Explainable and Efficient Randomized Voting Rules

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Abstract

With a rapid growth in the deployment of AI tools for making critical decisions (or aiding humans in doing so), there is a growing demand to be able to explain to the stakeholders how these tools arrive at a decision. Consequently, voting is frequently used to make such decisions due to its inherent explainability. Recent work suggests that using randomized (as opposed to deterministic) voting rules can lead to significant efficiency gains measured via the distortion framework. However, rules that use intricate randomization can often become too complex to explain to the stakeholders; losing explainability can eliminate the key advantage of voting over black-box AI tools, which may outweigh the efficiency gains.

We study the efficiency gains which can be unlocked by using voting rules that add a simple randomization step to a deterministic rule, thereby retaining explainability. We focus on two such families of rules, randomized positional scoring rules and random committee member rules, and show, theoretically and empirically, that they indeed achieve explainability and efficiency simultaneously to some extent.

1 Introduction

In the past decade, AI and machine learning solutions have been deployed ubiquitously to make increasingly critical decisions that affect human lives. Consequently, there is a growing demand for these models and their decisions to be *explainable* [1, 2]. The literature makes a distinction between two types of explanations: *outcome explanations*, which explain to the stakeholders why the chosen outcome was selected in a given instance, and *procedural explanations*, which explain to the stakeholders the procedure of choosing outcomes across all possible instances.[1] Much of the explainable AI (XAI) literature focuses on outcome explanations because many black-box AI solutions used in practice are too complex to admit simple procedural explanations [3].

However, there are several drawbacks of outcome explanations. First, it opens up the possibility of post-hoc explanations for why an outcome was selected. These are susceptible to adversarial reasoning that hides biases [4]. Also, psychological research suggests that people's perception of fairness of an outcome depends not only on the outcome itself, but also on the process by which the outcome is selected [5, 6], and the same outcome may be perceived as fair or unfair depending on the process used [6]. This motivates the need for *procedural explanations*. Note that an intuitive explanation of the procedure to select outcomes already serves as a rudimentary justification for why a given outcome was selected.

To that end, we turn our attention to voting. While explainability is a nascent demand in the AI ecosystem, voting rules, historically deployed for political decision-making, have always battled

[1] There exists a similar distinction between outcome fairness and procedural fairness.

with the need to be able to explain to the voters how the winner of an election is chosen. Thus, most prominent voting rules admit intuitive procedural explanations. Due to this key advantage over black-box AI solutions, they have been used to automate decision-making in a variety of applications such as designing recommender systems [7, 8], information extraction [9], collaborative filtering [10], ensemble learning [11], and game-playing by AI agents [12]. The advantage is more apparent when the decisions at hand are more significant. For example, Noothigattu et al. [13] and Kahng et al. [14] propose the design of a voting-based virtual democracy system that can automate ethical decision-making in AI systems. When Lee et al. [15] applied this framework to automate the distribution of food donations, they found that their stakeholders appreciated the fact that voting-based decision-making *"embodied democratic values"* and being able to provide easy explanations *"allowed them to understand how algorithmic recommendations were made"*.

However, most prominent voting rules are deterministic because their primary use case is making infrequent, high-stakes democratic decisions, for which randomization is generally unpalatable [16]. Aside from selected applications such as forming citizens' assemblies, juries, and independent redistricting commissions, lottery is seldom used to select representatives [17]. But in AI applications, it is common to make frequent, low-stakes decisions, for which randomization is well-suited.

Research on voting theory suggests that allowing the voting rule to randomize has numerous benefits. It can be essential for avoiding the tyranny of the majority and guarantee minority representation [18, 19]. Randomization also acts as a barrier to manipulations by strategic agents by circumventing the Gibbard-Satterthwaite impossibility [20, 21]. Most importantly, it can unlock significant efficiency gains over using deterministic voting rules [22, 23, 19]. Unfortunately, randomized voting rules designed to optimize for efficiency can be highly complex and rely on intricate mathematical results such as the minimax theorem [19], making them difficult to explain to the end users.

In view of this, we explore the use of *explainable randomized voting rules* for improving the efficiency of automated decision-making. To ensure explainability, one possibility is to use only extremely simple randomized rules such as random dictatorship, where the most preferred option of a randomly chosen agent is selected. But this may leave significant possible efficiency gains on the table. Instead, we propose a hybrid approach that adds a simple (and thus, explainable) randomization step to well-understood deterministic decision-making processes. This drives our main research question:

> *What efficiency gains can be unlocked by explainable randomized voting rules which add a simple randomization step to deterministic voting rules?*

As a yardstick for efficiency, we turn to the *distortion framework* [24]. Proposed by Procaccia and Rosenschein [25], this framework posits that votes submitted by agents, typically rankings over a set of alternatives, are induced by more expressive preferences underneath, typically cardinal utility functions over the alternatives. In this framework, the goal of a randomized voting rule is to choose a lottery over the alternatives that minimizes *distortion*, the worst-case ratio between maximum social welfare (total utility to the agents) and that of the chosen lottery.

We study the distortion of two families of randomized voting rules, which we refer to as *randomized positional scoring rules* and *random committee member rules*. The former builds on the widely-popular family of (deterministic) positional scoring rules, under which each agent assigns a score to each alternative based on its position in her ranking. But instead of deterministically selecting an alternative with the highest total score, each alternative is selected with probability proportional to its score. In contrast to picking alternatives with varying probabilities, the latter family utilizes the simplicity of uniform randomization by picking, uniformly at random, a member of a subset of alternatives chosen deterministically. Our inspiration for these two families stems from the use of such rules by Boutilier et al. [22] and Ebadian et al. [19].

Let us illustrate an example rule from each family based on the popular Borda count method, in which scores of $m-1, \ldots, 0$ are assigned to ranks $1, \ldots, m$ respectively. A rule from the first family would pick each alternative with probability proportional to its total Borda score, while a rule from the second family may select uniformly at random among the k alternatives with the highest Borda scores, for some fixed k. We select these two families of randomized voting rules because they admit straightforward procedural explanations. For instance, the aforementioned rules based on Borda count can be explained as follows (using version (a) for the former and (b) for the latter using $k=3$).

Table 1: Distortion and minimum welfare of common randomized positional scoring rules.

	Plurality	Borda	Harmonic	Veto	k-Approvals $k \in [1, m^{1/3}]$	k-Approvals $k \in [m^{1/3}, \sqrt{m}]$	k-Approvals $k \in [\sqrt{m}, m]$
dist	$\Theta(m\sqrt{m})$	$\Theta(m^{5/4})$	$\Theta(\sqrt{m}H_m)$	$\Theta(m)$	$\Theta(\frac{m\sqrt{m}}{k\sqrt{k}})$	$\Theta(m)$	$\Theta(k\sqrt{m-k+1})$
min-sw	$\Theta\left(\frac{n}{m^2}\right)$	$\Theta\left(\frac{n}{m\sqrt{m}}\right)$	$\Theta\left(\frac{n}{mH_m}\right)$	$\Theta\left(\frac{n}{m}\right)$	$\Theta\left(\frac{nk}{m^2}\right)$		$\Theta\left(\frac{n}{k(m-k+1)}\right)$

> *"Each user gives zero points to their least preferred option, one point to the next best option, two points to the next best option, and so on. Points are tallied and...*
> *(a) the chances of each option being selected are proportional to its total points.*
> *(b) the three options with the highest total points are selected with an equal chance."*

Unlike prior work on distortion, which is often focused on identifying the *most efficient* rule, we provide a refined analysis that characterizes the distortion of many interesting rules in these families, allowing the system designer to pick the one most suited to the application at hand.

1.1 Our Results

Randomized positional scoring rules. We develop a whole swathe of novel techniques for analyzing distortion, and use them to obtain tight distortion (dist) bounds for randomized versions of well-known positional scoring rules such as plurality, Borda count, harmonic, veto, and k-approval, presented in Table 1. Along the way, we also obtain tight bounds for another useful efficiency metric, *minimum welfare* (min-sw), defined in Section 3.1. In the supplementary material, we apply our novel techniques to analyze the distortion of *randomized multi-level approval rules*, which are uniform mixtures of different randomized k-approval rules. We demonstrate the strength of this result by using it to derive tight distortion bounds for a recently studied randomized positional scoring rule due to Gkatzelis et al. [26].

Our distortion bound for the randomized plurality rule (better known as random dictatorship) may be of independent interest because it is a widely studied voting rule [27–29]. It is also fascinating that the distortion of randomized k-approval is highly non-monotone: first decreasing from $\Theta(m\sqrt{m})$ to $\Theta(m)$ when k grows from 1 to $m^{1/3}$, then staying $\Theta(m)$ when k grows further to $\sqrt{m}$, then increasing again to $\Theta(m\sqrt{m})$ by $k = m - \Theta(m)$, and finally decreasing again to $\Theta(m)$ by $k = m$.

Random committee member rules. For the family of random committee member rules, we design a novel voting rule, which selects a random member of a *top-biased stable k-committee*, and achieves a distortion of $O(\max\{k, m^2/(k\sqrt{k})\})$. We complement this with a lower bound of $\Omega(\max\{k, m^2/k^2\})$ on the distortion of any rule in this class.

Experiments. Our experiments with synthetic data indicate that while the (worst-case) distortion of various rules in both families is not significantly better than the optimal deterministic distortion of $\Theta(m^2)$ and often even worse than the $\Theta(m)$ distortion of the trivial rule selecting a uniformly random alternative, rules from both families (almost always) significantly outperform deterministic voting rules and randomized positional scoring rules (almost always) outperform the uniform-random rule as well. This suggests that one should strongly consider replacing deterministic rules with explainable randomized rules from these two families in order to achieve significantly improved efficiency.

1.2 Additional Related Work

Outcome explanations in voting. We focus on (randomized) voting rules with procedural explanations because that is, in our view, the key advantage of voting over black-box AI solutions. Nonetheless, there is also compelling literature on producing outcome explanations for voting rules. Classical work that seeks voting rules satisfying qualitative axioms such as Condorcet consistency can be viewed in this light. However, these axioms often provide a justification only in limited instances with a special structure. Cailloux and Endriss [30] propose a method for producing a justification on any given instance by starting from an axiomatic justification on a nearby special instance and using a chain of explanations relating adjacent pairs of instances to arrive at the given instance. Peters et al. [31] bound the length of such chains, focusing especially on positional scoring rules, while Boixel and Endriss [32] and Boixel et al. [33] study computational aspects of finding them.

On randomized positional scoring rules. The family of randomized positional scoring rules was first proposed by Barbera [34], under the name of point-voting schemes. He establishes several appealing properties of these rules, including strategyproofness (i.e., no agent can ever strictly benefit from misreporting her preferences). Note that the outcome of a randomized positional scoring rule can be computed by first selecting an agent uniformly at random, and then, for each k, selecting her k-th most preferred alternative with probability proportional to score assigned to position k. In this sense, implementing a randomized positional scoring rule requires little elicitation.

2 The Setting

Let us formally introduce our setting and define the notion of distortion. We will define each class of explainable voting rules in the section in which we will study it. We use the terminology of *elections* for consistency with the literature, but our setting captures a general decision-making scenario in which one of several alternatives must be chosen by aggregating conflicting preferences or opinions.

Basic notation. Let $[t] = \{1, 2, \ldots, t\}$ for $t \in \mathbb{N}$, and define $\Delta(S)$ to be the probability simplex over the finite set S. For a vector $\vec{s} = (s_1, \ldots, s_t)$, denote its ℓ^1-norm by $\|\vec{s}\|_1 = \sum_{i\in[t]} s_i$.

Utilitarian voting. A (single-winner) election consists of sets $N = [n]$ of n agents and $A = [m]$ of m alternatives. Each agent $i \in N$ has a personal *cardinal* utility function $u_i \colon A \mapsto \mathbb{R}_{\geqslant 0}$, where $u_i(a)$ is the value associated by agent i to alternative a. Following the convention in the literature (e.g., see [22, 19]), we adopt the *unit-sum* normalization of utility functions: for every $i \in N$, let $\sum_{a\in A} u_i(a) = 1$. Aziz [35] provides several compelling justifications for using unit-sum utility functions. For a utility profile $\vec{u} = (u_1, \ldots, u_n)$ and a subset of agents $T \subseteq N$, define the *social welfare* of an alternative $a \in A$ with respect to T as $\mathsf{sw}_T(a, \vec{u}) = \sum_{i\in T} u_i(a)$. We write sw_N simply as sw, and drop $\vec{u}$ when it is clear from the context. As an extension, for a distribution $p \in \Delta(A)$ over the alternatives, define $u_i(p) = \mathbb{E}_{a\sim p}[u_i(a)]$ and its social welfare as $\mathsf{sw}(p, \vec{u}) = \sum_{i\in N} u_i(p)$. Our goal is to find a distribution over alternatives with high social welfare. We will sometimes construct and analyze a *partial utility profile*, where the utilities of each agent sum to *at most* 1.

Ordinal preferences and voting rules. We consider *voting rules* that have access only to the *ordinal* preferences induced by the utilities. This is because a ranking of alternatives can often be elicited with less cognitive burden or estimated more accurately than exact numerical utilities for each alternative.

