 | 1.6274 | 8.0478 | 4.2728 | 1.3566 | 7.5829 |
| $\zeta_3$ | | | 2.04 | 1.87 | 2.09 | 2.05 | 1.86 | 2.10 |

Table 5.3: Value of the 1-year spread option with $K = 10$ at different values of $(S_1(0), S_2(0), \rho(0))$, in three successive grid refinements, using the PDE in price space formulation (3.1). The domain is $\Omega^{\Delta} = [0,200] \times [0,200] \times [-1,1]$. Nodes are placed uniformly, with $n_1$ (resp. $n_2, n_3$) being the number of subintervals in the $S_1$ (resp. $S_2, \rho$) direction.

To validate our approaches, we compare the numerical PDE prices with those obtained by the asymptotic solution (Section 4) and Monte Carlo (MC) simulations. To obtain more accurate PDE prices, the PDE solutions from Tables 5.2 and 5.3 are extrapolated using Richardson extrapolation, with convergence exponent 2, as the method is supposed and has demonstrated to achieve. The numerical results are given in Table 5.4. They show good agreement among solutions under the various approaches. In particular, the PDE and asymptotic solutions all lie in the MC's 95% confidence intervals (CIs). Here, for MC simulation, 50000 scenarios and 200 timesteps are used.

| | (50,50,-0.2) | (40,50,-0.2) | (50,40,-0.2) | (50,50,0.2) | (40,50,0.2) | (50,40,0.2) |
|---|---|---|---|---|---|---|
| PDE (3.3) extrap | 4.7420 | 1.6346 | 8.0486 | 4.2730 | 1.3659 | 7.5822 |
| PDE (3.1) extrap | 4.7545 | 1.6338 | 8.0646 | 4.2848 | 1.3627 | 7.5988 |
| asymptotic (4.7) | 4.7592 | 1.6412 | 8.0672 | 4.2666 | 1.3546 | 7.5797 |
| MC 95% CI | [4.6772, 4.8543] | [1.5965, 1.6919] | [7.9666, 8.1867] | [4.2106, 4.3744] | [1.3266, 1.4111] | [7.5011, 7.7095] |

Table 5.4: Value comparison for the 1-year spread option with $K = 10$ at different values of $(S_1(0), S_2(0), \rho(0))$. Both sets of PDE prices are extrapolated from respective data in Tables 5.2-5.3, using Richardson extrapolation, assuming quadratic convergence.

### 5.1.2 *Effect of truncated boundary*

In this section, we verify numerically, that the truncated boundary in the $S_1$- and $S_2$-directions is far enough, so that the quality of the approximation is not affected. We price again the spread option of Section 5.1.1 by solving the price space formulation PDE (3.1), this time with the truncated boundary in the $S_1$- and $S_2$-directions double as far, and with double the number of grid points in these two directions. Table 5.5 presents the results. As it can be seen, the differences between the results of Tables 5.5 and 5.3 are approximately at the level of $10^{-4}$ and they are decreasing as the number of grid points increases. Furthermore, the accuracy of the results of Table 5.3 seems to be at most at the level of $10^{-2}$. These results indicate that the truncated boundary chosen for the experiment of Table 5.3 does not compromise the quality of the numerical PDE approximation.

| $n_1, n_2$ | $n_3$ | $\Delta t$ | (50,50,-0.2) | (40,50,-0.2) | (50,40,-0.2) | (50,50,0.2) | (40,50,0.2) | (50,40,0.2) |
|---|---|---|---|---|---|---|---|---|
| 40 | 10 | 0.05 | 4.5504<br>4.8e-03 | 1.5372<br>1.3e-04 | 7.7888<br>6.0e-03 | 4.0927<br>4.9e-03 | 1.2723<br>-3.6e-04 | 7.3378<br>6.3e-03 |
| 80 | 20 | 0.025 | 4.7046<br>1.2e-03 | 1.6081<br>1.4e-04 | 7.9987<br>1.4e-03 | 4.2379<br>1.2e-03 | 1.3385<br>1.5e-05 | 7.5367<br>1.5e-03 |
| 160 | 40 | 0.0125 | 4.7420<br>2.9e-04 | 1.6274<br>4.1e-05 | 8.0481<br>3.4e-04 | 4.2731<br>3.0e-04 | 1.3566<br>8.0e-06 | 7.5833<br>3.6e-04 |

Table 5.5: Value of the 1-year spread option with $K = 10$ at different values of $(S_1(0), S_2(0), \rho(0))$, using the PDE in price space formulation (3.1). The domain is $\Omega^{\Delta} = [0, 400] \times [0, 400] \times [-1, 1]$. Below each line of values, the differences from the respective values of Table 5.3 are also presented.

### 5.1.3 *Effect of mean reversion rate on accuracy of asymptotic solution*

The development of the asymptotic solution in Section 4 suggests that the accuracy of the asymptotic solution improves as the mean reversion rate $\lambda$ increases. In this section, we focus on the effect of $\lambda$ on the accuracy of the asymptotic solution. For this reason, we consider pricing spread options with the parameters of Table 5.1, except that we vary $\lambda$ from 0.5 to 2.5, and we scale $\tilde{\sigma_\rho}$ accordingly, so that ${\sigma_\rho}^2/\lambda$ remains constant. We consider the solutions by the price space PDE (3.1), the Richardson-extrapolated price space PDE, the asymptotic (4.7), and the MC simulation. Table 5.6 shows the results. We have run PDE experiments for grid sizes with $n_1, n_2$ and $n_3$ as in Section 5.1.1, but, for brevity, we only display the PDE solution for $n_1 = n_2 = 80$, $n_3 = 40$.

In Figure 5.1, we plot the solutions at one point versus $\lambda$. The results of Table 5.6 and Figure 5.1 indicate that, once $\lambda$ is above 1.5 approximately, the asymptotic solution falls within the 95% confidence interval of the MC simulation. We also notice monotonic convergence of the asymptotic to both the PDE and extrapolated PDE solutions as $\lambda$ increases. The agreement of the asymptotic to the PDE solutions varies from approximately one to three digits as $\lambda$ varies from 0.5 to 3. Clearly, as long as $\lambda$ is reasonably large, the asymptotic solution (4.7) is a powerful alternative to the PDE solution.

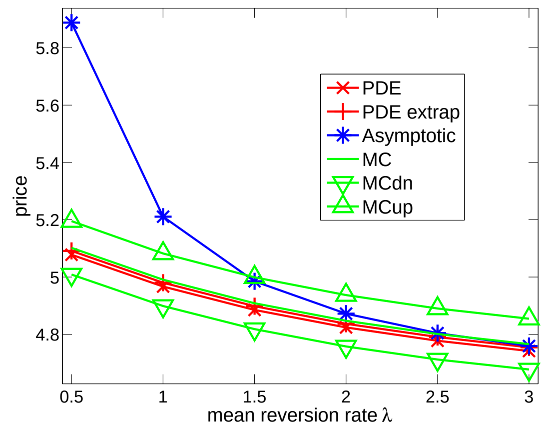


Figure 5.1: Comparison of 1-year spread option prices with $K = 10$ at $(S_1, S_2, \rho) = (50, 50, -0.2)$ versus $\lambda$ for different numerical solution methods. Parameters as in Table 5.1, except that $\lambda$ varies as shown and $\sigma_\rho = \sqrt{\lambda/12}$.

