Question 1. [15 MARKS] Nash Equilibria

Consider the following two normal-form games between Alice (the row player) and Bob (the column player), who each choose between H and T. In each cell, the first number is Alice's reward and the second number is Bob's reward. The two games differ only in Alice's reward when both players choose H. You are guaranteed that each game has exactly one Nash equilibrium, and in that equilibrium, both players strictly randomize (play each action with a positive probability). So, in your calculations below, you do not need to consider and rule out any other cases.

Game 1

	H	T
H	$(1,-1)$	$(-1,1)$
T	$(-1,1)$	$(1,-1)$

Game 2

	H	T
H	$(\mathbf{3},-1)$	$(-1,1)$
T	$(-1,1)$	$(1,-1)$

Part (a) [5 MARKS] Find the Nash equilibrium of Game 1 using the indifference principle. Show your calculation for full marks.

SOLUTION

Suppose Alice plays H w.p. p and Bob plays H w.p. q, where $p, q \in (0,1)$. By the indifference principle for Alice, we need $q - (1-q) = -q + (1-q) \Leftrightarrow q = 1/2$. By the indifference principle for Bob, we need $-p + (1-p) = p - (1-p) \Leftrightarrow p = 1/2$. Hence, in the Nash equilibrium, both players play H and T w.p. $1/2$ each.

Part (b) [5 MARKS] Find the Nash equilibrium of Game 2 using the indifference principle. Show your calculation for full marks.

SOLUTION

Suppose Alice plays H w.p. p and Bob plays H w.p. q, where $p, q \in (0,1)$. By the indifference principle for Alice, we need $3q - (1-q) = -q + (1-q) \Leftrightarrow 4q - 1 = 1 - 2q \Leftrightarrow q = 1/3$. Bob's rewards are the same as in Game 1, so the indifference principle for Bob again gives $p = 1/2$. Hence, in the Nash equilibrium, Alice plays H and T w.p. $1/2$ each, while Bob plays H w.p. $1/3$ and T w.p. $2/3$.

Part (c) [5 MARKS] Compared to Game 1, which player's strategy changed in the Nash equilibrium of Game 2? Why did the other player's strategy not change? Justify your answer.

SOLUTION

Bob's strategy changed: he plays H w.p. $1/3$ in Game 2 instead of $1/2$. Alice's strategy did not change: she plays H and T w.p. $1/2$ each in both games.

Why: in an equilibrium in which both players randomize, Alice's strategy must make Bob indifferent between his two actions, so it is determined by Bob's rewards alone. Bob's rewards are the same in both games, so Alice's strategy does not change. In contrast, Bob's strategy must make Alice indifferent between her two actions, and Alice's rewards did change: H became more attractive to her when Bob plays H, so Bob must play H less often to keep her indifferent.

Question 2. [20 MARKS] Price of Stability

Consider a congestion game with n players and m resources. Each player i chooses a set of resources P_i. If $n_j(P)$ players use resource j under the pure strategy profile $P = (P_1, \ldots, P_n)$, each of them pays $f_j(n_j(P))$ for it, and a player's cost is the sum of what she pays for the resources she uses.

Suppose all cost functions are linear, i.e., $f_j(k) = a_j \cdot k$ for some constant $a_j \geqslant 0$. Let $C(P)$ denote the social cost (the sum of the costs of all players), and recall the potential function $\Phi(P) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(P)} f_j(k)$.

Part (a) [10 MARKS] Show that $C(P) = \sum_{j=1}^{m} a_j \cdot n_j(P)^2$ and $\Phi(P) = \sum_{j=1}^{m} a_j \cdot \frac{n_j(P)\,(n_j(P)+1)}{2}$.

SOLUTION

Each of the $n_j(P)$ players using resource j pays $f_j(n_j(P)) = a_j \cdot n_j(P)$ for it. Swapping the order of summation,

$$C(P) = \sum_{i=1}^{n} \sum_{j \in P_i} f_j(n_j(P)) = \sum_{j=1}^{m} n_j(P) \cdot f_j(n_j(P)) = \sum_{j=1}^{m} a_j \cdot n_j(P)^2.$$

For the potential function, $\Phi(P) = \sum_{j=1}^{m} a_j \sum_{k=1}^{n_j(P)} k = \sum_{j=1}^{m} a_j \cdot \frac{n_j(P)(n_j(P)+1)}{2}$.

Part (b) [5 MARKS] Use part (a) to show that $\Phi(P) \leqslant C(P) \leqslant 2\,\Phi(P)$ for every pure strategy profile P. You can solve this part even if you did not solve part (a).

SOLUTION

Fix a resource j. For every integer $n_j(P) \geqslant 0$, we have

$$\frac{n_j(P)\,(n_j(P)+1)}{2} \leqslant n_j(P)^2 \leqslant n_j(P)\,(n_j(P)+1) = 2 \cdot \frac{n_j(P)\,(n_j(P)+1)}{2},$$

where the first inequality holds because $n_j(P) + 1 \leqslant 2\,n_j(P)$ when $n_j(P) \geqslant 1$ (and both sides are 0 when $n_j(P) = 0$). Multiplying by $a_j \geqslant 0$ and summing over all resources j, part (a) gives $\Phi(P) \leqslant C(P) \leqslant 2\,\Phi(P)$.