Each agent $i \in N$ submits a preference ranking $\sigma_i \colon [m] \mapsto A$ of the alternatives. We use $\mathsf{rank}_i(a) = \sigma_i^{-1}(a)$ to denote the rank of alternative a in agent i's preference ranking (the most preferred alternative has rank 1), and $a \succ_i a'$ to denote that agent i prefers a to a' (i.e., $\mathsf{rank}_i(a) < \mathsf{rank}_i(a')$). We assume that σ_i is consistent with agent i's utility function u_i, i.e., $a \succ_i a'$ implies $u_i(a) \geqslant u_i(a')$ for all $a, a' \in A$; ties can be broken arbitrarily without affecting our distortion upper bounds.

Let $\vec{\sigma} = (\sigma_i)_{i\in N}$ be a *preference profile* and $\mathcal{C}(\vec{\sigma})$ denote the set of utility profiles $\vec{u}$ such that σ_i is consistent with u_i for each agent $i \in N$. A voting rule f takes a preference profile $\vec{\sigma}$ as input and returns a distribution p over alternatives.

Distortion. The distortion of a distribution $p \in \Delta(A)$ over alternatives with respect to a utility profile $\vec{u}$ is defined as

$$\mathsf{dist}(p, \vec{u}) = \frac{\max_{a\in A} \mathsf{sw}(a, \vec{u})}{\mathsf{sw}(p, \vec{u})}.$$

The distortion of a voting rule f is defined as its worst-case distortion over all instances: $\mathsf{dist}_m(f) = \sup_{\vec{\sigma}, \vec{u}\in\mathcal{C}(\vec{\sigma})} \mathsf{dist}(f(\vec{\sigma}), \vec{u})$, where the supremum is taken over all instances with m alternatives and any number of agents. For simplicity, we drop m and write $\mathsf{dist}(f)$.

3 Distortion of Randomized Positional Scoring Rules

The first class of explainable randomized voting rules we study is randomized positional scoring rules, or point-voting schemes [34]. This builds on the popular class of (deterministic) positional scoring rules, which assign scores to alternatives based on their positions in agents' preference rankings, and adds an easy-to-explain randomization step where each alternative is chosen with probability proportional to its score instead of deterministically choosing the one with the highest score.

Positional scoring rules. A *scoring vector* $\vec{s} = (s_1, \ldots, s_m)$ assigns a score s_r to each position $r \in [m]$ and satisfies $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_m \geqslant 0$. For an alternative $a \in A$, let $\mathsf{score}_i(c, \vec{s}) = s_{\mathsf{rank}_i(a)}$ be the score a obtains from agent i, and for $N' \subseteq N$, $\mathsf{score}_{N'}(a, \vec{s}) = \sum_{i \in N'} \mathsf{score}_i(a, \vec{s})$. Note that $\sum_{a \in A} \mathsf{score}_N(a, \vec{s}) = n \cdot \|\vec{s}\|_1$. We drop $\vec{s}$ when it is clear from the context. For a scoring vector $\vec{s}$, we can define the following rules.

- The *deterministic positional scoring rule* $f^{\mathsf{det}}_{\vec{s}}$ selects the top scored alternative (breaking ties arbitrarily), i.e., $\quad f^{\mathsf{det}}_{\vec{s}}(\vec{\sigma}) = \arg\max_{a \in A} \mathsf{score}_N(a, \vec{s})$.
- The *randomized positional scoring rule* $f^{\mathsf{rand}}_{\vec{s}}$ selects every alternative $a \in A$ with probability proportional to its score, i.e., $\quad \Pr[f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}) = a] = \mathsf{score}_N(a, \vec{s}) \,/\, (n \cdot \|\vec{s}\|_1)$.

The deterministic rules introduced above include several well-known voting rules such as *plurality*, *Borda*, k*-approval*, *veto*, and *harmonic* defined, by the following scoring vectors, respectively:

$$\vec{s}_{\mathsf{plu}} = (1, 0, \ldots, 0), \qquad \vec{s}_{\mathsf{Borda}} = (m-1, m-2, \ldots, 0), \quad \vec{s}_{k\text{-}\mathsf{approval}} = (\underbrace{1, \ldots, 1}_{k \text{ ones}}, 0, \ldots, 0),$$
$$\vec{s}_{\mathsf{veto}} = (1, 1, \ldots, 1, 0), \quad \vec{s}_{\mathsf{harmonic}} = (1, 1/2, 1/3, \ldots, 1/m).$$

We refer to the randomized versions of these rules as "randomized f", where $f \in$ {plurality, Borda, harmonic, veto}, and extend this terminology to any positional scoring rule $f_{\vec{s}}$. Note that "randomized plurality" is more widely known as *random dictatorship* (see, e.g., [27, 28]).

3.1 High-Level Distortion Analysis and Novel Insights

Logarithmic rounding of the scores. Our first useful insight is that we can reduce the number of distinct scores by rounding any score down to the nearest power of $1+\alpha$,for a constant $\epsilon > 0$, and this only changes the distortion of the rule by a factor of at most $1+\alpha$.

Lemma 1 (Rounding Down Scores). *Let $\alpha \geqslant 0$, and $\vec{s}, \vec{s'}$ be scoring vectors such that $s'_j \leqslant s_j \leqslant (1+\alpha)s'_j$ for all $j \in [m]$. Then, for every preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$,*

$$\frac{1}{1+\alpha} \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s'}}(\vec{\sigma}), \vec{u}) \leqslant \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \leqslant (1+\alpha) \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s'}}(\vec{\sigma}), \vec{u}),$$

and consequently, $\frac{1}{1+\alpha} \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s'}}) \leqslant \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant (1+\alpha) \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s'}})$.

By applying this transformation, scores in the range $[\|\vec{s}\|_1/(4m^2), \|\vec{s}\|_1]$ can be reduced to $O(\log m)$ distinct values, which we will find helpful in the subsequent sections. In the supplementary material, we show that by ignoring the remaining scores by reducing any $s_j \leqslant \|\vec{s}\|_1/(4m^2)$ to 0 changes the distortion further by another factor of at most two. Hence, we can limit our focus to scoring vectors that contain $O(\log m)$ distinct scores, resulting in only a constant factor loss in the distortion analysis.

High-level distortion analysis. After reducing the scoring vector to $O(\log m)$ distinct scores, we partition the agents into $O(\log m)$ groups based on where they rank the optimal alternative a^*. This is only for the analysis; the voting rule does not know the optimal alternative. More formally, let $0 = \ell_0 < \ell_1 < \ldots < \ell_q = m$ be the indices where the score changes in the reduced scoring vector, where, for each $r \in [q]$, we have $s_i = s_{\ell_r}$ for all $i \in [\ell_{r-1}+1, \ell_r]$, and $s_{\ell_1} > \ldots > s_{\ell_q}$. Furthermore, let N_r be the set of agents who rank a^* among positions $[\ell_{r-1}+1, \ell_r]$.

Next, borrowing an insight from the prior distortion literature [36–38], we round the agent utilities to the nearest power of two and ignore utilities below $1/m^2$, reducing the number of distinct utility values to $O(\log m)$ while losing at most a constant factor in the analysis. This allows us to we subdivide voters in each group N_r into $O(\log m)$ subgroups such that every agent in a subgroup with the same utility, say τ, for a^*. We then employ three strategies to bound the distortion within each subgroup. Finally, we show that the overall distortion can be upper bounded, up to logarithmic factors, by the worst of these $O(\log^2 m)$ distortion bounds across all subgroups of all the N_r groups.

Strategy 1 (Welfare above $\mathbf{a}^*$). Voters in a subgroup of N_r who have utility τ for a^* also have utility at least τ for their top ℓ_{r-1} alternatives. This helps us derive social welfare guarantees for the randomized positional scoring rule.

Strategy 2 (Probability of $\mathbf{a}^*$). We use the well-known observation (see, e.g., [22]) that the distortion is always upper bounded by the inverse of the probability of selecting a^*.

Strategy 3 (Absolute Welfare Lower Bounds). Another novel insight from our work is that proving an absolute lower bound on the welfare achieved by a rule across all instances can be useful in bounding the distortion, even though the latter needs to compare the welfare achieved in each instance to the optimum welfare in that instance. We carefully analyze and approximate, up to logarithmic factors, the minimum welfare achieved by the randomized positional scoring rules we study, and use it to bound its distortion. A similar idea has been used in other domains (see, e.g. [39]), but to the best of our knowledge, we are the first to successfully apply it to distortion analysis.

Definition 1 (Minimum Welfare)**.** *Define the* minimum welfare *of a distribution over alternatives $p \in \Delta(A)$ on a preference profile $\vec{\sigma}$ as $\textsf{min-sw}(p, \vec{\sigma}) = \inf_{\vec{u} \in \mathcal{C}(\vec{\sigma})} \textsf{sw}(p, \vec{u})$, which is the minimum social welfare of p across all consistent utility profiles. The* minimum welfare *of a voting rule f is the minimum welfare of its output, minimized over all preference profiles: $\textsf{min-sw}_{n,m}(f) = \min_{\vec{\sigma}} \textsf{min-sw}(f(\vec{\sigma}), \vec{\sigma})$, where the minimum is taken over all preference profiles with n agents and m alternatives. We drop n and m when clear from the context.*

Due to $\mathcal{C}(\vec{\sigma})$ being compact, the infimum in the $\textsf{min-sw}(p, \vec{\sigma})$ definition is indeed attained. In the supplementary material, we make an structural observation that for any preference profile and distribution $p \in \Delta(A)$, the minimum welfare is at most n/m, attained at a *dichotomous utility profile*. Furthermore, every randomized positional scoring rule $f^{\textsf{rand}}_{\vec{s}}$ satisfies $\textsf{min-sw}(f^{\textsf{rand}}_{\vec{s}}) \in [n/(4m^2), n/m]$. We also show how to approximate minimum welfare better (up to constants or logarithmic terms); see Table 1 for tight bounds for the rules induced by common scoring vectors.

To this end, we present our most intricate technical lemma to derive a generic welfare lower bound, which we use to apply Strategies 1 and 3. Instead of focusing only on $\Pr[f^{\textsf{rand}}_{\vec{s}}(\vec{\sigma}) = a^*]$, it bounds the overall welfare expression $\sum_{a \in A} \Pr[f^{\textsf{rand}}_{\vec{s}}(\vec{\sigma}) = a] \cdot \textsf{sw}_T(a)$; the product of probability of selection and welfare of an alternative leads to a quadratic program, where the variables encode the worst case, and this is analytically solved using the Karush-Kuhn-Tucker (KKT) conditions.

Lemma 2. *Fix any scoring vector $\vec{s}$, preference profile $\vec{\sigma}$, subset of agents $T \subseteq N$, threshold $\tau \geqslant 0$, and rank $\ell \in [m]$. For a partial utility profile $\vec{u}$ in which every agent in T has utility at least τ for each of her top ℓ alternatives and all other utilities are 0, we have:*

$$\textsf{sw}_T(f^{\textsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \geqslant \tau \cdot \frac{|T|\ell}{2n\|\vec{s}\|_1} \min_{h \in [m]} \frac{1}{h}\left(2s_\ell \cdot |T|\ell + (n - |T|) \cdot \sum_{j=1}^{h} s_{m-j+1}\right).$$

Instead of tediously explaining the lemma, we will later show how its straightforward application wondrously gives us the desired welfare lower bound for the example of randomized Borda rule.

3.2 Analyzing Common Rules

We are ready to present our main result, which uses the aforementioned insights to pinpoint the asymptotic distortion of common randomized positional scoring rules.

Theorem 2. *For $f \in$ {plurality, Borda, harmonic, veto, k-approvals}, the minimum welfare* ($\textsf{min-sw}$) *and the distortion* ($\textsf{dist}$) *of the 'randomized f' rule are as shown in Table 1.*

Due to space limitations, we only provide a proof for the distortion upper bound of the randomized Borda rule, and defer the rest to the supplementary material. For conciseness, we use the notation $\textsf{Borda}(a) \triangleq \textsf{score}(a, \vec{s}_{\textsf{Borda}})$. First, we need the following lower bound on its minimum welfare.

Lemma 3. *The minimum welfare of the randomized Borda rule is $\textsf{min-sw}(f^{\textsf{rand}}_{\vec{s}_{\textsf{Borda}}}) = \Omega(\frac{n}{m\sqrt{m}})$.*

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Our goal is to show that $\textsf{sw}(f^{\textsf{rand}}_{\vec{s}_{\textsf{Borda}}}(\vec{\sigma}), \vec{u}) = \Omega(\frac{n}{m\sqrt{m}})$. First, we make a few modifications to the scoring vector and the preference profile that are guaranteed to not increase the welfare, and then invoke Lemma 2.

Simplify the scores. Let us consider the scoring vector $\vec{s}\,'$ which is equal to $\vec{s}_{\textsf{Borda}}$ except the top $m/2$ scores are all equal to $m/2$. Note that $s'_j \leqslant (s_{\textsf{Borda}})_j \leqslant 2s'_j$ for all $j \in [m]$. Hence, invoking Lemma 1 with $\alpha = 1$ yields $\textsf{sw}(f^{\textsf{rand}}_{\vec{s}_{\textsf{Borda}}}(\vec{\sigma}), \vec{u}) \geqslant 1/2 \cdot \textsf{sw}(f^{\textsf{rand}}_{\vec{s}\,'}(\vec{\sigma}), \vec{u})$.