| | (50,50,-0.2) | (40,50,-0.2) | (50,40,-0.2) | (50,50,0.2) | (40,50,0.2) | (50,40,0.2) |
|---|---|---|---|---|---|---|
| $\lambda = 0.5, \quad \sigma_\rho = 0.2041$ | | | | | | |
| PDE (3.1) | 5.0778 | 1.8282 | 8.3783 | 3.9122 | 1.1558 | 7.2227 |
| PDE (3.1) extrap | 5.0911 | 1.8350 | 8.3957 | 3.9234 | 1.1617 | 7.2373 |
| asymptotic (4.7) | 5.8880 | 2.3569 | 9.1614 | 2.9325 | 0.6376 | 6.2366 |
| MC 95% CI | [5.0087, 5.1950] | [1.7928, 1.8957] | [8.2899, 8.5181] | [3.8105, 3.9624] | [1.1051, 1.1802] | [7.0812, 7.2791] |
| $\lambda = 1.0, \quad \sigma_\rho = 0.2887$ | | | | | | |
| PDE (3.1) | 4.9669 | 1.7620 | 8.2691 | 4.0303 | 1.2221 | 7.3405 |
| PDE (3.1) extrap | 4.9800 | 1.7687 | 8.2864 | 4.0418 | 1.2280 | 7.3555 |
| asymptotic (4.7) | 5.2107 | 1.9275 | 8.5049 | 3.7329 | 1.0678 | 7.0425 |
| MC 95% CI | [4.8989, 5.0822] | [1.7275, 1.8279] | [8.1834, 8.4089] | [3.9275, 4.0831] | [1.1699, 1.2479] | [7.1970, 7.3980] |
| $\lambda = 1.5, \quad \sigma_\rho = 0.3536$ | | | | | | |
| PDE (3.1) | 4.8851 | 1.7132 | 8.1887 | 4.1178 | 1.2708 | 7.4279 |
| PDE (3.1) extrap | 4.8981 | 1.7198 | 8.2058 | 4.1295 | 1.2768 | 7.4432 |
| asymptotic (4.7) | 4.9849 | 1.7843 | 8.2860 | 3.9997 | 1.2112 | 7.3111 |
| MC 95% CI | [4.8181, 4.9991] | [1.6797, 1.7783] | [8.1046, 8.3282] | [4.0140, 4.1722] | [1.2179, 1.2980] | [7.2830, 7.4863] |
| $\lambda = 2.0, \quad \sigma_\rho = 0.4082$ | | | | | | |
| PDE (3.1) | 4.8240 | 1.6767 | 8.1286 | 4.1836 | 1.3073 | 7.4936 |
| PDE (3.1) extrap | 4.8369 | 1.6832 | 8.1456 | 4.1954 | 1.3133 | 7.5092 |
| asymptotic (4.7) | 4.8720 | 1.7127 | 8.1766 | 4.1332 | 1.2829 | 7.4454 |
| MC 95% CI | [4.7577, 4.9371] | [1.6440, 1.7412] | [8.0458, 8.2679] | [4.0790, 4.2391] | [1.2541, 1.3357] | [7.3483, 7.5533] |
| $\lambda = 2.5, \quad \sigma_\rho = 0.4564$ | | | | | | |
| PDE (3.1) | 4.7776 | 1.6489 | 8.0830 | 4.2338 | 1.3351 | 7.5439 |
| PDE (3.1) extrap | 4.7904 | 1.6554 | 8.0999 | 4.2457 | 1.3411 | 7.5596 |
| asymptotic (4.7) | 4.8043 | 1.6698 | 8.1109 | 4.2132 | 1.3260 | 7.5260 |
| MC 95% CI | [4.7121, 4.8903] | [1.6171, 1.7133] | [8.0011, 8.2220] | [4.1289, 4.2904] | [1.2817, 1.3645] | [7.3984, 7.6047] |

Table 5.6: Value comparison for the 1-year spread option with $K = 10$ at different values of $(S_1(0), S_2(0), \rho(0))$ and several values of $\lambda$. The PDE solution is obtained with $n_1 = n_2 = 80$ and $n_3 = 40$, while the extrapolated PDE solution is obtained using PDE solutions with two grid sizes, the above and a coarser one, assuming quadratic convergence.

### 5.1.4 *Basket options*

We also consider an equal-weighted basket call option whose payoff is

$$g(S_1, S_2) = \max(S_1 + S_2 - K, 0).$$

In Tables 5.7 and 5.8, we present convergence results for the two versions of PDE for different values of $(S_1(0), S_2(0), \rho(0))$ with the strike $K = 100$. Again, while a monotone discretization scheme is no longer guaranteed in price space, numerically we do not observe a problem with this approach. Comparison of values using the PDE, Monte Carlo and asymptotic methods is presented in Table 5.9. Given the symmetry of the problem, the solution should be symmetric in $(S_1(0), S_2(0))$. The discrepancy in Monte Carlo solutions is due to randomness from the simulations.

| $n_1, n_2$ | $n_3$ | $\Delta t$ | (50,50,-0.2) | (40,50,-0.2) | (50,40,-0.2) | (50,50,0.2) | (40,50,0.2) | (50,40,0.2) |
|---|---|---|---|---|---|---|---|---|
| 20 | 10 | 0.1 | 13.6654 | 7.8450 | 7.8450 | 14.0295 | 8.1965 | 8.1965 |
| 40 | 20 | 0.05 | 11.4502 | 6.0209 | 6.0209 | 11.9017 | 6.4390 | 6.4390 |
| 80 | 40 | 0.025 | 10.8984 | 5.5584 | 5.5584 | 11.3750 | 5.9966 | 5.9966 |
| $\zeta_3$ | | | 2.01 | 1.98 | 1.98 | 2.01 | 1.99 | 1.99 |

Table 5.7: Value of the 1-year equal-weighted basket call option with strike $K = 100$ at different values of $(S_1(0), S_2(0), \rho(0))$, in three successive grid refinements, using the log-price space formulation PDE (3.3). The domain is $\Omega^{\Delta,\log} = [1, 5] \times [1, 5] \times [-1, 1]$. Nodes are placed uniformly, with $n_1$ (resp. $n_2, n_3$) being the number of subintervals in the $x = \log(S_1)$ (resp. $y = \log(S_2), \rho$) direction.

| $n_1, n_2$ | $n_3$ | $\Delta t$ | (50,50,-0.2) | (40,50,-0.2) | (50,40,-0.2) | (50,50,0.2) | (40,50,0.2) | (50,40,0.2) |
|---|---|---|---|---|---|---|---|---|
| 20 | 10 | 0.1 | 10.4788 | 5.2020 | 5.2020 | 10.9576 | 5.6262 | 5.6262 |
| 40 | 20 | 0.05 | 10.6710 | 5.3601 | 5.3601 | 11.1537 | 5.8006 | 5.8006 |
| 80 | 40 | 0.025 | 10.7165 | 5.3985 | 5.3985 | 11.2001 | 5.8428 | 5.8428 |
| $\zeta_3$ | | | 2.08 | 2.04 | 2.04 | 2.08 | 2.05 | 2.05 |

Table 5.8: Value of the 1-year equal-weighted basket call option with strike $K = 100$ at different values of $(S_1(0), S_2(0), \rho(0))$, in three successive grid refinements, using the price space formulation PDE (3.1). The domain is $\Omega^{\Delta} = [0, 200] \times [0, 200] \times [-1, 1]$. Nodes are placed uniformly, with $n_1$ (resp. $n_2, n_3$) being the number of subintervals in the $S_1$ (resp. $S_2, \rho$) direction.

| | (50,50,-0.2) | (40,50,-0.2) | (50,40,-0.2) | (50,50,0.2) | (40,50,0.2) | (50,40,0.2) |
|---|---|---|---|---|---|---|
| PDE (3.3) extrap | 10.7145 | 5.4043 | 5.4043 | 11.1992 | 5.8491 | 5.8491 |
| PDE (3.1) extrap | 10.7317 | 5.4113 | 5.4113 | 11.2156 | 5.8569 | 5.8569 |
| asymptotic (4.7) | 10.7131 | 5.3945 | 5.3945 | 11.2199 | 5.8616 | 5.8616 |
| MC 95% CI | [10.5006, 10.7668] | [5.2276, 5.4151] | [5.2429, 5.4309] | [10.9719, 11.2542] | [5.6614, 5.8629] | [5.6782, 5.8804] |

Table 5.9: Value comparison for the 1-year equal-weighted basket call option with $K = 100$ at different values of $(S_1(0), S_2(0), \rho(0))$. Both sets of PDE prices are extrapolated from respective data in Tables 5.7-5.8 using Richardson extrapolation, assuming quadratic convergence.