Part (c) [5 MARKS] Use part (b) to prove that the price of stability of this game with respect to pure Nash equilibria is at most 2, i.e., there is a pure Nash equilibrium P^* with $C(P^*) \leqslant 2 \cdot C(P)$ for every pure strategy profile P. You can solve this part even if you did not solve part (b).

SOLUTION

Let P^* be a strategy profile minimizing Φ (it exists as there are finitely many profiles). As seen in class, P^* is a pure Nash equilibrium: if a player deviates, the change in her cost equals the change in Φ, which is nonnegative because $\Phi(P^*)$ is the smallest possible. Now let P be any pure strategy profile. Then,

$$C(P^*) \leqslant 2\,\Phi(P^*) \leqslant 2\,\Phi(P) \leqslant 2\,C(P),$$

where the first inequality is the second inequality of part (b) applied to P^*, the second holds as P^* minimizes Φ, and the third is the first inequality of part (b) applied to P. Taking P to be a social optimum, the price of stability is at most 2.

Question 3. [15 MARKS] Stackelberg Games

Consider the following 2×2 game between Alice (row player) and Bob (column player). Each cell shows Alice's reward, then Bob's reward.

	L	R
T	$(0, 1)$	$(3, 0)$
B	$(4, 1)$	$(1, 3)$

Part (a) [5 MARKS] Using the indifference principle, find the Nash equilibrium of this game and Alice's expected reward in it. Show your calculation for full marks. You are guaranteed that this game has exactly one Nash equilibrium, in which both players strictly randomize (play each action with a positive probability), so you do not need to rule out any other cases.

SOLUTION

Suppose Alice plays T w.p. p and Bob plays L w.p. q, where $p, q \in (0, 1)$. By the indifference principle for Alice, we need $0 \cdot q + 3(1 - q) = 4q + 1 \cdot (1 - q) \Leftrightarrow 3 - 3q = 1 + 3q \Leftrightarrow q = 1/3$. By the indifference principle for Bob, we need $1 \cdot p + 1 \cdot (1 - p) = 0 \cdot p + 3(1 - p) \Leftrightarrow 1 = 3 - 3p \Leftrightarrow p = 2/3$. Hence, in the Nash equilibrium, Alice plays T w.p. $2/3$ and B w.p. $1/3$, while Bob plays L w.p. $1/3$ and R w.p. $2/3$. Since Alice is indifferent between her actions, her expected reward is her reward from T: $3 \cdot (1 - 1/3) = 2$.

Part (b) [10 MARKS] Suppose now that Alice can commit to a *pure* strategy, and Bob then plays a best response to it. Which pure strategy should Alice commit to, and what would her reward be? Is this reward lower or higher than her reward in the Nash equilibrium from part (a)? Justify your answer.

SOLUTION

If Alice commits to T, Bob's reward is 1 from L and 0 from R, so he responds with L and Alice's reward is 0. If Alice commits to B, Bob's reward is 1 from L and 3 from R, so he responds with R and Alice's reward is 1. So Alice should commit to B, and her reward would be 1, which is *lower* than her reward of 2 in the Nash equilibrium.

Justification: the equilibrium reward of 2 relies on Alice randomizing. A pure commitment is perfectly predictable, so Bob best responds to it, and the best response to each of Alice's pure strategies hurts her (0 against T and 1 against B). (In class we saw that if Alice could commit to a *mixed* strategy, she could never be worse off than in a Nash equilibrium, since she could commit to her equilibrium strategy. This is not true for pure commitments.)

Bonus. [10 MARKS] Congestion Game Returns

Consider the congestion game with linear cost functions from Question 2. Design an example game of this form in which no social optimum is a Nash equilibrium, proving that the price of stability with respect to pure Nash equilibria can be strictly greater than 1. Show your calculations for full marks.

[Hint: There is such a game with two players $\{1, 2\}$ and two resources $\{x, y\}$, in which each player chooses either $\{x\}$ or $\{y\}$. In every social optimum, one player uses x and the other uses y, but then the player using y can reduce her own cost by switching to x. Your task is to find the two cost function parameters, a_x and a_y, which will make this happen.]

SOLUTION

Consider two players and two resources $\{x, y\}$ with $a_x = 1$ and $a_y = 2.5$ (i.e., $f_x(k) = k$ and $f_y(k) = 2.5k$), where each player chooses either $\{x\}$ or $\{y\}$. The social costs of the four profiles (player 1, player 2) are as follows.

- (x, x): each player pays $1 \cdot 2 = 2$, so the social cost is 4.
- (x, y) and (y, x): one player pays 1 and the other pays 2.5, so the social cost is 3.5.
- (y, y): each player pays $2.5 \cdot 2 = 5$, so the social cost is 10.

Hence, the social optima are (x, y) and (y, x), with social cost 3.5. Neither is a Nash equilibrium, because the player using y pays 2.5 but would pay only $1 \cdot 2 = 2$ after switching to x. Since no social optimum is a Nash equilibrium, every pure Nash equilibrium has social cost strictly greater than 3.5, so the price of stability is strictly greater than 1.