Simplify the preference and utility profiles. Next, let us lower bound $\textsf{sw}(f^{\textsf{rand}}_{\vec{s}\,'}(\vec{\sigma}), \vec{u})$. For each agent i, let A_i be the set of top $m/2$ alternatives in σ_i and $a_i \in \arg\min_{a \in A_i} \textsf{score}(a, \vec{s}\,')$ be the alternative in A_i with the lowest score (equivalently, probability of selection under $f^{\textsf{rand}}_{\vec{s}\,'}(\vec{\sigma})$). Now,

$$u_i(f^{\textsf{rand}}_{\vec{s}\,'}(\vec{\sigma})) \geqslant \sum_{a \in A_i} \frac{\textsf{score}(a, \vec{s}\,')}{n\|\vec{s}\,'\|_1} \cdot u_i(a) \overset{(1)}{\geqslant} \frac{\textsf{score}(a_i, \vec{s}\,')}{n\|\vec{s}\,'\|_1} \cdot \left(\sum_{a \in A_i} u_i(a)\right) \overset{(2)}{\geqslant} \frac{\textsf{score}(a_i, \vec{s}\,')}{n\|\vec{s}\,'\|_1} \cdot \frac{1}{2},$$

where (1) follows from the definition of a_i and (2) uses the fact that each agent has a total utility of at least $1/2$ for her top $m/2$ alternatives (due to the pigeonhole principle).

Invoking Lemma 2. The final expression above can be written as $u_i'(f_{\vec{s}'}^{\mathsf{rand}}(\vec{\sigma}'))$, where $\vec{\sigma}'$ is a preference profile in which each agent i ranks a_i first, and $\vec{u}'$ is a partial utility profile in which each agent i has utility $1/2$ for her top alternative and 0 for the rest. Summing the above for all agents, we have $\mathsf{sw}(f_{\vec{s}'}^{\mathsf{rand}}(\vec{\sigma}), \vec{u}) \geqslant \mathsf{sw}(f_{\vec{s}'}^{\mathsf{rand}}(\vec{\sigma}', \vec{u}'))$. Thus, to lower bound it, we invoke Lemma 2 with $\vec{s} \leftarrow \vec{s}'$, $\vec{\sigma} \leftarrow \vec{\sigma}'$, T being an arbitrary subset of $n/2$ agents, $\tau \leftarrow 1/2$, and $\ell \leftarrow 1$. Using $\|\vec{s}'\|_1 = \frac{m}{2} \cdot \frac{m}{2} + \binom{m/2}{2} = \frac{m(3m-2)}{8}$, this gives us

$$\mathsf{sw}(f_{\vec{s}'}^{\mathsf{rand}}(\vec{\sigma}'), \vec{u}') \overset{(1)}{\geqslant} \frac{1}{2} \cdot \frac{\frac{n}{2} \cdot 1}{2n \cdot \frac{m(3m-2)}{8}} \cdot \min_{h \in [m/2]} \frac{1}{h} \left(2 \cdot \frac{m}{2} \cdot \frac{n}{2} + \frac{n}{2} \cdot \frac{h(h-1)}{2} \right)$$

$$= \frac{n}{4m(3m-2)} \cdot \min_{h \in [m/2]} \left(\frac{2m}{h} + h - 1 \right) \overset{(2)}{\geqslant} \frac{n \cdot (2\sqrt{2m} - 1)}{4m(3m-2)} \overset{(3)}{\geqslant} \frac{n}{6m\sqrt{m}},$$

where the restriction to $h \in [m/2]$ in (1) is based on the fact that the bound would be $\Omega(n/m)$ when $h > m/2$, (2) is due to the AM-GM inequality, and (3) uses $m \geqslant 2$. Connecting the dots, we have

$$\mathsf{sw}(f_{\vec{s}_{\mathsf{Borda}}}^{\mathsf{rand}}(\vec{\sigma}), \vec{u}) \geqslant \frac{1}{2} \cdot \mathsf{sw}(f_{\vec{s}'}^{\mathsf{rand}}(\vec{\sigma}), \vec{u}) \geqslant \frac{1}{2} \cdot \mathsf{sw}(f_{\vec{s}'}^{\mathsf{rand}}(\vec{\sigma}'), \vec{u}') \geqslant \frac{n}{12m\sqrt{m}}. \qquad \square$$

To translate the welfare lower bound from Lemma 3 into a distortion upper bound, we need the following relation between the Borda score of an alternative and its social welfare.

Lemma 4. *For any preference profile $\vec{\sigma}$, consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$, and alternative $a \in A$, we have $\mathsf{sw}(a) \leqslant (\mathsf{Borda}(a) + n)/m$.*

The desired distortion upper bound can now be derived using a standard analysis. The crux of our proof for the randomized Borda rule lies in our intricate derivation of its minimum welfare in Lemma 3, for which Lemma 2 does the heavy lifting.

Lemma 5. *The distortion of the randomized Borda rule is $O(m^{5/4})$.*

Proof sketch. Fix a pair of preference profile $\vec{\sigma}$ and $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Let $a^* \in \arg\max_{a \in A} \mathsf{sw}(a, \vec{u})$ be an optimal alternative. By Strategy 2, we know the distortion is at most $\frac{n\|\vec{s}_{\mathsf{Borda}}\|_1}{\mathsf{Borda}(a^*)}$. Following Strategy 3 and by Lemma 3, we have that distortion is at most $\frac{\mathsf{sw}(a^*)}{n/(12m\sqrt{m})}$, which, using Lemma 4, is at most $12\sqrt{m} \cdot \left(\mathsf{Borda}(a^*)/n + 1\right)$. Putting everything together, we show that the distortion is $O(m^{5/4})$. $\square$

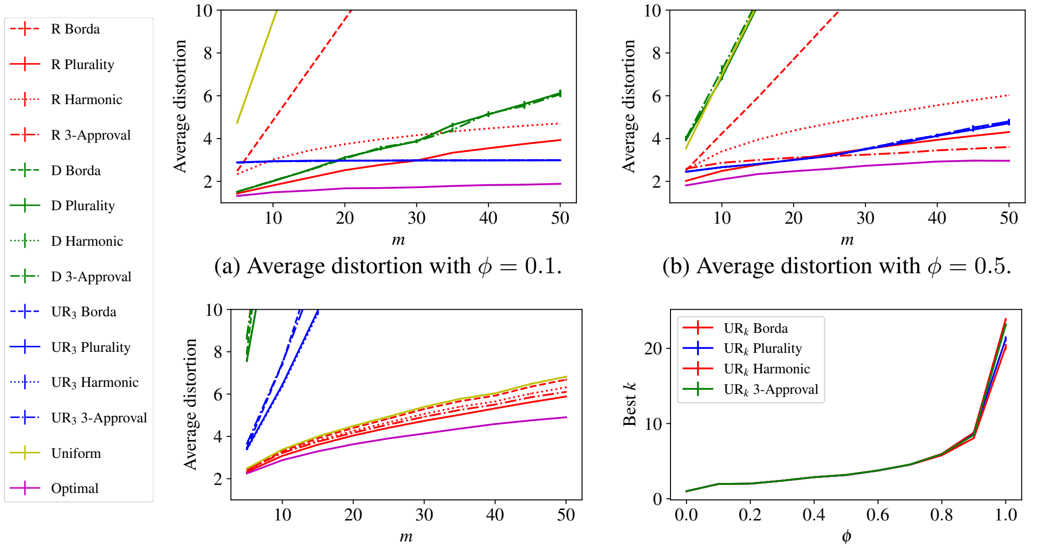
4 Random Committee Member Rules

Next, we focus on our second class of explainable randomized voting rules that select an alternative uniformly at random from a shortlisted committee of size $k \in [m]$. We call them *random k-committee member* rules. For $k = 1$, we are left with deterministic rules, among which plurality achieves the optimal distortion of $\Theta(m^2)$. For $k = m$, we are left with uniform selection among all alternatives, which has distortion $\Theta(m)$. Is it possible that, for some intermediate value of k, we in fact achieve sublinear distortion? Could we achieve distortion at most logarithmic factors worse than the optimal $\Theta(\sqrt{m})$, like with randomized positional scoring rules? We answer the former positively but the latter negatively. First, we present a lower bound proving that any random k-committee member rule, for any value of k, incurs a distortion of at least $\Omega(m^{2/3})$.

Theorem 3. *For $k \in [m]$, the random k-committee member rule incurs $\Omega(\max(k, m^2/k^2))$ distortion. This lower bound is at least $\Omega(m^{2/3})$ for all k.*

As a result, this class of rules is less powerful than the class of randomized positional scoring rules, and thus, the class of all randomized rules. However, especially for small values of k, we gain the benefit of randomizing over a small support, which could translate to greater explainability.

To derive upper bounds, one might be tempted to turn again to positional scoring rules, and consider selecting uniformly at random from the k alternatives with the highest score according to some scoring vector. In the supplementary material, we show that using plurality scoring vector yields $\Theta(m(m-k+1))$ distortion. While it nicely interpolates between the extremes of $\Theta(m^2)$ at $k = 1$



(a) Average distortion with $\phi = 0.1$.
(b) Average distortion with $\phi = 0.5$.
(c) Average distortion with $\phi = 1$.
(d) Best value of k based on ϕ, for $m = 25$.

Figure 1: All figures show results averaged over 150 runs along with the standard error. Figures 1a to 1c share the legend on the left.

and $\Theta(m)$ at $k = m$, it fails to achieve sublinear distortion, which we prove to be achievable. What about other scoring vectors? Unfortunately, it is relatively easy to see that using scoring vectors such as Borda, harmonic, or veto results in unbounded distortion. Despite the disappointing worst-case performance, we show in Section 5 that these rules perform relatively well empirically.

Next, we design a novel random k-committee member rule, which, with the right value of k, allows us to achieve sublinear distortion.

Theorem 4. *There is a polynomial-time computable random k-committee member rule with distortion $O(\max\{k, m^2/(k\sqrt{k})\})$. This is minimized at $k = m^{4/5}$, where the bound becomes $O(m^{4/5})$.*

To achieve the above, we concoct a three-way mixture of an *approximately stable committee* [40], a powerful notion which has been used to derive optimal randomized rules [19], alternatives with high plurality scores, and alternatives picked carefully to guarantee high minimum welfare from sufficiently many agents. We refer to the committee thus formed as a *top-biased stable k-committee*. The rule that returns this committee is presented as Algorithm 1 in the supplementary material.

Theorems 3 and 4 leave open the question of the optimal distortion that can be achieved by a random committee member rule, sandwiching this value between $O(m^{4/5})$ and $\Omega(m^{2/3})$. It is also interesting to wonder which value of k is optimal. Our upper bound is optimized at $k = m^{4/5}$, and our lower bound implies that the optimal k must be in $[m^{3/5}, m^{4/5}]$ as the distortion outside of that range is $\Omega(m^{4/5})$. See Section 5 for an empirical evaluation of the optimal k.

5 Experiments

Next, we empirically evaluate the efficiency of explainable rules studied in the previous sections.

Rules. We consider three classes of rules: deterministic positional scoring rules, randomized positional scoring rules from Section 3, and, from Section 4, rules that select uniformly at random from the k alternatives with the highest scores (henceforth, *uniform random k-positional scoring rules*). We consider four representative scoring vectors $f \in \{\text{Plurality}, \text{Borda}, \text{Harmonic}, \text{3-Approval}\}$, and denote the corresponding rules in the three classes by 'D f', 'R f', and 'UR$_k f$', respectively. Thus, overall, we test 12 voting rules. As benchmarks, we also add the *Uniform* rule, which selects an alternative uniformly at random from the set of all alternatives, and the *Instance Optimal* rule, which selects the lottery over alternatives minimizing distortion on the preference profile. Boutilier et al. [22] show how to use linear programming to compute the latter in polynomial time.

Data Generation. We generate preference profiles by sampling n rankings over m alternatives iid from the Mallows model [41], which is widely used in machine learning and statistics. The model takes as input an underlying reference ranking σ^* (which can be set arbitrarily) and a *dispersion parameter* $\phi \in [0, 1]$. When $\phi = 1$, the model converges to a uniform distribution over all $m!$ rankings (also known as *impartial culture*), whereas $\phi \to 0$ converges to the point distribution where σ^* is sampled with probability 1, so all the agents have the same preference ranking in the sampled profile. For a precise definition of the model and an efficient algorithm to sample from it (which we use in our experiments), see the work of Lu and Boutilier [42]. For each combination of $n = 100$ agents, $m \in \{5, 10, \ldots, 50\}$ alternatives, and dispersion parameter $\phi \in \{0, 0.1, \ldots, 1\}$, we sample 150 instances, and report averages along with the standard error.

Evaluation. For each rule f under consideration and each instance $\vec{\sigma}$, we evaluate the efficiency of the output $f(\vec{\sigma})$ by measuring its instance-specific distortion $\mathsf{dist}(f(\vec{\sigma}), \vec{\sigma})$. Note that this still takes a worst case over the utility profiles consistent with $\vec{\sigma}$, but unlike in Sections 3 and 4 where we also take a worst case over $\vec{\sigma}$, here we compute the expected distortion over $\vec{\sigma}$ drawn from the Mallows model. Again, we compute $\mathsf{dist}(f(\vec{\sigma}), \vec{\sigma})$ using the LP-based approach of Boutilier et al. [22].