We conclude this subsection by noting that, although call payoffs do not satisfy the boundedness required in Proposition 3.1, numerically, we do not observe any a problem, and good agreement is achieved among different approaches. Also see Remark 3.2.

### 5.2 Quanto options

In this section, we consider the pricing of a quanto option under stochastic correlation as in model (2.9) and (2.12). Similar to previous experiments, we assume that the parameters are calibrated so that $\Lambda \equiv 0$. As an illustration, we price a 5-year quanto call option with payoff

$$g(S(T)) = \max(S(T) - K, 0),$$

where $K = 100$. The parameters to the model are given in Table 5.10.

| | |
|---|---|
| Volatility of price $\sigma_S$ | 30 % |
| Volatility of exchange rate $\sigma_R$ | 10 % |
| Domestic risk-free rate $r_d$ | 5 % |
| Foreign risk-free rate $r_f$ | 3 % |
| Mean reversion rate of correlation $\lambda$ | 3.0 |
| Mean reversion level of correlation $\eta$ | -0.1 |
| Volatility of correlation $\sigma_\rho$ | 30 % |

Table 5.10: Market parameters for quanto option

In this case, the pricing PDE is (2.10). Because there is no cross term in the PDE, by discretizing the first derivatives carefully in the upwind direction when necessary, a positive discretization can be obtained without grid-size restrictions. Therefore, techniques such as non-uniform spacing can be used without sacrificing a positive discretization. In our experiment, we use a non-uniform grid that concentrates around the strike value 100 (see, e.g. [6]).

Results in Table 5.11 show that the PDE method has numerically quadratic convergence. Table 5.12 shows good agreement among solutions obtained from different techniques.

| $n_1$ | $n_3$ | $\Delta t$ | $(100, -0.2)$ | $(120, -0.1)$ | $(90, 0.0)$ |
|---|---|---|---|---|---|
| 50 | 10 | 0.5 | 29.9609 | 43.9209 | 23.4803 |
| 100 | 20 | 0.25 | 29.9843 | 43.9513 | 23.5008 |
| 200 | 40 | 0.125 | 29.9894 | 43.9578 | 23.5052 |
| 400 | 800 | 0.0625 | 29.9907 | 43.9595 | 23.5064 |
| $\zeta_3$ | | | 2.21 | 2.22 | 2.22 |
| $\zeta_4$ | | | 1.91 | 1.90 | 1.91 |

Table 5.11: Value of a 5-year quanto call option with strike $K = 100$ at different values of $(S(0), \rho(0))$, in four successive grid refinements, using PDE (2.10). The domain is $\Omega^\Delta = [0, 500] \times [-1, 1]$. There are $n_1$ (resp. $n_3$) subintervals in the $S$ (resp. $\rho$) direction.

| | (100,-0.2) | (120, -0.1) | (90, 0.0) |
|---|---|---|---|
| PDE (2.10) | 29.9910 | 43.9599 | 23.5067 |
| asymptotic solution | 29.9674 | 43.8848 | 23.4952 |
| MC 95% CI | [29.6126, 30.6397] | [43.3529, 44.6529] | [23.1074, 23.9883] |

Table 5.12: Value comparison for the 5-year quanto call option with strike $K = 100$ at different values of $(S(0), \rho(0))$. The set of PDE prices is extrapolated from respective data in Table 5.11 using Richardson extrapolation, assuming quadratic convergence.

In Table 5.13, we present values of selected partial derivatives of the option price with respect to the underlying asset price $S$ and the correlation factor $\rho$. Note that, although second order convergence is not guaranteed due to the choice of upwind differencing, in most cases we obtain second order convergence.

| $n_1$ | $n_3$ | $\Delta t$ | $(100, -0.2)$ | $(120, -0.1)$ | $(90, 0.0)$ |
|---|---|---|---|---|---|
| $\partial^2 V/\partial S^2$ | | | | | |
| 50 | 10 | 0.5 | $4.5994 \times 10^{-3}$ | $3.1606 \times 10^{-3}$ | $5.5495 \times 10^{-3}$ |
| 100 | 20 | 0.25 | $4.6107 \times 10^{-3}$ | $3.1625 \times 10^{-3}$ | $5.5434 \times 10^{-3}$ |
| 200 | 40 | 0.125 | $4.6129 \times 10^{-3}$ | $3.1630 \times 10^{-3}$ | $5.5444 \times 10^{-3}$ |
| 400 | 800 | 0.0625 | $4.6134 \times 10^{-3}$ | $3.1631 \times 10^{-3}$ | $5.5445 \times 10^{-3}$ |
| $\zeta_3$ | | | 2.31 | 2.03 | 2.63 |
| $\zeta_4$ | | | 2.36 | 1.76 | 2.73 |
| $\partial V/\partial \rho$ | | | | | |
| 50 | 10 | 0.5 | $-6.6204 \times 10^{-1}$ | $-8.8469 \times 10^{-1}$ | $-5.4737 \times 10^{-1}$ |
| 100 | 20 | 0.25 | $-6.6191 \times 10^{-1}$ | $-8.8515 \times 10^{-1}$ | $-5.4806 \times 10^{-1}$ |
| 200 | 40 | 0.125 | $-6.6194 \times 10^{-1}$ | $-8.8526 \times 10^{-1}$ | $-5.4817 \times 10^{-1}$ |
| 400 | 800 | 0.0625 | $-6.6196 \times 10^{-1}$ | $-8.8529 \times 10^{-1}$ | $-5.4818 \times 10^{-1}$ |
| $\zeta_3$ | | | 1.80 | 2.05 | 2.70 |
| $\zeta_4$ | | | 0.94 | 2.01 | 2.78 |

Table 5.13: Selected sensitives of the option price with respect to $S$ and $\rho$.

### 5.3 Effects of model parameters

In this section, the effect of correlation model parameters on option prices is studied. We will focus on the max option, which has payoff

$$V(t = T, S_1(T), S_2(T)) = \max(S_1(T), S_2(T)). \tag{5.1}$$

For options of this form, it is not necessary to solve the full three-dimensional PDE (3.1). Instead, a similarity reduction is possible because of the nature of the payoff. For $\tau = T - t > 0$, define $W(\tau, S_1, S_2)$ by

$$V(\tau, S_1, S_2) = S_1 W(\tau, S_1, S_2).$$

Introduce the similarity reduction $\xi = S_2/S_1$, corresponding to a change of numéraire. It is now straightforward to see that

$$W_\tau = \frac{(\sigma_{S_1}^2 + \sigma_{S_2}^2 - 2\rho\sigma_{S_1}\sigma_{S_2})\xi^2}{2} W_{\xi\xi} + \frac{\beta^2}{2} W_{\rho\rho} + \alpha W_\rho, \tag{5.2}$$

where $\alpha, \beta$ are chosen as in (2.12). The reduced problem has terminal condition

$$W(\tau = 0, \xi) = \max(1, \xi).$$

The diffusion coefficient in (5.2) is non-negative because $\sigma_{S_1}^2 + \sigma_{S_2}^2 - 2\rho\sigma_{S_1}\sigma_{S_2} \geq 0$. It should be noted that $r$, the risk-free rate, factors out of the pricing problem naturally. This similarity reduction can be easily seen to be the PDE equivalent to a measure change from the $T$-forward measure to the $S_1$-measure.