Results. Figures 1a to 1c show the average distortion of different rules for $\phi \in \{0.1, 0.5, 1\}$, respectively, fixing $m = 25$. For large ϕ (impartial culture), randomized positional scoring rules outperform deterministic and random committee member rules as well as the uniform benchmark. In this case, it is more efficient to give a chance of winning to each alternative. As ϕ decreases to 0.5 and agent rankings become somewhat correlated, random committee member rules start to outperform some of the randomized positional scoring rules (though randomized plurality and randomized 3-approval still perform quite well). But crucially, both families of rules still outperform deterministic rules and the improved performance of random committee member rules now allows them to outperform the uniform benchmark as well. At $\phi = 0.1$, when agent preferences are highly correlated, deterministic rules gain some traction. Nonetheless, at least one rule from one of the two randomized classes still outperforms all deterministic rules (randomized plurality for low m and any random committee member rule for high m). Overall, the evidence suggests that one can almost always choose an explainable randomized rule that achieves better efficiency than deterministic rules, though the choice of the rule may have to depend on the setting at hand. A detailed comparison of rules in each class can be found in the supplementary material.

In the above experiments, for random committee member rules (specifically, the uniform random k-positional scoring rules), we use a committee size of $k = 3$. Figure 1d shows the best value of k (one that yields the minimum distortion) for different scoring vectors as a function of ϕ. It turns out that the best k is indeed very small ($\leqslant 5$) unless ϕ is really close to 1. Thus, $k = 3$ is a reasonable choice that helps our random committee member rules achieve high efficiency. Still, it is possible to further optimize the efficiency of these rules by pairing them with their corresponding optimal value of k; the resulting average distortion is presented in the supplementary material.

6 Limitations and Future Work

Explainability. We focus on the families of randomized positional scoring rules and random committee member rules as two examples of explainable randomized voting rules. While we argue in the introduction that these families admit intuitive procedural explanations and provide example explanations, checking whether stakeholders find these explanations reasonable and satisfactory in the context of a real-life application requires an in-depth investigation, possibly via user studies. Our work also treats explainability as a qualitative attribute, but different rules — even within the same family — may differ in the *degree* to which they are explainable. Quantifying the degree of explainability, both theoretically and empirically, remains to be tackled.

Efficiency. Our work uses distortion as a yardstick for efficiency, leaving open exciting technical questions. While our analysis of randomized multi-level approval rules in the supplementary material takes a step towards characterizing the distortion of all randomized positional scoring rules, it still remains an unresolved challenge. For random committee member rules, even the more basic question of identifying the optimal distortion they can achieve remains open, though we are able to pinpoint it to be between $\Omega(m^{2/3})$ and $O(m^{4/5})$. Taking a step back, while distortion is a reasonable theoretical measure for efficiency, it remains to be seen whether it is also correlated for other measures of efficiency one may care about in practice. For example, in the context of food donations, Lee et al. [15], who use the deterministic Borda rule to make decisions, suggest a number of important decision factors other than the social welfare of the stakeholders, such as whether the donations are distributed equitably, how long the drivers have to travel to deliver donations, and whether organizations with higher poverty rates, lower median incomes, and worse food access are receiving sufficient donations. An important next step would be to measure the efficiency of explainable randomized voting rules in real-life applications such as food donation.

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Appendix

In this supplementary material, we provide the details and proofs omitted from the main text. The structure of the supplementary material resembles the structure of the main text.

In Appendix A, we present the results on randomized positional scoring rules. Specifically, we first present novel insights (A.1), use them to prove distortion upper bounds for common randomized positional scoring rules (A.2), prove generic lower bounds that prove tightness of these results (A.3), and finally extend our results to randomized multi-level approval rules (A.4) as the first step towards characterizing the distortion of every randomized positional scoring rule.

In Appendix B, we present the results on random k-committee member rules, first presenting a lower bound (B.1) and then an algorithm achieving a non-trivial upper bound (B.2).

Finally, in Appendix C, we present additional experiments, such as analyzing how the distortion of explainable randomized rules depends on the Mallows noise parameter ϕ for fixed values of m (as opposed to the results in the main text, which analyze the dependence on m for fixed values of ϕ), and analyzing how the optimal k (and the distortion achieved at this optimal k for random k-committee member rules changes with m.

A Randomized Positional Scoring Rules

In this section, we expand on our discussion of randomized positional scoring rules.

A.1 High-Level Distortion Analysis and Novel Insights

Next, we elaborate on the high-level distortion analysis and novel insights presented in Section 3.1 as well as provide additional novel insights.

A.1.1 Absolute Welfare is Minimized at Dichotomous Utilities

As mentioned in Strategy 3 of Section 3.1, a novel insight of our work is that proving an absolute lower bound on the welfare achieved by a rule across all instances can be useful in bounding the distortion, even though the latter needs to compare the welfare achieved in each instance to the optimum welfare in that instance. First, recall the definition of minimum welfare.

Definition 1 (Minimum Welfare)**.** *Define the* minimum welfare *of a distribution over alternatives $p \in \Delta(A)$ on a preference profile $\vec{\sigma}$ as $\text{min-sw}(p, \vec{\sigma}) = \inf_{\vec{u} \in \mathcal{C}(\vec{\sigma})} \text{sw}(p, \vec{u})$, which is the minimum social welfare of p across all consistent utility profiles. The* minimum welfare *of a voting rule f is the minimum welfare of its output, minimized over all preference profiles: $\text{min-sw}_{n,m}(f) = \min_{\vec{\sigma}} \text{min-sw}(f(\vec{\sigma}), \vec{\sigma})$, where the minimum is taken over all preference profiles with n agents and m alternatives. We drop n and m when clear from the context.*

First, we show that this minimum welfare is in fact attained at a *dichotomous utility profile*.

Definition 5 (Dichotomous Utilities)**.** *For $k \in [m]$, define the k-dichotomous utility function for agent i to be*

$$\mathbb{1}_{i,k} = \begin{cases} 1/k & \text{rank}_i(a) \leqslant k, \\ 0 & o.w.. \end{cases}$$

That is, agent i is indifferent between her top k alternatives, but does not value any other alternative. We call $\vec{u}$ a dichotomous utility profile *if, for each $i \in N$, we have $u_i = \mathbb{1}_{i,k}$ for some $k \in [m]$.*

Lemma 6. *For any preference profile $\vec{\sigma}$ and distribution over alternatives $p \in \Delta(A)$, there exists a dichotomous utility profile $\vec{u}^* \in \arg\min_{\vec{u} \in \mathcal{C}(\vec{\sigma})} \text{sw}(p, \vec{u})$ at which minimum welfare is achieved, and this minimum welfare is bounded as $\text{min-sw}(p, \vec{\sigma}) = \text{sw}(p, \vec{u}^*) \leqslant n/m$.*

Proof. Let $p = f(\vec{\sigma})$ be the distribution returned by the rule f. Take the utility profile $\vec{u} = \arg\min_{\vec{u} \in \mathcal{C}(\vec{\sigma})} \text{sw}(p, \vec{u})$. Suppose by contradiction that there exists an agent $i \in N$ such that u_i is not dichotomous. Since u_i is a unit-sum utility vector, u_i can be represented as $u_i = \sum_{k \in [m]} \alpha_k \mathbb{1}_{i,k}$ for a unique list of α_k's subject to $\sum_k \alpha_k = 1$. Then, we have

$$u_i(p) = \sum\nolimits_{k \in [m]} \alpha_k \cdot \mathbb{1}_{i,k}(p).$$

By the linearity of the expression above, we can assume, without loss of generality, that it is minimized at one of the $\mathbb{1}_{i,k}$'s.

To show that $\mathsf{min\text{-}sw}(p, \vec{\sigma}) \leqslant \frac{n}{m}$, take the utility profile $u_i = \mathbb{1}_{i,m}$ for all i (all agents are indifferent). Then, $u_i(p) = \sum_{a \in A} \frac{p_a}{m} = \frac{1}{m}$ and $\mathsf{sw}(p) = \frac{n}{m}$. ☐

Next, we show that the minimum welfare of any distribution that is returned by a randomized positional scoring rule on any preference profile is not much lower.

Lemma 7. *For any randomized positional scoring rule $f^{\mathsf{rand}}_{\vec{s}}$ and preference profile $\vec{\sigma}$, the minimum welfare is bounded as* $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{\sigma}) \geqslant n/(4m^2)$.

Proof. Normalize $\vec{s}$ such that $\|\vec{s}\|_1 = 1$. Following Lemma 11, for all $k \in [m]$, $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \frac{n}{4m^2}$. Furthermore, any scoring vector can be uniquely rewritten as $\vec{s} = \sum_{k \in [m]} \alpha_k \cdot \frac{\vec{s}_{k\text{-approval}}}{k}$ with $\alpha_k \geqslant 0$ and $\sum_{k \in [m]} \alpha_k = 1$. Note that scaling a scoring vector does not affect its distortion or obtained social welfare. Therefore,

$$\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}}) \geqslant \sum_{k \in [m]} \alpha_k \cdot \mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \frac{n}{4m^2} \cdot \sum_{k \in [m]} \alpha_k = \frac{n}{4m^2}.$$ ☐

From Lemmas 6 and 7, we have the following.

Corollary 6. *Every randomized positional scoring rule $f^{\mathsf{rand}}_{\vec{s}}$ satisfies* $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}}) \in [n/(4m^2), n/m]$.

See Table 1 for tight bounds for the rules induced by common scoring vectors.

A.1.2 Logarithmic Rounding of the Scores

Let us begin by providing a proof of the following result stated in Section 3.1.

Lemma 1 (Rounding Down Scores)**.** *Let $\alpha \geqslant 0$, and $\vec{s}, \vec{s'}$ be scoring vectors such that $s'_j \leqslant s_j \leqslant (1+\alpha)s'_j$ for all $j \in [m]$. Then, for every preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$,*

$$\frac{1}{1+\alpha} \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}) \leqslant \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \leqslant (1+\alpha) \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}),$$

and consequently, $\frac{1}{1+\alpha} \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s'}}) \leqslant \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant (1+\alpha) \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s'}})$.

Proof. Fix a preference profile $\vec{\sigma}$ and a consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Since $s'_j \leqslant s_j \leqslant (1+\alpha)s'_j$ for all $j \in [m]$, we have $\mathsf{score}(a, \vec{s}) \geqslant \mathsf{score}(a, \vec{s}')$ for all alternatives $a \in A$ as well as $\|\vec{s}\|_1 \leqslant (1+\alpha) \cdot \|\vec{s}'\|_1$. Hence,

$$\begin{aligned}\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) = \sum_{a \in A} \mathsf{sw}(a, \vec{u}) \cdot \frac{\mathsf{score}(a, \vec{s})}{n\|\vec{s}\|_1} &\geqslant \sum_{a \in A} \mathsf{sw}(a, \vec{u}) \cdot \frac{\mathsf{score}(a, \vec{s}')}{n(1+\alpha)\|\vec{s}'\|_1} \\ &= \frac{1}{1+\alpha} \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}).\end{aligned}$$

Hence,

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \leqslant (1+\alpha) \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}).$$

Since the above holds for all $\vec{\sigma}$ and $\vec{u} \in \mathcal{C}(\vec{\sigma})$, we have $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant (1+\alpha) \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'})$.

The other direction follows similarly using $\mathsf{score}(a, \vec{s}) \leqslant (1+\alpha) \cdot \mathsf{score}(a, \vec{s}')$ and $\|\vec{s}\|_1 \geqslant \|\vec{s}'\|_1$. ☐

As stated in Section 3.1, applying this transformation to scores in the range $[\|\vec{s}\|_1/(4m^2), \|\vec{s}\|_1]$ allows us to reduce them to $O(\log m)$ distinct values. The next result shows that the remaining small scores can be ignored by reducing them to 0 while only changing the distortion by another factor of at most two.

Lemma 8 (Ignoring Small Scores)**.** *Let $\vec{s}$ be a scoring vector. Let $\vec{s'}$ be $\vec{s}$ except that $s_j \leqslant \frac{\|\vec{s}\|_1}{4m^2} \Rightarrow s'_j = 0$ for all $j \in [m]$. Then,*

$$\frac{1}{2} \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s'}}) \leqslant \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant 2 \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s'}}).$$

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. For each $a \in A$, define $\delta(a) = \mathsf{score}(a, \vec{s}) - \mathsf{score}(a, \vec{s}')$. Then,

$$\begin{aligned} \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) &= \sum_{a \in A} \mathsf{sw}(a, \vec{u}) \cdot \left(\frac{\mathsf{score}(a, \vec{s}') + \delta(a)}{n\|\vec{s}\|_1} \right) \\ &= \frac{\|\vec{s}'\|_1}{\|\vec{s}\|_1} \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}) + \sum_{a \in A} \mathsf{sw}(a, \vec{u}) \cdot \frac{\delta(a)}{n\|\vec{s}\|_1}. \end{aligned} \tag{1}$$

Note that $\|\vec{s}'\|_1 \geqslant \|\vec{s}\|_1 - (m-1) \cdot \frac{\|\vec{s}\|_1}{4m^2} \geqslant (1 - \frac{1}{4m+1}) \cdot \|\vec{s}\|_1$. Thus, in Equation (1), we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \geqslant \left(1 - \frac{1}{4m+1}\right) \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}).$$

This implies

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \leqslant \frac{4m+1}{4m} \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}).$$

Since this holds for all $\vec{\sigma}$ and $\vec{u} \in \mathcal{C}(\vec{\sigma})$, we have $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant (1 + \frac{1}{4m}) \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'})$.