For illustration purposes we have restricted to non-dividend-paying assets. For dividend-paying assets, the same variable transformation can be carried out. In that case, a PDE two-dimensional in space similar to (5.2) will be obtained, with a convection and a discounting term.

The discretization of (5.2) is less restricted than that of (3.1). A positive discretization is ensured with usual central differences of the second derivatives and careful discretization of the first derivatives in the upwind direction where necessary. In our experiment, we solve (5.2) on a uniform grid of $801 \times 81$ nodes in $[0, 5] \times [-1, 1]$, with timestep 0.025.

In Figure 5.2, we show the effect of the long-term mean reversion level $\eta$ of correlation on the price of the max option. With higher $\eta$, the expected value of the total correlation experienced during the life of the option is increased. Naturally this leads to a lower value of the optionality. This effect is captured also by the asymptotic solution (4.5), which we restate here:

$$V^{\epsilon,1} = V^{(0)} + \epsilon(\rho - \eta)\sigma_{S_1}\sigma_{S_2} D_{1,1}V^{(0)} + \epsilon(T - t)\frac{(1 - \eta^2)\tilde{\sigma}_\rho^2 \sigma_{S_1}^2 \sigma_{S_2}^2}{2 + \tilde{\sigma}_\rho^2} D_{1,1}^2 V^{(0)}.$$

Recall $V^{(0)}$ is the Black-Scholes price with constant correlation $\eta$. The Black-Scholes sensitivity $\frac{\partial V^{(0)}}{\partial \eta}$ is negative, and the dominant zeroth order term decreases in value as $\eta$ increases, for $\epsilon \ll 1$. The effect of the spot-correlation is present in the first order correction (second term in the above equation), and is of order $\epsilon$ for fast mean reversion. The sensitivity $D_{1,1}V^{(0)}$ of the Black-Scholes price is negative as it is a positive multiple of $\frac{\partial V^{(0)}}{\partial \eta}$. This explains the three decreasing set of prices for different values of $\rho(0) = \rho_0$.

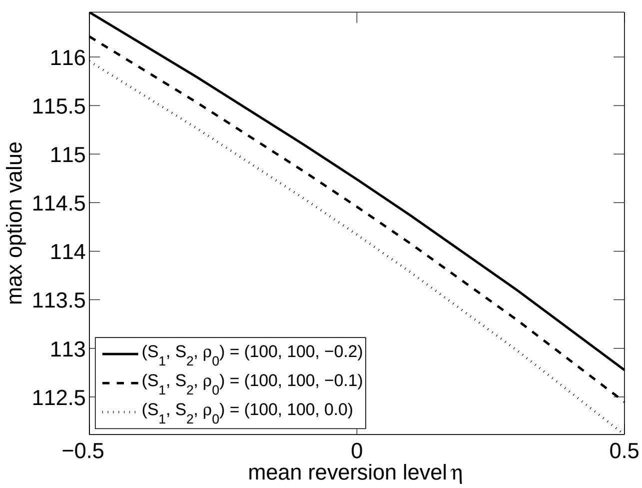


Figure 5.2: Effect of $\eta$ on max option prices. Other parameters: $\sigma_{S_1} = 0.2$, $\sigma_{S_2} = 0.3$, $\lambda = 2.0$, $\sigma_\rho = 1.0$, maturity is 1 year.

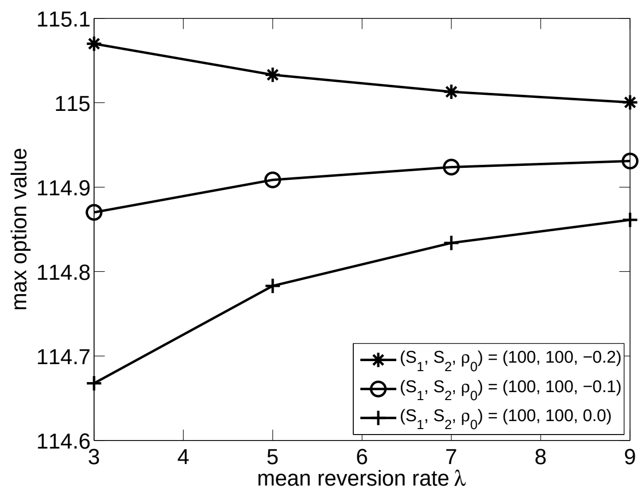


Figure 5.3: Effect of $\lambda$ on max option prices. Other parameters: $\sigma_{S_1} = 0.2$, $\sigma_{S_2} = 0.3$, $\eta = -0.1$, $\sigma_\rho = 1.0$, maturity is 1 year.

In Figure 5.3, we show the effect of the mean reversion speed $\lambda$ on the prices of the max option. As $\lambda$ increases, it is plausible that any deviation from the mean of the correlation is more heavily punished with the stronger convection, and one should expect a price closer to the Black-Scholes price with constant correlation equal to the long-term mean. Once again this is encoded in (4.5). The terms involving $\epsilon$ decrease in absolute value and convergence towards $V^{(0)}$, the Black-Scholes price with constant correlation $\eta$, is expected.

When $\rho = \eta = -0.1$, the sensitivity $D_{1,1}^2 V^{(0)}$ is negative. Therefore, while the second term in (4.5) vanishes, an upward trend is still predicted by the formula (as $\epsilon$ decreases), albeit of a much smaller magnitude due to the further diminishing effects of $\sigma_{S_1}^2 \sigma_{S_2}^2$.

## 6 Conclusions

In this paper, we have studied the problem of option pricing in the presence of stochastic correlation from a computational viewpoint. Starting with the derivation of the pricing PDE, we have developed two approaches to computing option values in this setting.

The first approach is a numerical PDE method. Unique to this problem is the specification of the boundary *behaviour* when the correlation $\rho$ is $\pm 1$. The boundary condition is necessary when one uses such numerical techniques as the finite difference method. Following the findings in [17, 18], we propose using a "PDE condition" on the $\rho$-boundaries. Furthermore, we discuss other important numerical issues such as meshing, discretization of the cross term and numerical stability of the numerical scheme (with this somewhat unusual boundary condition).

When the correlation process exhibits fast mean reversion, a second approach, based on singular perturbation [21], is developed. The asymptotic solution involves a correction to the (multi-asset) Black-Scholes price under a constant correlation. For options where analytic solutions or derivatives for a constant correlation under the Black-Scholes multi-dimensional framework are not available or are difficult to obtain, we have studied a quadrature method based on the asymptotic density. Exact calculations of such density corrections are provided. Our numerical experiments indicate that this approach is in great agreement with Monte Carlo and PDE pricers. Furthermore, this first order correction is able to capture effects of model parameters on prices, as shown in our numerical experiments.

## A Proof of Theorem 2.1

Consider the European option price given by

$$v(t,x,y,\rho) = e^{-r(T-t)}\mathbf{E}_{t,x,y,\rho}\left[g(S_1(T),S_2(T))\right],$$

where we take expectation in the risk-neutral measure induced by (2.12). We follow [17] and start by first proving that the option price is continuous on $(0,T)\times(0,\infty)^2\times(-1,1)$.

Let $(t_n,x_n,y_n,\rho_n)$ be a sequence of points converging to $(t,x,y,\rho)$. Let $S_1^n$, $S_2^n$, and $\rho^n$ be solutions to their corresponding SDEs (2.1) and (2.2) (having parameter choices (2.12)) with initial conditions $S_1^n(t_n)=x_n$, $S_2^n(t_n)=y_n$, and $\rho^n(t_n)=\rho_n$, respectively. The explicit expressions for $S_1^n(T)$, $S_2^n(T)$ are

$$\begin{aligned} S_1^n(T) &= x_n \exp\left(\frac{\left(r-\sigma_{S_1}^2\right)(T-t_n)}{2} + \sigma_{S_1}\int_{t_n}^T dW_1(s)\right) \\ S_2^n(T) &= y_n \exp\left(\frac{\left(r-\sigma_{S_2}^2\right)(T-t_n)}{2} + \sigma_{S_2}\int_{t_n}^T \rho^n(s)dW_1(s) + \sigma_{S_2}\int_{t_n}^T \sqrt{1-(\rho^n(s))^2}dW_2(s)\right) \end{aligned} \tag{A.1}$$

where $dW_1(s)$ and $dW_2(s)$ are independent Brownian motions adapted to the same filtrations generated by $dB_1(s)$ and $dB_2(s)$. Similarly, define $S_1$, $S_2$, and $\rho$ accordingly with the initial conditions $S_1(t)=x$, $S_2(t)=y$, and $\rho(t)=\rho$, respectively. Note that $S_1(T)$, $S_2(T)$ can also be expressed in the same form as (A.1).

By the Yamada-Watanabe theorem, pathwise uniqueness holds for our choice of the correlation model (2.12). It follows from [1] that

$$\mathbf{E}\left[\sup_{s\le T}\left|\rho^n(s)-\rho(s)\right|^2\right] \longrightarrow 0 \quad \text{as} \quad n\to\infty. \tag{A.2}$$

Therefore, using Itô isometry,

$$\begin{aligned}
\mathbf{E}\Big[\Big(\int_{t_n}^{T}\rho^n(s)dW_1(s) \;&-\; \int_t^T \rho(s)dW_1(s)\Big)^2\Big] \\
&\le\; 2\mathbf{E}\Big[\Big(\int_{t_n}^{T}(\rho^n(s)-\rho(s))dW_1(s)\Big)^2\Big] + 2\mathbf{E}\Big[\Big(\int_t^{t_n}\rho(s)dW_1(s)\Big)^2\Big] \\
&=\; 2\mathbf{E}\Big[\int_{t_n}^{T}(\rho^n(s)-\rho(s))^2 ds\Big] + 2\mathbf{E}\Big[\int_t^{t_n}\rho(s)^2 ds\Big] \longrightarrow 0
\end{aligned}$$

as $n \to \infty$. We also have

$$\begin{aligned}
&\mathbf{E}\Big[\Big(\int_{t_n}^{T}\sqrt{1-(\rho^n(s))^2}dW_2(s) - \int_t^T \sqrt{1-(\rho(s))^2}dW_2(s)\Big)^2\Big] \\
\le\;& \mathbf{E}\Big[\Big(\int_{t_n}^{T}\sqrt{|(\rho^n(s))^2-(\rho(s))^2|}dW_2(s)\Big)^2\Big] + \mathbf{E}\Big[\Big(\int_t^{t_n}\sqrt{1-(\rho(s))^2}dW_2(s)\Big)^2\Big] \\
\le\;& 2\mathbf{E}\Big[\Big(\int_{t_n}^{T}|\rho^n(s)-\rho(s)|\,ds\Big)\Big] + \mathbf{E}\Big[\Big(\int_{t_n}^{t}\left(1-(\rho(s))^2\right)ds\Big)\Big] \longrightarrow 0,
\end{aligned}$$

as $n \to \infty$. This is due to Itô isometry, Lipschitz continuity of the function $f(y) = y^2$ on the domain $[-1, 1]$, and Jensen's inequality. In view of (A.1) above, it follows that $S_1^n(T)$ and $S_2^n(T)$ respectively converge to $S_1(T)$ and $S_2(T)$ in probability, as $n \to \infty$. Since $g(\cdot,\cdot)$ is bounded, it follows that

$$\mathbf{E}_{t_n,x_n,y_n,\rho_n}\left[g(S_1^n(T), S_2^n(T))\right] \to \mathbf{E}_{t,x,y,\rho}\left[g(S_1(T), S_2(T))\right], \quad \text{as} \quad n \to \infty.$$

Hence, the option price $v$ is continuous.

Next, following Theorem 2.7 in [26], and since $v$ is a continuous (stochastic) solution, it can be shown that $v$ is also a classical solution (in $C^{1,2,2,2}$) that satisfies (2.7).

# B Discretization

We define a $(n_3+1)\times(n_3+1)$ matrix $\omega$ in the $\rho$-direction by first defining the three $(n_3+1)\times(n_3+1)$ matrices

$$\omega_{kl}^{(c)} = \begin{cases} \frac{\sigma_\rho^2(1-\rho_k^2)}{2h_3^2} - \frac{\lambda(\eta-\rho_k)}{2h_3} & \text{for } l = k-1,\ 0 < k < n_3 \\ -\frac{\sigma_\rho^2(1-\rho_k^2)}{h_3^2} & \text{for } l = k,\ 0 < k < n_3 \\ \frac{\sigma_\rho^2(1-\rho_k^2)}{2h_3^2} + \frac{\lambda(\eta-\rho_k)}{2h_3} & \text{for } l = k+1,\ 0 < k < n_3 \\ 0 & \text{else} \end{cases} \tag{B.1}$$

$$\omega_{kl}^{(b)} = \begin{cases} \frac{\sigma_\rho^2(1-\rho_k^2)}{2h_3^2} - \frac{\lambda(\eta-\rho_k)}{h_3} & \text{for } l = k-1,\ 0 < k \le n_3 \\ -\frac{\sigma_\rho^2(1-\rho_k^2)}{h_3^2} + \frac{\lambda(\eta-\rho_k)}{h_3} & \text{for } l = k,\ 0 < k \le n_3 \\ \frac{\sigma_\rho^2(1-\rho_k^2)}{2h_3^2} & \text{for } l = k+1,\ 0 < k < n_3 \\ 0 & \text{else} \end{cases} \tag{B.2}$$

$$\omega_{kl}^{(f)} = \begin{cases} \frac{\sigma_\rho^2(1-\rho_k^2)}{2h_3^2} & \text{for } l = k-1,\ 0 \le k < n_3 \\ -\frac{\sigma_\rho^2(1-\rho_k^2)}{h_3^2} - \frac{\lambda(\eta-\rho_k)}{h_3} & \text{for } l = k,\ 0 \le k < n_3 \\ \frac{\sigma_\rho^2(1-\rho_k^2)}{2h_3^2} + \frac{\lambda(\eta-\rho_k)}{h_3} & \text{for } l = k+1,\ 0 \le k < n_3 \\ 0 & \text{else} \end{cases} \tag{B.3}$$

Then, starting with $\omega = \omega^{(c)}$, rows of $\omega$ are modified one-by-one (chosen from any of the three $\omega^{(\cdot)}$) so that off-diagonals are non-negative. The first row of $\omega$ (corresponding to $k = 0$) should be based on forward differencing while the last row (corresponding to $k = n_3$) should be based on backward differencing.