For the other direction, we use the facts that $\|\vec{s}'\|_1 \leqslant \|\vec{s}\|_1$ and $\frac{\delta(a)}{n\|\vec{s}\|_1} \leqslant \frac{1}{n\|\vec{s}\|_1} \cdot n \cdot \frac{\|\vec{s}\|_1}{4m^2} = \frac{1}{4m^2}$. Thus, in Equation (1), we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \leqslant \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}) + \sum_{a \in A} \mathsf{sw}(a, \vec{u}) \cdot \frac{1}{4m^2} = \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}) + \frac{n}{4m^2} \leqslant 2 \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}),$$

where the last inequality is due to Lemma 7. Using the same argument as above, this results in $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'}) \leqslant 2 \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}})$. □

Using Lemma 1 with $\alpha = 1$ and combining with Lemma 8, we get the following.

Corollary 7 (Scoring Vector Reduction)**.** *Given any scoring vector $\vec{s}$, there exists a scoring vector $\vec{s}'$ with $O(\log m)$ distinct positive scores such that $(1/4) \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'}) \leqslant 4 \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}})$.*

A.1.3 Helpful Technical Lemmas

Before proving Lemma 2, we show a weaker lemma which follows from a simpler proof. This is useful in particular for analyzing randomized k-approval rules with $k \in [1, m - \Omega(m)]$ (which includes the randomized dictatorship rule) and randomized approval mixtures rules where again we mix some k-approvals with $k \in [1, m - \Omega(m)]$.

Lemma 9. *Fix any preference profile $\vec{\sigma}$, subset of agents $T \subset N$, threshold $\tau \geqslant 0$, and rank $\ell \in [m]$. For a partial utility profile $\vec{u}$ in which every agent in T has utility at least τ for each of her top ℓ alternatives and all other utilities are 0, we have*

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \tau \cdot \frac{|T|^2}{nm} \cdot \frac{(\min\{k, \ell\})^2}{k}.$$

Proof. Let $t = \min\{k, \ell\}$. For all $a \in A$ let x_a denote the number of appearances of a among the top t votes of T. Then,

$$\begin{aligned} \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) &= \sum_{a \in A} \Pr[a] \cdot \mathsf{sw}(a) \\ &\geqslant \sum_{a \in A} \frac{x_a}{nk} \cdot (x_a \cdot \tau) \\ &= \frac{\tau}{nk} \cdot \sum_{a \in A} x_a^2 \\ &\geqslant \frac{\tau}{nk} \cdot \frac{\left(\sum_{a \in A} x_a\right)^2}{m} && \text{(by AM-QM inequality)} \\ &= \frac{\tau}{nk} \cdot \frac{(|T| \cdot t)^2}{m} = \tau \cdot \frac{|T|^2}{nm} \cdot \frac{(\min\{k, \ell\})^2}{k}. \end{aligned}$$

□

The key limitation of the lemma above is that the probability of selecting alternative $\Pr[a]$ is lower bounded by $\frac{x_a}{nk}$ (the score a gets only from the top ℓ alternatives of T), while this could be higher due to the score that agents in $N \setminus T$ give to a. As a result, in the transition that uses the AM-QM inequality, we divide the top ℓ alternatives of T among all m alternatives. If the scores obtained from $N \setminus T$ is also considered, it may be the case that the top ℓ alternatives of T is divided among fewer number of alternatives (sublinear in m), which enables stronger lower bounds. To this end, we need a more complicated analysis provided in the following lemma.

Lemma 2. *Fix any scoring vector $\vec{s}$, preference profile $\vec{\sigma}$, subset of agents $T \subseteq N$, threshold $\tau \geqslant 0$, and rank $\ell \in [m]$. For a partial utility profile $\vec{u}$ in which every agent in T has utility at least τ for each of her top ℓ alternatives and all other utilities are 0, we have:*

$$\mathsf{sw}_T(f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma}), \vec{u}) \geqslant \tau \cdot \frac{|T|\ell}{2n\|\vec{s}\|_1} \min_{h \in [m]} \frac{1}{h}\Big(2s_\ell \cdot |T|\ell + (n-|T|) \cdot \sum_{j=1}^{h} s_{m-j+1}\Big).$$

Proof. For all alternative a, denote by x_a the number of appearances of a among the top ℓ votes of T. Note that $\sum_{a \in A} x_a = |T| \cdot \ell$. We have

$$\begin{aligned}
\mathsf{sw}_T(f_{\vec{s}}^{\mathsf{rand}}) &= \sum_{a \in A} \mathsf{sw}_T(a) \cdot \frac{\mathsf{score}(a)}{n\|\vec{s}\|_1} \\
&\geqslant \sum_{a \in A} \tau \cdot x_a \cdot \frac{\mathsf{score}_T(a) + \mathsf{score}_{N\setminus T}(a)}{n\|\vec{s}\|_1} \\
&\geqslant \sum_{a \in A} \tau \cdot x_a \cdot \frac{x_a \cdot s_\ell + \mathsf{score}_{N\setminus T}(a)}{n\|\vec{s}\|_1}. \qquad (2)
\end{aligned}$$

Rename the candidates such that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_m$.

Worst-case Preference Ranking of $N \setminus T$. The contribution of $N \setminus T$ to Equation (2) is $\sum_a x_a \cdot \mathsf{score}_{N\setminus T}(a)$ which can be decomposed to across agents, i.e. the contribution of each agent $i \in N \setminus T$ is $\sum_{a \in A} x_a \cdot \mathsf{score}_i(a)$. Since x_a's are only depends on agents in T, to obtain a lower bound, we may assume without loss of generality that their preference ranking is $m \succ_i m-1 \succ_i \ldots \succ_i 1$, since $x_m \leqslant x_{m-1} \leqslant \ldots \leqslant x_1$ and $\mathsf{score}_i(\sigma_i(1)) \geqslant \ldots \geqslant \mathsf{score}_i(\sigma_i(m))$.

Forming a Quadratic Program. Recall that we renamed the alternatives in decreasing order by x_a. Following the observation of worst-case preference ranking of $N \setminus T$, to obtain a lower bound, for $a \in [m]$, we have $\mathsf{score}_{N\setminus T}(a) = (n - |T|) \cdot s_{m-a+1}$. Now, we can rewrite Equation (2) as follows,

$$\frac{\tau}{n\|\vec{s}\|_1 \cdot s_\ell} \cdot \sum_{a=1}^{m} s_\ell \cdot x_a \cdot (s_\ell \cdot x_a + (n-|T|) \cdot s_{m-i+1}). \qquad (3)$$

Now, we simplify this expression to form a quadratic program and analyze its minimum. Since $\frac{\tau}{n\|\vec{s}\|_1 \cdot s_\ell}$ is a constant value, we focus on the summation. For conciseness, define $y_a = s_\ell \cdot x_a$, $\gamma_a = (n-|T|) \cdot s_{m-a+1}$, and $\beta = s_\ell \cdot |T| \cdot \ell = \sum_{a=1}^{m} y_a$ (holds due to $\sum_a x_a = |T| \cdot \ell$). Then, we have the following quadratic program with variables $\mathbf{y} = \{y_a\}_{a \in A}$,

$$\begin{aligned}
\min \quad & \sum_{a=1}^{m} y_a \cdot (y_a + \gamma_a) \\
\text{s.t.} \quad & \sum_{a=1}^{m} y_a = \beta \\
& y_a \geqslant 0 \qquad\qquad \forall a \in [m]. \qquad (4)
\end{aligned}$$

Applying the KKT Conditions. Our objective is convex in y_a's, and it is easy to check that this program satisfies the Slater's condition for $y_a = \beta/m$. Hence, we can apply the KKT conditions to find the minimizer of this program. The Lagrangian for $f(\mathbf{y}) = \sum_a y_a(y_a + \gamma_a)$, $g(\mathbf{y}) = \beta - \sum_a y_a$, and $h_a(\mathbf{y}) = -y_a$, is

$$\mathcal{L}(\mathbf{y}, \lambda, \boldsymbol{\mu}) = f(\mathbf{y}) + \lambda g(\mathbf{y}) + \sum_{a=1}^{m} \mu_a \cdot h_a(\mathbf{y}),$$

where, in the dual program, μ_a is the variable for $y_a \geqslant 0$ conditions and λ is for the single equality condition. From the KKT conditions, we have y_a's are the minimizer of this function if

C1. (stationarity) $\forall a \in [m],\ 2 \cdot y_a + \alpha_a - \mu_a - \lambda = 0$

C2. (primal feasibility) (1) $\forall a \in [m], y_a \geqslant 0$ and (2) $\sum_{a=1}^{m} = \beta$

C3. (dual feasibility) $\forall a \in [m], \mu_a \geqslant 0$

C4. (complementary slackness) $\forall a \in [m], \mu_a \cdot y_a = 0$.

By C1 and C3, we have

$$y_a \leqslant \frac{1}{2}\left(\lambda - \alpha_a\right).$$

For $y_a \neq 0$, we can multiply C1 by y_a and using C4 (i.e. $y_a \cdot \mu_a = 0$), we get

$$y_a \neq 0 \ \Rightarrow \ y_a = \frac{1}{2}\left(\lambda - \alpha_a\right)$$

Therefore, $y_a = \max\{0, \frac{1}{2}(\lambda - \alpha_a)\}$. Let $|C^*|$ be the number of non-zero y_a's (i.e. the number of candidates with non-zero appearances among the top-ℓ votes of T). Furthermore, by C2, we have

$$\sum_{a=1}^{m} y_a = \sum_{a=1}^{m} \frac{1}{2} \max\{0, \lambda - \alpha_a\} = \beta \quad \Rightarrow \quad \lambda = \frac{1}{|C^*|}\left(2\beta + \sum_{a=1}^{|C^*|} \alpha_a\right) \tag{5}$$

Since $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_m$, the final y_a's will form a decreasing series $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{|C^*|} > y_{|C^*|+1} = \cdots y_m = 0$ with $|C^*|$ many non-zero values. This is similar to a water-filling argument. Initial levels are α_a, and the water fills up from the bottom to level λ with a total water amount of 2β.

Deriving a Lower Bound. Now, we use the findings above to get a lower bound as follows

$$\begin{aligned}
\sum_{a=1}^{m} y_a(y_a + \alpha_a) &\geqslant \sum_{a=1}^{m} y_a(y_a + \alpha_a/2) \\
&\geqslant \sum_{a=1}^{|C^*|} y_a \left(\frac{1}{2}(\lambda - \alpha_a) + \alpha_a/2\right) \\
&\geqslant \frac{\lambda}{2} \sum_{a=1}^{|C^*|} y_a = \frac{\lambda \cdot \beta}{2} \\
&= \frac{\beta}{2} \cdot \frac{1}{|C^*|}\left(2\beta + \sum_{a=1}^{|C^*|} \alpha_a\right). \\
&\geqslant \min_{|C^*| \in [m]} \ \frac{\beta}{2|C^*|} \cdot \left(2\beta + \sum_{a=1}^{|C^*|} \alpha_a\right). \\
&= \min_{|C^*| \in [m]} \ \frac{s_\ell \cdot |T|\ell}{2|C^*|} \cdot \left(2 s_\ell \cdot |T|\ell + (n - |T|) \cdot \sum_{a=1}^{|C^*|} s_{m-i+1}\right).
\end{aligned}$$

Combined with Equation (2), we have

$$\mathsf{sw}_T(f_{\vec{s}}^{\mathsf{rand}}) \geqslant \tau \cdot \frac{|T|\ell}{2n\|s\|_1} \min_{|C^*| \in [m]} \frac{1}{|C^*|}\left(2 s_\ell \cdot |T|\ell + (n - |T|) \cdot \sum_{a=1}^{|C^*|} s_{m-i+1}\right). \qquad \square$$

A.2 Distortion of Common Scoring Rules

Here, we use the insights and high-level strategies laid out above to analyze common randomized positional scoring rules. At first, we derive only a lower bound on their minimum welfare and an upper bound on their distortion, and later we show that our bounds are tight.

A.2.1 Randomized Plurality

Lemma 10. *The minimum welfare of the randomized plurality (randomized dictatorship) rule is* $\text{min-sw}(f^{\text{rand}}_{\vec{s}_{\text{plu}}}) \geqslant \frac{n}{m^2}$.

Proof. For $a \in A$, let $\text{plu}(a) = \text{score}(a, \vec{s}_{\text{plu}})$ be the score of alternative a. Then, $\text{sw}(a) \geqslant \text{plu}(a) \cdot \frac{1}{m}$, since each voter deems a utility of at least $\frac{1}{m}$ for their top alternative. Then,

$$\text{sw}(f_{\text{plu}}) = \sum_{a\in A} \frac{\text{plu}(a)}{n} \cdot \text{sw}(a) \geqslant \frac{1}{n} \sum_{a\in A} \frac{\text{plu}(a)^2}{m} \overset{(1)}{\geqslant} \frac{1}{n} \left(\sum_{a\in A} \frac{\text{plu}(a)}{m} \right)^2 \geqslant \frac{n}{m^2},$$

where inequality (1) holds by the AM-QM inequality,and we used $\sum_{a\in A} \text{plu}(a) = n$ in the last inequality.[2] $\square$

Theorem 8. *The distortion of the randomized plurality rule is* $O(m\sqrt{m})$.