Denote $u_{i,j,k} = u(x_i, y_j, \rho_k)$. Now following [7], the cross derivatives are discretized as follows:

$$\frac{\partial^2 u}{\partial x \partial y}(x_i, y_j, \rho_k) \approx \frac{2u_{i,j,k}+u_{i+1,j+1,k}+u_{i-1,j-1,k}-u_{i+1,j,k}-u_{i-1,j,k}-u_{i,j+1,k}-u_{i,j-1,k}}{2h_1h_2} \text{ for } \rho_k \ge 0, \tag{B.4}$$

$$\frac{\partial^2 u}{\partial x \partial y}(x_i, y_j, \rho_k) \approx \frac{-2u_{i,j,k}-u_{i+1,j-1,k}-u_{i-1,j+1,k}+u_{i+1,j,k}+u_{i-1,j,k}+u_{i,j+1,k}+u_{i,j-1,k}}{2h_1h_2} \text{ for } \rho_k < 0. \tag{B.5}$$

Intuitively, the case for $\rho \ge 0$ corresponds to the discretization of the cross derivative operator

$$\frac{1}{2}(\partial_{x-}\partial_{y-} + \partial_{x+}\partial_{y+}),$$

where $\pm$ indicates the direction of the one-sided difference. Similarly, the case for $\rho < 0$ corresponds to the discretization scheme

$$\frac{1}{2}(\partial_{x+}\partial_{y-} + \partial_{x-}\partial_{y+}).$$

Denote by $\chi^{(p)}$ the induced cross derivative discretization matrix from the $\rho \ge 0$ case, and similarly for $\chi^{(m)}$ for the $\rho < 0$ case. The cross derivative discretization can be written as $\frac{\partial^2 u}{\partial x \partial y}(x_i, y_j, \rho_k) \approx \sum_{i',j'} \chi_{ij,i'j'}^{(*)} u_{i',j',k}$. We now describe the discretization in $x$ and $y$ directions.

Denote $\mu_i = r - \frac{\sigma_{S_i}^2}{2}$, $i = 1, 2$. With the discretized domain $\Omega^\Delta$ as in Section 3.3 define the following $(n_1+1) \times (n_1+1)$ matrices:

$$\phi_{il}^{(c)} = \begin{cases} \frac{\sigma_{S_1}^2}{2h_1^2} - \frac{\mu_1}{2h_1} & \text{for } l = i-1,\ 1 \le i < n_1 \\ -\frac{\sigma_{S_1}^2}{h_1^2} & \text{for } l = i,\ 1 \le i < n_1 \\ \frac{\sigma_{S_1}^2}{2h_1^2} + \frac{\mu_1}{2h_1} & \text{for } l = i+1,\ 1 \le i < n_1 \\ 0 & \text{else} \end{cases} \tag{B.6}$$

$$\phi_{il}^{(b)} = \begin{cases} \frac{\sigma_{S_1}^2}{2h_1^2} - \frac{\mu_1}{h_1} & \text{for } l = i-1,\ 1 \le i < n_1 \\ -\frac{\sigma_{S_1}^2}{h_1^2} + \frac{\mu_1}{h_1} & \text{for } l = i,\ 1 \le i < n_1 \\ \frac{\sigma_{S_1}^2}{2h_1^2} & \text{for } l = i+1,\ 1 \le i < n_1 \\ 0 & \text{else} \end{cases} \tag{B.7}$$

$$\phi_{il}^{(f)} = \begin{cases} \frac{\sigma_{S_1}^2}{2h_1^2} & \text{for } l = i-1,\ 1 \le i < n_1 \\ -\frac{\sigma_{S_1}^2}{h_1^2} - \frac{\mu_1}{h_1} & \text{for } l = i,\ 1 \le i < n_1 \\ \frac{\sigma_{S_1}^2}{2h_1^2} + \frac{\mu_1}{h_1} & \text{for } l = i+1,\ 1 \le i < n_1 \\ 0 & \text{else} \end{cases} \tag{B.8}$$

The matrix $\phi^{(c)}$ corresponds to the spatial discretization of terms involving $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial}{\partial x}$ using central differences. The matrices $\phi^{(f)}$ and $\phi^{(b)}$ correspond to the discretization of the same terms using forward and backward differences of the first derivatives, respectively.

For each $\rho$, we define a $(n_1+1) \times (n_1+1)$ matrix $\phi(\rho)$ which is equal to either $\phi^{(c)}$, $\phi^{(b)}$ or $\phi^{(f)}$ such that non-negativity of the off-diagonals of $\mathcal{L}^{\Delta}$ will be preserved. We can define correspondingly a matrix, say $\psi(\rho)$, for the $S_2$ direction.

The discretization matrix $A$ is specified by the action

$$\begin{aligned} (Au)_{i,j,k} &= \sum_{i'=0}^{n_1} \phi_{ii'}(\rho_k) u_{i',j,k} + \sum_{j'=0}^{n_2} \psi_{jj'}(\rho_k) u_{i,j',k} + \sigma_{S_1}\sigma_{S_2}\rho_k \Big(\mathbf{1}_{\rho_k \ge 0} \sum_{i',j'} \chi_{ij,i'j'}^{(p)} u_{i',j',k} \\ &+ \mathbf{1}_{\rho_k < 0} \sum_{i',j'} \chi_{i',j'}^{(m)} u_{i',j',k}\Big) + \sum_{k'=0}^{n_3} \omega_{kk'} u_{i,j,k'} - r u_{i,j,k}, \end{aligned} \tag{B.9}$$

for $i \ne 0, n_1$ and $j \ne 0, n_2$. For $i = 0, n_1$ or $j = 0, n_2$, one can specify $(Au)_{i,j,k} = 0$ and modify the right-side of the time-stepping equation to the imposed Dirichlet condition. From this discretization one obtains a sufficient condition for (3.10) to hold. Consider the coefficients of $u_{i+1,j,k}$ and $u_{i-1,j,k}$ in (B.9). Clearly, if

$$\frac{\sigma_{S_1}^2}{2}\frac{1}{h_1^2} - \frac{\sigma_{S_1}\sigma_{S_2}|\rho_k|}{2h_1h_2} \ge 0,$$

then one can choose between central, backward and forward differences in the $x$-direction such that the coefficients of $u_{i+1,j,k}$ and $u_{i-1,j,k}$ are non-negative. This is the case if we space the grid in such a way that

$$\frac{h_1}{h_2} \le \frac{\sigma_{S_1}}{\sigma_{S_2}}.$$

By considering the non-negativity of the coefficients of $u_{i,j+1,k}$ and $u_{i,j-1,k}$, we get, as a sufficient condition,

$$\frac{\sigma_{S_2}^2}{2}\frac{1}{h_2^2} - \frac{\sigma_{S_1}\sigma_{S_2}|\rho_k|}{2h_1h_2} \ge 0,$$

which is ensured by

$$\frac{h_1}{h_2} \ge \frac{\sigma_{S_1}}{\sigma_{S_2}}.$$

In a similar way, we can study the sign of the other coefficients in (B.9). In summary, if $\frac{h_1}{h_2} = \frac{\sigma_{S_1}}{\sigma_{S_2}}$, and central, backward or forward differences are chosen appropriately in each direction, and cross-derivatives are discretized by (B.4) or (B.5) appropriately, conditions (3.9)-(3.10) are satisfied. It is also easy to see that (3.11) is trivially satisfied for $r > 0$, irrespectively of the central, backward or forward differences in each direction, and the cross-derivative discretization.

## C Properties of the correlation process

The density $\Phi$ of the invariant distribution satisfies

$$\frac{\partial}{\partial\rho}(\lambda(\eta-\rho)\Phi) = \frac{1}{2}\frac{\partial^2}{\partial\rho^2}(\sigma_\rho^2(1-\rho^2)\Phi). \tag{C.1}$$

The solution that satisfies $\int\Phi = 1$ is given by

$$\Phi(x) = \frac{1}{2^{\frac{2\lambda}{\sigma_\rho^2}-1}}\frac{\Gamma(\frac{2\lambda}{\sigma_\rho^2})}{\Gamma(\frac{\lambda(1-\eta)}{\sigma_\rho^2})\Gamma(\frac{\lambda(1+\eta)}{\sigma_\rho^2})}(1-x)^{\frac{\lambda(1-\eta)}{\sigma_\rho^2}-1}(1+x)^{\frac{\lambda(1+\eta)}{\sigma_\rho^2}-1}. \tag{C.2}$$

The moments of the process (where subscript $t$ here indicates dependency on time, $\rho_t = \rho(t)$) can be evaluated as follows:

$$\mathbf{E}(\rho_t) = \int_0^t \lambda(\eta - \mathbf{E}(\rho_u))du.$$

The solution is given by

$$\mathbf{E}(\rho_t) = \rho_0 e^{-\lambda t} + \eta(1-e^{-\lambda t}). \tag{C.3}$$

Taking limit $t\to\infty$, the mean with respect to the invariant distribution is $\eta$. This can also be verified numerically by direct integration with (C.2). Similarly, by Itô's lemma,

$$\mathbf{E}(\rho_t^2) = \int_0^t \left(2(\lambda\eta\mathbf{E}(\rho_u)) - (2\lambda+\sigma_\rho^2)\mathbf{E}(\rho_u^2) + \sigma_\rho^2\right)du.$$

The ODE that arises from this can be solved analytically given $\mathbf{E}(\rho_t)$ above. As $t\to\infty$, the second moment with respect to the invariant distribution is $\frac{\sigma_\rho^2+2\lambda\eta^2}{2\lambda+\sigma_\rho^2}$. Therefore, the variance is $\frac{\sigma_\rho^2(1-\eta^2)}{2\lambda+\sigma_\rho^2}$.

## D Proof of Theorem 4.1

Define

$$\mathcal{A}^\epsilon V^\epsilon \doteq \mathcal{A}_1 V^\epsilon + \frac{1}{\epsilon}\mathcal{A}_0 V^\epsilon = 0,$$

and $V^\epsilon = V^{(0)} + \epsilon V^{(1)} + \epsilon^2 V^{(2)} + \dots$. We write $Z^\epsilon = V^\epsilon - V^{(0)} - \epsilon V^{(1)} - \epsilon^2 V^{(2)}$, where $V^{\epsilon,1}$ is as defined in (4.5). At terminal time, we have

$$Z^\epsilon(T,S_1,S_2,\rho) = -\epsilon(\rho-\eta)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)}(T,S_1,S_2) - \epsilon^2 V^{(2)}(T,S_1,S_2,\rho),$$

as we specified that $C(T,S_1,S_2)=0$. In addition, we have

$$\begin{aligned}
\mathcal{A}^\epsilon Z^\epsilon &= (\mathcal{A}_1 + \frac{1}{\epsilon}\mathcal{A}_0)(V^\epsilon - V^{(0)} - \epsilon V^{(1)} - \epsilon^2 V^{(2)}) \\
&= -\frac{1}{\epsilon}\mathcal{A}_0 V^{(0)} - (\mathcal{A}_1 V^{(0)} + \mathcal{A}_0 V^{(1)}) - \epsilon(\mathcal{A}_1 V^{(1)} + \mathcal{A}_0 V^{(2)}) - \epsilon^2\mathcal{A}_1 V^{(2)} \\
&= -\epsilon^2 \mathcal{A}_1 V^{(2)},
\end{aligned}$$

where we have used $\mathcal{A}^\epsilon V^\epsilon = 0$ and equations (4.1) to (4.3). Therefore, the probabilistic representation of the solution is

$$\begin{aligned}
Z^\epsilon(t,S_1,S_2,\rho) &= -\epsilon\mathbf{E}\Big[e^{-r(T-t)}(\rho_T^\epsilon-\eta)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)}(T,S_{1,T}^\epsilon,S_{2,T}^\epsilon) + \epsilon V^{(2)}(T,S_{1,T}^\epsilon,S_{2,T}^\epsilon,\rho_T^\epsilon) \\
&\quad -\epsilon\int_t^T e^{-r(s-t)}\mathcal{A}_1 V^{(2)}(s,S_{1,s}^\epsilon,S_{2,s}^\epsilon,\rho_s^\epsilon)ds \mid S_{1,t}^\epsilon = S_1, S_{2,t}^\epsilon = S_2, \rho_t^\epsilon = \rho\Big],
\end{aligned}$$

where subscripts $t, s, T$ indicate time dependence. The dependence of the processes on $\epsilon$ is emphasized by superscripts. We bound each of the terms in the following lemmas. These bounds conclude the proof of Theorem 4.1.

**Lemma 1.** *We have*

$$\left|\mathbf{E}\left[e^{-r(T-t)}(\rho_T^\epsilon-\eta)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)}(T,S_{1,T}^\epsilon,S_{2,T}^\epsilon)\mid S_{1,t}^\epsilon=S_1,S_{2,t}^\epsilon=S_2,\rho_t^\epsilon=\rho\right]\right|\le C_1e^{-C_2\frac{1}{\epsilon}},$$

*where $C_1, C_2 > 0$ are independent of $\epsilon$.*

*Proof.* By an argument similar to Lemmas A.1 and A.3 in [20], we can prove that

$$\sup_{\epsilon\le 1,t\le s\le T}\mathbf{E}\left[|D_{1,1}V^{(0)}(s,S_{1,s}^\epsilon,S_{2,s}^\epsilon)||S_{1,t}^\epsilon=S_1,S_{2,t}^\epsilon=S_2,\rho_t^\epsilon=\rho\right]\le C',$$

where $C'$ is independent of $\epsilon$. The proof consists of proving that the derivative $D_{1,1}V^{(0)}$ of the Black-Scholes price $V^{(0)}$ is of polynomial growth in $S_1$ and $S_2$, which is a consequence of the assumption on the payoff. The bound is obtained from boundedness of the marginal moments of $S_1$ and $S_2$, independent of correlation. More details can be found in [20]. The lemma follows directly from this bound because

$$\begin{aligned}&\left|\mathbf{E}\left[e^{-r(T-t)}(\rho_T^\epsilon-\eta)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)}(T,S_{1,T}^\epsilon,S_{2,T}^\epsilon)\mid S_{1,t}^\epsilon=S_1,S_{2,t}^\epsilon=S_2,\rho_t^\epsilon=\rho\right]\right|\\ \le\ &\mathbf{E}\left[e^{-r(T-t)}|\rho-\eta|e^{-\frac{T-t}{\epsilon}}\sigma_{S_1}\sigma_{S_2}|D_{1,1}V^{(0)}(T,S_{1,T}^\epsilon,S_{2,T}^\epsilon)|\mid S_{1,t}^\epsilon=S_1,S_{2,t}^\epsilon=S_2\right]\\ \le\ &2\sigma_{S_1}\sigma_{S_2}e^{-r(T-t)}e^{-\frac{T-t}{\epsilon}}\mathbf{E}\left[|D_{1,1}V^{(0)}(T,S_{1,T}^\epsilon,S_{2,T}^\epsilon)|\mid S_{1,t}^\epsilon=S_1,S_{2,t}^\epsilon=S_2\right].\end{aligned}$$

Note that in the second inequality we have used (C.3). □

**Lemma 2.** *We have*

$$\mathbf{E}\left[|V^{(2)}(T,S_{1,T}^\epsilon,S_{2,T}^\epsilon,\rho_T^\epsilon)|+|\int_t^T e^{-r(s-t)}\mathcal{A}_1V^{(2)}(s,S_{1,s}^\epsilon,S_{2,s}^\epsilon,\rho_s^\epsilon)ds|\mid S_{1,t}^\epsilon=S_1,S_{2,t}^\epsilon=S_2,\rho_t^\epsilon=\rho\right]\le C_3,$$