Proof. For an alternative $a \in A$, let $N_a^+ \subseteq N$ be the set of agents whose top alternative is a. Then, we have $|N_a^+| = \text{plu}(a)$, and

$$\text{sw}_{N_a^+}(a^*) = \sum_{i\in N_a^+} u_i(a^*) \leqslant |N_a^+| = \text{plu}(a),$$

where the inequality comes from the unit-sum assumption that utilities are at most 1. Moreover, $\text{sw}_{N_a^+}(a) \geqslant \text{sw}_{N_a^+}(a^*)$, since voters in N_a^+ prefer a to a^*. Thus,

$$\begin{aligned} \text{sw}(f^{\text{rand}}_{\vec{s}_{\text{plu}}}) = \sum_{a\in A} \frac{\text{plu}(a)}{n} \cdot \text{sw}(a) &\geqslant \frac{1}{n} \sum_{a\in A} \text{sw}_{N_a^+}(a^*) \cdot \text{sw}_{N_a^+}(a^*) \\ &\geqslant \frac{1}{n} \cdot \frac{1}{m} \left(\sum_{a\in A} \text{sw}_{N_a^+}(a^*) \right)^2 = \frac{\text{sw}(a^*)^2}{n \cdot m}, \end{aligned}$$

where the second inequality holds by the AM-QM inequality. Consequently,

$$\text{dist}(f_{\text{plu}}) \leqslant \frac{n \cdot m}{\text{sw}(a^*)}.$$

Furthermore, by Lemma 10 we have $\text{dist}(f^{\text{rand}}_{\vec{s}_{\text{plu}}}) \leqslant \frac{\text{sw}(a^*)}{n/m^2}$. Combining the two bounds, we have

$$\text{dist}(f^{\text{rand}}_{\vec{s}_{\text{plu}}}) \leqslant \min \left\{ \frac{m^2 \cdot \text{sw}(a^*)}{n}, \frac{n \cdot m}{\text{sw}(a^*)} \right\} \leqslant \sqrt{\frac{m^2 \cdot \text{sw}(a^*)}{n} \cdot \frac{n \cdot m}{\text{sw}(a^*)}} \leqslant m\sqrt{m}. \qquad \square$$

A.2.2 Randomized Borda

The following lemma is useful to analyze the distortion of the randomized Borda rule.

Lemma 4. *For any preference profile $\vec{\sigma}$, consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$, and alternative $a \in A$, we have* $\text{sw}(a) \leqslant (\text{Borda}(a) + n)/m$.

Proof. Fix an alternative a and agent i. By the unit-sum assumption, we have

$$u_i(a) \leqslant \frac{1}{\text{rank}_i(a)} \leqslant \frac{m - \text{rank}_i(a) + 1}{m} = \frac{\text{Borda}(a, i) + 1}{m}.$$

By summing over all agents, we get $\text{sw}(a) \leqslant \frac{\text{Borda}(a)+n}{m}$. $\square$

Lemma 5. *The distortion of the randomized Borda rule is* $O(m^{5/4})$.

[2] This is an instantiation of Lemma 9 for $k = \ell = 1$, $\tau = \frac{1}{m}$, $|T| = n$.

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Let $a^* \in \arg\max_{a \in A} \mathsf{sw}(a, \vec{u})$ be an optimal alternative. By Strategy 2 from Section 3.1, we know that the distortion is at most $\frac{n\|\vec{s}_{\mathsf{Borda}}\|_1}{\mathsf{Borda}(a^*)}$. Following Strategy 3 from Section 3.1 and the minimum welfare analysis in Lemma 3, we have that distortion is at most $\frac{\mathsf{sw}(a^*)}{n/(12m\sqrt{m})}$, which, using Lemma 4, is at most $12\sqrt{m} \cdot \left(\mathsf{Borda}(a^*)/n + 1\right)$. Putting everything together, and using the fact that $\min\{a, b\} \leqslant \sqrt{ab}$ for all $a, b \in \mathbb{R}_{\geqslant 0}$, the distortion is upper bounded by

$$
\begin{aligned}
&\min\left\{\frac{n\|\vec{s}_{\mathsf{Borda}}\|_1}{\mathsf{Borda}(a^*)}, \frac{\mathsf{Borda}(a^*) \cdot 12\sqrt{m}}{n} + 12\sqrt{m}\right\} \\
&\quad \leqslant 12\sqrt{m} + \sqrt{\frac{n \cdot m(m-1)/2}{\mathsf{Borda}(a^*)} \cdot \frac{\mathsf{Borda}(a^*) \cdot 12\sqrt{m}}{n}} \leqslant 8\sqrt{m} + \sqrt{6}m^{5/4} = O(m^{5/4}). \quad \square
\end{aligned}
$$

A.2.3 Randomized k-Approval

Lemma 11. *The minimum welfare of the randomized k-approval rule is,*

- *when $k \leqslant \sqrt{m}$,* $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \frac{n}{4m} \cdot \frac{k}{m}$,
- *and when $k > \sqrt{m}$,* $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{Borda}}}) \geqslant \frac{n}{4(m-k+1)} \cdot \frac{1}{k}$.

Proof. When $k = m$, $f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}$ is equivalent to selecting an alternative uniformly at random, which achieves a social welfare of exactly $\frac{n}{m}$. Now, suppose $k \leqslant m - 1$.

Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. First, similar to the analysis of randomized Borda in Lemma 3, we make a few modifications to the preference profile that are guaranteed to not increase the welfare, and then invoke Lemma 2. For conciseness, let $\vec{s} = \vec{s}_{k\text{-approval}}$, and $f = f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}$.

Simplify the preference and utility profiles. By Lemma 6, to obtain a lower bound, we assume without loss of generality that each agent i has a dichotomous utility, i.e. $u_i \in \{\mathbb{1}_{i,\ell}\}_{\ell \in [m]}$. Partition $N = N_1 \cup N_2$, where $N_1 = \{u_i = \mathbb{1}_{i,\ell} \mid i \in N, \ell \leqslant k\}$ and $N_2 = \{u_i = \mathbb{1}_{i,\ell} \mid i \in N, \ell > k\}$.

Case $|N_1| \geqslant |N_2|$. Fix an agent $i \in N_1$. Let A_i be the set of top ℓ alternatives in σ_i and $a_i \in \arg\min_{a \in A_i} \mathsf{score}(a, \vec{s})$ be the alternative in A_i with the lowest score (equivalently, with the lowest probability of selection under f). Note that $\sum_{a \in A_i} u_i(a) = 1$. Now,

$$
u_i(f) \geqslant \sum_{a \in A_i} \frac{\mathsf{score}(a, \vec{s})}{n\|\vec{s}\|_1} \cdot u_i(a) \overset{(1)}{\geqslant} \frac{\mathsf{score}(a_i, \vec{s})}{n\|\vec{s}\|_1} \cdot \left(\sum_{a \in A_i} u_i(a)\right) \geqslant \frac{\mathsf{score}(a_i, \vec{s})}{n\|\vec{s}\|_1} \cdot 1,
$$

where (1) follows from the definition of a_i. This can be rewritten as $u'_i(f(\vec{\sigma}'))$, where $\vec{\sigma}'$ is a preference profile in which each agent $i \in N_1$ ranks a_i first, and $\vec{u}'$ is a partial utility profile in which each agent $i \in N_1$ has utility of 1 for her top alternative and 0 for the rest. For $i \in N_2$, we assume they have 0 utility for all alternatives. Summing the above for all agents, we have $\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \geqslant \mathsf{sw}(f(\vec{\sigma}'), \vec{u}')$. Now, to lower bound it, we invoke Lemma 2 with $\vec{s} \leftarrow \vec{s}_{k\text{-approval}}$, $\vec{\sigma} \leftarrow \vec{\sigma}'$, $\vec{u} \leftarrow \vec{u}'$, $T = N_1$, $\tau \leftarrow 1$, and $\ell \leftarrow 1$. Using $\|\vec{s}\|_1 = k$, this gives us

$$
\begin{aligned}
\mathsf{sw}(f(\vec{\sigma}'), \vec{u}') &\geqslant 1 \cdot \frac{\frac{n}{2} \cdot 1}{2n \cdot k} \cdot \min_{h \in [m]} \frac{1}{h}\left(2 \cdot \frac{n}{2} \cdot 1 + \frac{n}{2} \cdot \max\{0, h - (m-k)\}\right) \\
&= \frac{n}{8k} \cdot \min_{h \in [m]} \left(\frac{2 + \max\{0, h - (m-k)\}}{h}\right) \\
&\geqslant \frac{n}{8k} \cdot \min\left\{\frac{2}{m-k}, \frac{3}{m-k+1}, \frac{4}{m-k+2}, \dots, \frac{k+2}{m}\right\} \\
&\overset{(1)}{\geqslant} \frac{n}{8k} \cdot \frac{2}{m-k+1} = \frac{n}{4k(m-k+1)},
\end{aligned}
$$

where (1) holds for $k \in [m-1]$.

Case $|N_1| \leqslant |N_2|$. Any agent $i \in N_2$ has a utility of at least $\frac{1}{\ell} \geqslant \frac{1}{m}$ for each of her top k alternatives. This can be considered as a utility profile $\vec{u}'$ where agents $i \in N_2$ have a utility of $\frac{1}{m}$ for their top k alternatives and 0 for the rest, and agents $i \in N_1$ have a 0 utility for all alternatives. Now, we invoke Lemma 9 with $k \leftarrow k$, $\vec{\sigma} \leftarrow \vec{\sigma}$, $\vec{u} \leftarrow \vec{u}'$, $T \leftarrow N_2$, $\tau \leftarrow \frac{1}{m}$, $\ell \leftarrow k$. Since $\|\vec{s}\| = k$, we have

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \geqslant \frac{1}{m} \cdot \frac{\left(\frac{n}{2}\right)^2}{nm} \cdot \frac{k^2}{k} = \frac{nk}{4m^2}.$$

Since either of the two cases must hold,

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \geqslant \min\left\{\frac{n}{4k(m-k+1)}, \frac{nk}{4m^2}\right\},$$

which completes the proof. □

Lemma 12. *The distortion of the randomized k-approvals rule is*

- $O\left(\frac{m\sqrt{m}}{k\sqrt{k}}\right)$ *when* $k \leqslant m^{1/3}$,
- $O(m)$ *when* $m^{1/3} \leqslant k \leqslant \sqrt{m}$,
- *and* $O(k\sqrt{m-k+1})$ *when* $k > \sqrt{m}$.

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Let $a^* \in \arg\max_{a \in A} \mathsf{sw}(a, \vec{u})$ be an optimal alternative. Partition the agents $N = N^+ \cup N^-$ based on whether they approve a^* or not, i.e. $N^+ = \{i \in N \mid \mathsf{rank}_i(a^*) \leqslant k\}$ and $N^- = \{i \in N \mid \mathsf{rank}_i(a^*) > k\}$. Furthermore, $\mathsf{sw}(a^*) = \mathsf{sw}_{N^+}(a^*) + \mathsf{sw}_{N^-}(a^*)$. We derive distortion bounds for two cases based on the comparison of $\mathsf{sw}_{N^+}(a^*)$ and $\mathsf{sw}_{N^-}(a^*)$, and report the maximum of the two bounds as the upper bound. Before the case analysis, recall that by following strategy 3 and Lemma 11, we have

$$\begin{aligned} &\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \text{min-}\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \min\left\{\frac{nk}{4m^2}, \frac{n}{4k(m-k+1)}\right\} = g(n,m) \\ \Rightarrow\quad &\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \frac{\mathsf{sw}(a^*)}{g(n,m)}. \end{aligned} \tag{6}$$

Case $\mathsf{sw}_{N^+}(a^*) \geqslant \mathsf{sw}_{N^-}(a^*)$. Since $\mathsf{sw}_{N^+}(a^*) \leqslant |N^+|$, it follows that $\mathsf{sw}(a^*) \leqslant 2 \cdot \mathsf{sw}_{N^+}(a^*) \leqslant 2|N^+|$. By Equation (6), we have

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \frac{2|N^+|}{g(n,m)}.$$

By strategy 2 and that $\Pr[a^* \in f(\vec{\sigma})] = \frac{|N^+|}{nk}$, we get

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \frac{nk}{|N^+|}.$$

Putting the two together we have

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \min\left\{\frac{2|N^+|}{g(n,m)}, \frac{nk}{|N^+|}\right\} \overset{(1)}{\leqslant} \sqrt{\frac{2|N^+|}{g(n,m)} \cdot \frac{nk}{|N^+|}} = \sqrt{\frac{2nk}{g(n,m)}},$$

where (1) follows from the inequality $\min\{a,b\} \leqslant \sqrt{a \cdot b}$ for $a, b \geqslant 0$. By expanding $g(n,m)$, we have