*where $C_3$ is independent of $\epsilon$.*

*Proof.* By (4.3), and (4.4), we have

$$\mathcal{A}_0V^{(2)}=-\mathcal{A}_1V^{(1)}=-(\mathcal{A}_1V^{(1)}-\langle\mathcal{A}_1V^{(1)}\rangle).$$

Recall that $V^{(1)}=(\rho-\eta)\sigma_{S_1}\sigma_{S_2}D_{1,1}V^{(0)}+(T-t)\frac{(1-\eta^2)\tilde\sigma_\rho^2\sigma_{S_1}^2\sigma_{S_2}^2}{2+\tilde\sigma_\rho^2}D_{1,1}^2V^{(0)}$. As $V^{(0)}$ is independent of $\rho$, we have

$$\mathcal{A}_1V^{(1)}-\langle\mathcal{A}_1V^{(1)}\rangle=(\rho(\rho-\eta)-\langle\rho(\rho-\eta)\rangle)\mathcal{K}(V^{(0)}),$$

where $\mathcal{K}(V^{(0)})$ is a linear combination of derivatives of $V^{(0)}$, which similarly has polynomial growth and thereby bounded conditional expectation as in the lemma above. We now write $V^{(2)}=-\psi(\rho)\mathcal{K}(V^{(0)})$, where $\psi$ solves the equation $\mathcal{A}_0\psi=\rho(\rho-\eta)-\langle\rho(\rho-\eta)\rangle$. Denote $g(\rho)=\rho(\rho-\eta)-\langle\rho(\rho-\eta)\rangle$. Observe that $\int_{-1}^1 g(\rho)\Phi(\rho)=\langle g\rangle=0$. Making use of (C.1), we derive

$$\psi'=\frac{2}{\tilde\sigma_\rho^2(1-\rho^2)\Phi}\int_{-1}^\rho g(u)\Phi(u)du.$$

We prove $\psi'$ is bounded as $\rho \to -1$. This is seen from the l' Hôpital's rule, where we apply (C.1) once again:

$$\lim_{\rho \to -1} \frac{2}{\tilde{\sigma}_\rho^2 (1-\rho^2)\Phi} \int_{-1}^{\rho} g(u)\Phi(u)du = \lim_{\rho \to -1} \frac{g(\rho)}{(\eta - \rho)}.$$

Similarly $\psi' = \frac{2}{\tilde{\sigma}_\rho^2(1-\rho^2)\Phi} \int_{-1}^{\rho} g(u)\Phi(u)du = \frac{2}{\tilde{\sigma}_\rho^2(1-\rho^2)\Phi} \int_{\rho}^{1} g(u)\Phi(u)du$ is bounded as $\rho \to 1$. As a result, $\psi$ is bounded on $(-1, 1)$, where the bound is independent of $\epsilon$, i.e.

$$\psi(\rho) \le C_4, \rho \in (-1, 1). \tag{D.1}$$

Finally $\mathcal{A}_1 V^{(2)}$ and $V^{(2)}$ can be written as a polynomial combination of derivatives of $V^{(0)}$ times the bounded functions $\psi(\rho)$ and $\rho\psi(\rho)$. Hence its conditional expectation is bounded independently of $\epsilon$ as desired. □

## E Density calculations

We denote by $p(T, S_1(T), S_2(T)|t, S_1, S_2)$, $t \le T$, the joint transition density function in the case of constant correlation $\bar{\rho} = \eta$ of the terminal prices $S_1(T)$ and $S_2(T)$, given asset prices $S_1$, $S_2$ at an earlier time $t$. Note that $p(\cdot|\cdot)$ satisfies the backward Kolmogorov equation

$$\frac{\partial p}{\partial t} + \frac{\sigma_{S_1}^2 {S_1}^2}{2}\frac{\partial^2 p}{\partial S_1^2} + \frac{\sigma_{S_2}^2 {S_2}^2}{2}\frac{\partial^2 p}{\partial S_2^2} + \bar{\rho}\sigma_{S_1}\sigma_{S_2} S_1 S_2 \frac{\partial^2 p}{\partial S_1 \partial S_2} + rS_1 \frac{\partial p}{\partial S_1} + rS_2 \frac{\partial p}{\partial S_2} = 0$$

with the terminal condition $p(T, S_1(T), S_2(T)|T, S_1, S_2) = \delta(S_1(T) - S_1, S_2(T) - S_2)$, where $\delta$ denotes the Dirac delta function. By direct computation, it can be shown that

$$p(T, S_1(T), S_2(T)|t, S_1, S_2) = \frac{1}{2\pi\sqrt{\det \Sigma} S_1(T) S_2(T)} e^{-\frac{1}{2} v^T A v},$$

where

$$\begin{aligned} \Sigma &= (T-t)\begin{bmatrix} \sigma_{S_1}^2 \sigma_{S_1}\sigma_{S_2}\bar{\rho} & \\ \sigma_{S_1}\sigma_{S_2}\bar{\rho} & \sigma_{S_2}^2 \end{bmatrix}, \qquad A = \Sigma^{-1}, \\ v &\doteq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \log(\frac{S_1'}{S_1}) - (r - \frac{\sigma_{S_1}^2}{2})(T-t) \\ \log(\frac{S_2'}{S_2}) - (r - \frac{\sigma_{S_2}^2}{2})(T-t) \end{pmatrix}. \end{aligned}$$

By straightforward differentiation,

$$\begin{aligned} D_{1,0}(p) &= S_1 \frac{\partial p}{\partial S_1} = p \times (A_{1,1}v_1 + A_{1,2}v_2) \\ D_{0,1}(p) &= S_2 \frac{\partial p}{\partial S_2} = p \times (A_{2,1}v_1 + A_{2,2}v_2) \\ D_{1,1}(p) &= S_1 S_2 \frac{\partial^2 p}{\partial S_1 S_2} = p \times \left( (A_{1,1}v_1 + A_{1,2}v_2)(A_{2,1}v_1 + A_{2,2}v_2) + \frac{\bar{\rho}}{\sigma_{S_1}\sigma_{S_2}(T-t)(1-\bar{\rho}^2)} \right). \end{aligned}$$

Finally,

$$
\begin{aligned}
D_{1,1}^2(p) = D_{1,1}(D_{1,1}(p)) \;&=\; D_{1,1}(p) \times \left( (A_{1,1}v_1 + A_{1,2}v_2)(A_{2,1}v_1 + A_{2,2}v_2) + \frac{\bar{\rho}}{\sigma_{S_1}\sigma_{S_2}(T-t)(1-\bar{\rho}^2)} \right) \\
&+\; D_{1,0}(p) \times \Big( - A_{1,2}(A_{2,1}v_1 + A_{2,2}v_2) - A_{2,2}(A_{1,1}v_1 + A_{1,2}v_2) \Big) \\
&+\; D_{0,1}(p) \times \Big( - A_{1,1}(A_{2,1}v_1 + A_{2,2}v_2) - A_{2,1}(A_{1,1}v_1 + A_{1,2}v_2) \Big) \\
&+\; p \times \Big( A_{1,1}A_{2,2} + A_{1,2}A_{2,1} \Big).
\end{aligned}
$$

Incidentally, one can verify equalities such that $\frac{\partial p}{\partial \bar{\rho}} = \sigma_{S_1}\sigma_{S_2}(T-t)D_{1,1}(p)$.
It is then straightforward to apply the right-side of (4.5) to $p(\cdot|\cdot)$ to obtain $p_m^{\epsilon,1}(\cdot|\cdot)$ for use in (4.7).

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