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \begin{cases} \sqrt{\frac{2nk}{nk/(4m^2)}} = \sqrt{8} \cdot m, & \text{if } k \in [1, \sqrt{m}], \\ \sqrt{\frac{2nk}{n/(4k(m-k+1))}} = \sqrt{8} \cdot k\sqrt{m-k+1} & \text{if } k \in [\sqrt{m}, m]. \end{cases} \tag{7}$$

Case $\mathsf{sw}_{N^+}(a^*) < \mathsf{sw}_{N^-}(a^*)$. Similar to the previous case, $\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \leqslant 2 \cdot \mathsf{sw}_{N^-}(f(\vec{\sigma}), \vec{u})$. The key insight in this case is to apply strategy 2 (analyzing the welfare above a^* in N^-). First, construct a new utility profile by rounding down the utilities to the closest power of two, i.e., $u_i'(a^*) = 2^{\lfloor \log_2 u_i(a^*) \rfloor}$ and replace utilities less than $\frac{1}{4m^2}$ with 0. This way,

$$\mathsf{sw}(a^*) \leqslant 2 \cdot \mathsf{sw}_{N^-}(a^*, \vec{u}), \quad \text{and} \quad \mathsf{sw}_{N^-}(a^*, \vec{u}) \leqslant 2 \cdot \mathsf{sw}_{N^-}(a^*, \vec{u}') + \frac{n}{m^2}$$

Now, we subdivide N^- based on their utility for a^*. Since $\mathsf{rank}_i(a^*) \geqslant k+1$ for all $i \in N^-$, $u_i(a^*) \leqslant \frac{1}{k+1}$ (otherwise agent's total utility for her top $k+1$ alternatives exceeds one). For $z \in \left[\lfloor \log_2 \frac{1}{k+1} \rfloor, \lfloor \log_2 \frac{1}{m^2} \rfloor\right]$, let $N_z^- = \{i \in N_i \mid u_i'(a^*) = 2^z\}$. For each group, let $\vec{u}_z'$ be the utility profile where agents $i \in N_z^-$ have utility of 2^z for their top k votes and value the rest at 0, and agents $i \in N \setminus N_z^-$ have utility of 0 for all agents. Note that this is a decomposition of the $\vec{u}'$, and each nonzero utilities is only considered in one of the $\vec{u}_z'$'s. Now, we invoke Lemma 9 with $\vec{s} \leftarrow \vec{s}_{k\text{-approval}}$, $\vec{\sigma} \leftarrow \vec{\sigma}$, $\vec{u} \leftarrow \vec{u}_z'$, $\ell \leftarrow k$, $k \leftarrow k$, $T \leftarrow N_z^-$, $\tau \leftarrow 2^z$, and we have

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}_z') \geqslant 2^z \cdot \frac{|T|^2}{nm} \cdot \frac{k^2}{k} = 2^z \cdot \frac{|N_z^-|^2 \cdot k}{nm}.$$

Since the utilities above are disjoint, we can combine the bounds above and derive

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}') \geqslant \sum_z 2^z \cdot \frac{|N_z^-|^2 \cdot k}{nm}.$$

We use the above combined with the absolute welfare guarantee in Equation (6), to derive the distortion guarantee as follows.

$$\begin{aligned}
\mathsf{dist}(f(\vec{\sigma}), \vec{u}) &= \frac{\mathsf{sw}(a^*, \vec{u})}{\mathsf{sw}(f(\vec{\sigma}), \vec{u}))} \\
&\leqslant \frac{4 \cdot \mathsf{sw}_{N^-}(a^*) + \frac{2n}{m^2}}{\mathsf{sw}(f(\vec{\sigma})} \\
&\leqslant \frac{4 \cdot \sum_z 2^z \cdot |N_z^-| + \frac{2n}{m^2}}{\max\left\{\sum_z 2^z \cdot \frac{k}{nm} \cdot |N_z^-|^2 \ , \ g(n,m)\right\}} \\
&\leqslant \frac{2n}{m^2 \cdot g(n,m)} + 4 \sum_z \frac{2^z \cdot |N_z^-|}{\max\{2^z \cdot \frac{k}{nm} \cdot |N_z^-|^2, g(n,m)\}} \\
&\leqslant 8 + 4 \sum_z \frac{2^z \cdot |N_z^-|}{\max\{2^z \cdot \frac{k}{nm} \cdot |N_z^-|^2, g(n,m)\}} && (\text{since } g(n,m) \geqslant \frac{n}{4m^2}) \\
&\leqslant 8 + 4 \sum_z \min\left\{\frac{nm}{|N_z^-| \cdot k}, \frac{2^z |N_z^-|}{g(n,m)}\right\} \\
&\leqslant 8 + 4 \sum_z \sqrt{\frac{nm}{|N_z^-| \cdot k} \cdot \frac{2^z |N_z^-|}{g(n,m)}} && (\min\{a,b\} \leqslant \sqrt{ab}) \\
&\overset{(1)}{\leqslant} 8 + 4 \sqrt{\frac{nm}{k \cdot g(n,m)}} \cdot \sum\nolimits_{z=-\lceil \log_2(k+1) \rceil}^{\lceil \log_2(m^2) \rceil} (2^z)^{\frac{1}{2}} \\
&\leqslant 8 + \frac{4}{\sqrt{k}} \cdot \sqrt{\frac{nm}{g(n,m)}} \cdot \frac{4}{\sqrt{k+1}} && (2^z \leqslant \frac{1}{k+1}),
\end{aligned}$$

where (1) holds due to the following

$$\sum\nolimits_{z=-\lceil \log_2(k+1) \rceil}^{\lceil \log_2(m^2) \rceil} (2^z)^{\frac{1}{2}} \leqslant \frac{1}{\sqrt{k+1}} \cdot \sum_{j=0}^{\infty} 2^{-\frac{j}{2}} \leqslant \frac{1}{\sqrt{k+1}} \cdot 2 \sum_{j=0}^{\infty} 2^{-j} \leqslant \frac{4}{\sqrt{k+1}}.$$

By expanding $g(n,m)$, we have

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant 8 + \begin{cases} \frac{16}{k} \cdot \sqrt{\frac{nm}{nk/(m^2)}} = 16 \cdot \frac{m\sqrt{m}}{k\sqrt{k}}, & \text{if } k \in [1, \sqrt{m}], \\ \frac{16}{k} \cdot \sqrt{\frac{nm}{n/(4k(m-k+1))}} = 32 \cdot \frac{\sqrt{m(m-k+1)}}{\sqrt{k}}, & \text{if } k \in [\sqrt{m}, m]. \end{cases} \tag{8}$$

Now, by taking the pairwise maximum of Equations (7) and (8) we derive the following distortion upper bounds,

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \begin{cases} O(\frac{m\sqrt{m}}{k\sqrt{k}}) & \text{if } k \in [1, m^{1/3}], \\ O(m) & \text{if } k \in [m^{1/3}, \sqrt{m}], \\ O(k\sqrt{m-k+1}) & \text{if } k \in [\sqrt{m}, m]. \end{cases} \qquad \square$$

A.2.4 Randomized Harmonic

Boutilier et al. [22] proposed the rule that executes the randomized harmonic rule with probability $1/2$ and selects an alternative uniformly at random with the remaining probability $1/2$. They main contribution was to show that this rule achieves $O(\sqrt{m \log m})$ distortion (tightness of this bound was shown by Ebadian et al. [19]), very close to their lower bound of $\Omega(\sqrt{m})$. We show that the latter half (uniform selection) of this rule, which is often critized as impractical, is largely unnecessary. Simply executing the randomized harmonic rule achieves a distortion of $\Theta(\sqrt{m}\log m)$, which is only a $\Theta(\sqrt{\log m})$ factor larger. Let us first prove a lower bound on its minimum welfare.

Lemma 13. *The minimum welfare of the randomized harmonic rule is* $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}) \geqslant \frac{n}{mH_m}$.

Proof. Fix a preference profile $\vec{\sigma}$ and utility profile $\vec{u}$. Since all agents give a score of at least $\frac{1}{m}$ to all the alternatives, $\mathsf{score}(a) \geqslant \frac{n}{m}$. Therefore, by $\|\vec{s}_{\mathsf{harmonic}}\|_1 = H_m$, we have $\Pr[a \in f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}] = \frac{\mathsf{score}(a)}{n\|\vec{s}\|_1} \geqslant \frac{1}{mH_m}$, and $\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \geqslant \frac{1}{mH_m} \cdot \sum_{a \in A} \mathsf{sw}(a) = \frac{n}{mH_m}$. $\square$

Now, we show a well-known useful fact about harmonic scores.

Lemma 14 ([22])**.** *For any preference profile $\vec{\sigma}$, consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$, and alternative $a \in A$, we have* $\mathsf{sw}(a, \vec{u}) \leqslant \mathsf{score}(a, \vec{s}_{\mathsf{harmonic}})$.

Proof. Fix an agent $i \in N$. Then,

$$\mathsf{score}_i(a) = \frac{1}{\mathsf{rank}_i(a)} \geqslant u_i(a),$$

where the inequality is due to the unit-sum assumption, i.e. otherwise agent i's total utility for her top $\mathsf{rank}_i(a)$ alternatives exceeds one. Summing above for all agents yields the sought goal. $\square$

Lemma 15. *The distortion of the randomized harmonic rule is $O(\sqrt{m}H_m)$.*

Proof. Fix a preference profile $\vec{\sigma}$ and utility profile $\vec{u}$. Let $a^* \in \arg\max_{a \in A} \mathsf{sw}(a, \vec{u})$ be an optimal alternative. By strategy 3 (absolute welfare guarantee), we have

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}(\vec{\sigma}), \vec{u}) \leqslant \frac{\mathsf{sw}(a^*)}{n/(mH_m)} \leqslant \frac{mH_m \cdot \mathsf{score}(a^*)}{n},$$

where the last inequality holds due to Lemma 14. Furthermore, by strategy 2 (probability of a^*), we have

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}(\vec{\sigma}), \vec{u}) \leqslant \frac{nH_m}{\mathsf{score}(a^*)}.$$

Putting the two together, we have

$$\begin{aligned} \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}(\vec{\sigma}), \vec{u}) &\leqslant \min\left\{\frac{mH_m \cdot \mathsf{score}(a^*)}{n}, \frac{nH_m}{\mathsf{score}(a^*)}\right\} \\ &\leqslant \sqrt{\frac{mH_m \cdot \mathsf{score}(a^*)}{n} \cdot \frac{nH_m}{\mathsf{score}(a^*)}} = H_m\sqrt{m}, \end{aligned} \qquad \square$$

where the last inequality is due to $\min\{a, b\} \leqslant \sqrt{ab}$ for $a, b \geqslant 0$. Since this holds for all preference and utility profiles, $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}) \leqslant H_m \cdot \sqrt{m}$.

A.3 Lower Bounds

Next, we prove tightness of the distortion upper bounds obtained above. We do this by deriving a few general bounds.

Theorem 9. *The distortion of any randomized positional scoring rule $f_{\vec{s}}^{\mathsf{rand}}$ with $s_m \leqslant s_1/\sqrt{m}$ is $\Omega\big(\frac{\|\vec{s}\|_1}{s_1} \cdot \sqrt{t^*}\big)$, where $t^* = \arg\max_{t \in [m]} \big\{t \mid \sum_{j=1}^{t} s_{m-j+1} \leqslant s_1\big\}$.*

Proof. Consider the preference profile $\vec{\sigma}$ where alternative a^* appears as the top choice of $n/\sqrt{t^*}$ of the agents and the bottom of the list of the rest of them. Now consider set A_1 of t^* alternatives. Each member of A_1 appears as the top choice of $(n - n/\sqrt{t^*})/t^*$ agents. Whenever a member of A_1 is not the top choice of an agent, she appears in the bottom t^* places of his ranking. We create a symmetric setting among these alternatives. Now think of the utility profile $\vec{u}$ in which each agent has utility of 1 for his top choice and zero for the rest. We have $\mathsf{sw}(a^*, \vec{u}) = \frac{n}{\sqrt{t^*}}$, for $a \in A_1$, $\mathsf{sw}(a, \vec{u}) = \frac{n}{t^*}$ and the rest of the alternatives have zero social welfare. In addition for any positional scoring voting rule $f_{\vec{s}}^{\mathsf{rand}}$, we have

$$\Pr[f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma}) = a^*] = \frac{s_1 n/\sqrt{t^*} + s_m(n - n/\sqrt{t^*})}{n\|\vec{s}\|_1} \leqslant \frac{s_1 + s_m\sqrt{t^*}}{\|\vec{s}\|_1\sqrt{t^*}} \leqslant \frac{2s_1}{\|\vec{s}\|_1\sqrt{t^*}},$$

where the last inequality is due to the fact that $s_m\sqrt{t^*} \leqslant s_m\sqrt{m} \leqslant s_1$. In addition, for $a \in A_1$,

$$\Pr[f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma}) = a] = \frac{n\left(s_1 + \sum_{j=1}^{t} s_{m-j+1}\right)}{n\|\vec{s}\|_1|A_1|} \leqslant \frac{2s_1}{\|\vec{s}\|_1 t^*}.$$

That implies

$$\begin{aligned}
\mathbb{E}_{a \sim f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma})}[\mathsf{sw}(a, \vec{u})] &= \sum_{a \in A} \Pr[f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma}) = a] \cdot \mathsf{sw}(a, \vec{u}) \\
&= \Pr[f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma}) = a^*] \cdot \mathsf{sw}(a^*, \vec{u}) + \sum_{a \in A_1} \Pr[f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma}) = a] \cdot \mathsf{sw}(a, \vec{u}) \\
&\leqslant \frac{2s_1}{\|\vec{s}\|_1\sqrt{t^*}} \cdot \frac{n}{\sqrt{t^*}} + t^* \cdot \frac{2s_1}{\|\vec{s}\|_1 t^*} \cdot \frac{n}{t^*} \\
&\leqslant \frac{4n}{\|\vec{s}\|_1 t^*}.
\end{aligned}$$

$$\Rightarrow \mathsf{dist}(f_{\vec{s}}^{\mathsf{rand}}) \geqslant \mathsf{dist}(f_{\vec{s}}^{\mathsf{rand}}, \vec{\sigma}) \geqslant \frac{n/\sqrt{t^*}}{4ns_1/\|\vec{s}\|_1 t^*} \geqslant \frac{\|\vec{s}\|_1\sqrt{t^*}}{4s_1} = \Omega\left(\frac{\|\vec{s}\|_1}{s_1} \cdot \sqrt{t^*}\right). \qquad \square$$

Corollary 10. *The lower bounds on the distortion of common randomized positional scoring rules implied by Theorem 9 are shown in Table 2. These are tight for the randomized harmonic rule, the randomized k-approval rule with $k = \Omega(\sqrt{m})$, and the randomized veto rule.*

Next, we derive another general lower bound, which would help us establish the tightness for some more randomized positional scoring rules.

Theorem 11. *Let $\vec{s}$ be a scoring vector with at most k non-zero values, i.e. $\forall j \in [k+1, m], s_j = 0$. Then, $f_{\vec{s}}^{\mathsf{rand}}$ incurs a distortion of at least $\mathsf{dist}(f_{\vec{s}}^{\mathsf{rand}}) = \Omega\left(\frac{m\sqrt{m}}{k\sqrt{k}}\right)$.*

Proof. Assume that $m-1$ is divisible by 3. Let a^* be some alternative and partition the rest of the alternatives into three sets A_1, A_2, A_3, each of size $(m-1)/3$. In addition we partition the agents into two sets N_1 of size $n\sqrt{k/m}$ and $N_2 = N \setminus N_1$. Consider the preference profile where members of N_1 fill the top k positions of their rankings with members of A_1 and the rest of the agents have members of A_2 in the top k position of their rankings (the preference profile is symmetric among the members of each set, i.e., each member of A_i appears in the j-th position of $|N_i|/(|A_i| \cdot k)$ agents). Every agent has a^* in the $k+1$-th position, and all the members of A_3 after that (up to rank $k + 1 + m/3$). We do not care about the rest of the preference profile.

Table 2: Lower bounds on the distortion of common randomized positional scoring rules, achieved by Theorem 9.

Rule name	**Scoring vector** $\vec{s}$	$\|\vec{s}\|_1$	s_1	t^*	**Lower bound**
Harmonic	$(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m})$	H_m	1	$> \frac{m}{2}$	$\Omega\left(\sqrt{m}\log m\right)$
Veto	$(1, 1, \ldots, 1, 0)$	$m-1$	1	2	$\Omega\left(m\right)$
Half harmonic, half uniform	$(1+\frac{H_m}{m}, \frac{1}{2}+\frac{H_m}{m}, \ldots, \frac{H_m+1}{m})$	$2H_m$	$1+\frac{H_m}{m}$	$> \frac{m}{2H_m}$	$\Omega\left(\sqrt{m\log m}\right)$
k-approval	$(\underbrace{1, \ldots, 1}_{k \text{ ones}}, 0, \ldots, 0)$	k	1	$m-k+1$	$\Omega\left(k\sqrt{m-k+1}\right)$
Plurality	$(1, 0, \ldots, 0)$	1	1	m	$\Omega\left(\sqrt{m}\right)$
Borda	$(m-1, m-2, \ldots, 0)$	$\frac{m(m-1)}{2}$	$m-1$	$> \sqrt{m}$	$\Omega\left(m^{\frac{5}{4}}\right)$

Now consider the utility profile $\vec{u}$ where members of N_1 have a utility of $1/(k+1)$ for their top $k+1$ alternatives and members of N_2 have a utility of $1/(k+1+m/3)$ for each of their top $k+1+m/3$ alternatives. We have

$$\mathsf{sw}(a^*, \vec{u}) = |N_1|\frac{1}{k+1} + |N_2|\frac{1}{k+1+m/3} \geqslant \frac{n\sqrt{\frac{k}{m}}}{k+1} > \frac{n}{\sqrt{km}},$$

$$a_1 \in A_1 \implies \mathsf{sw}(a_1, \vec{u}) = \frac{|N_1|}{|A_1|}\frac{k}{k+1} = \frac{nk\sqrt{\frac{k}{m}}}{m(k+1)/3} < \frac{3n\sqrt{k}}{m\sqrt{m}},$$

$$a_2 \in A_2 \implies \mathsf{sw}(a_2, \vec{u}) = \frac{|N_2|}{|A_2|}\frac{k}{k+1+m/3} = \frac{nk\left(1-\sqrt{\frac{k}{m}}\right)}{m(k+1+m/3)/3} < \frac{9nk}{m^2}.$$

On the other hand, if we consider the probability given to each candidate we have:

$$\Pr_{a\sim f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma})}[a = a^*] = 0,$$

$$\Pr_{a\sim f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma})}[a \in A_1] = \frac{|N_1|}{n} = \sqrt{\frac{k}{m}},$$

$$\Pr_{a\sim f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma})}[a \in A_2] < 1,$$

which means

$$\mathbb{E}_{a\sim f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma})}[\mathsf{sw}(a, \vec{u})] \leqslant \sqrt{\frac{k}{m}} \cdot \frac{3n\sqrt{k}}{m\sqrt{m}} + 1 \cdot \frac{9nk}{m^2} = \frac{12nk}{m^2},$$

and that implies:

$$\mathsf{dist}(f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma}), \vec{u}) \geqslant \frac{\mathsf{sw}(a^*, \vec{u})}{\mathbb{E}_{a\sim f_{\vec{s}}^{\mathsf{rand}}(\vec{\sigma})}[\mathsf{sw}(a, \vec{u})]} \geqslant \frac{\frac{n}{\sqrt{km}}}{\frac{12nk}{m^2}} = \frac{m\sqrt{m}}{12k\sqrt{k}} \in \Omega\left(\frac{m\sqrt{m}}{k\sqrt{k}}\right). \qquad \square$$

The theorem above immediately shows that our analysis for randomized k-approvals rules with $k \in [m^{1/3}]$ in Lemma 12 is in fact tight.

Corollary 12. *The randomized k-approval rule with $k \in [m^{1/3}]$ incurs a distortion of $\Omega(\frac{m\sqrt{m}}{k\sqrt{k}})$.*

The results above do not match our upper bound of $O(m)$ for the randomized k-approval rule when $k \in [m^{1/3}, \sqrt{m}]$. In the following, we separately establish a matching lower bound of $\Omega(m)$ for this case as well.

Lemma 16. *The randomized k-approval rule with $k \in [m^{1/3}, \sqrt{m}]$ incurs a distortion of $\Omega(m)$.*

Proof. Suppose $m-1$ is divisible by 2. Let a^* be an alternative. Partition the rest of the alternatives into two subsets of A_1, A_2 each of size $(m-1)/2$. Then, construct a preference profile as follows. Let $\frac{nk}{m}$ agents N_1 have a^* as their top vote. Divide the other top $k-1$ top alternatives of N_1 and all the top k alternatives of $N \setminus N_1$ among A_1. This way,

$$\forall a \in A_1,\ \mathsf{score}(a) \leqslant \frac{nk}{|A_1|} \quad \Rightarrow \quad \Pr[a] \leqslant \frac{1}{|A_1|},$$

and $\Pr[a^*] = \frac{|N_1|}{nk}$. Divide the $k+1$ to $m/2$-th ranks of all voters among alternatives A_2 and fill the bottom of the ranking arbitrarily. This way, for all $a \in A_2$, $\Pr[a] = 0$ since they have a score of 0.

Furthermore, suppose N_1 have a utility of 1 for a^* and 0 for the rest, and $N \setminus N_1$ have a utility of $\frac{2}{m}$ for their top $\frac{m}{2}$ alternatives. This way,

$$\forall a \in A_1\ \mathsf{sw}(a) \leqslant \frac{nk}{|A_1|} \cdot \frac{2}{m}.$$

Then,

$$\begin{aligned}
\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) &\leqslant \Pr[a^*] \cdot \mathsf{sw}(a^*) + \sum_{a \in A_1} \Pr[a] \cdot \mathsf{sw}(a) + \sum_{a \in A_1} \Pr[a] \cdot \mathsf{sw}(a) \\
&\leqslant \frac{|N_1|}{nk} \cdot |N_1| + |A_1| \cdot \frac{1}{|A_1|} \cdot \frac{2nk}{|N_1| \cdot m}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}(\vec{\sigma})) &\geqslant \frac{|N_1|}{\frac{1}{nk} \cdot |N_1|^2 + \frac{2nk}{|N_1| \cdot m}} \\
&\geqslant \min\left\{ \frac{nk}{|N_1|}, \frac{|N_1| m}{2nk} \right\},
\end{aligned}$$

which for the choice of $|N_1| = \frac{nk}{m}$ gives a lower bound of $\Omega(m)$. □

Finally, it remains to show that our lower bounds on minimum welfare of these randomized positional scoring rules are also tight. This follows easily because our distortion upper bounds are essentially derived as a function of the minimum welfare bounds, and one can check that in each case an asymptotically better lower bound on minimum welfare would translate to an asymptotically better upper bound on distortion, which is not possible because we have already established tightness of our distortion bounds for the common randomized positional scoring rules.

Corollary 13. *The minimum welfare bounds presented in Table 1 for the randomized versions of plurality, Borda, harmonic, veto, and k-approval rules are asymptotically tight.*

A.4 Randomized Approval Mixture Rules

As a step towards analyzing the distortion of randomized positional scoring rules for any scoring vector, we present our distortion bounds for *approval mixture scores*, which are tight up to logarithmic factors.

Definition 14 (Approval Mixture Scores)**.** *For $k_1 < k_2 < \ldots < k_R \in [m]$, the* approval mixture score *denoted by $\{k_1, \ldots, k_R\}$*-mix-approval *is defined as*

$$\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}} = \frac{1}{R} \sum_{r=1}^{R} \frac{\vec{s}_{k_r\text{-approval}}}{\|\vec{s}_{k_r\text{-approval}}\|_1},$$

*that is the uniform mixture of the k_r-*approval *scores.*

This class of scores generalizes our results for randomized k-approvals (hence, plurality, veto).

A.4.1 Minimum Welfare Analysis

Lemma 17. *Fix any constant $\epsilon > 0$ and $k_1 < k_2 \ldots < k_R \in [(1-\epsilon)m]$. The minimum welfare of the randomized approval mixture rule with $\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}$ scoring vector is*

$$\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}}) = \Omega\left(\frac{n}{m}\cdot\frac{1}{R\log^2 m}\min\left\{\frac{1}{k_1},\sqrt{\frac{k_1}{k_2}},\sqrt{\frac{k_2}{k_3}},\cdots,\sqrt{\frac{k_{R-1}}{k_R}},\frac{k_R}{m}\right\}\right).$$

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. For conciseness, we use $\vec{s} = \vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}$.

Similar to the analysis in Lemma 11, to obtain a lower bound, by Lemma 6 we may assume without loss of generality that agents have dichotomous utilities, i.e. $u_i \in \{\mathbb{1}_{i,\ell}\}_{\ell\in[m]}$. To obtain a lower bound, we construct a new partial utility profile $\vec{u}'$ by rounding down the utilities to the nearest power of two, i.e. $u'_i \in \frac{1}{2}\cdot\{\mathbb{1}_{i,2^z}\}_{z\in[\lfloor\log_2 m\rfloor]}$. Now, we partition agents by their utility vectors to $\log m$ many groups, i.e. for $z \in [\lfloor\log_2 m\rfloor]$, define $N_z = \{i \mid u'_i = \mathbb{1}_{i,2^z}\}$. Furthermore, decompose $\vec{u}'$ to $\{\vec{u}'_z\}_{z\in[\log m]}$ where $\vec{u}'_z$ is $\vec{u}'$ except that agents in $N \setminus N_z$ have 0 utility for all alternatives. Now, we derive welfare lower bounds for each group by invoking Lemma 9 with $\vec{\sigma} \leftarrow \vec{\sigma}$, $\vec{u} \leftarrow \vec{u}'_z$, $\ell \leftarrow \frac{1}{2}\cdot\frac{1}{2^z}$, $T \leftarrow N_z$, $\tau \leftarrow 2^z$. We assign parameter k based on the different cases below.

Case $2^z \in [k_1]$. Invoking Lemma 9 with above and $k \leftarrow k_1$, we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_1\text{-approval}}},\vec{u}'_z) \geqslant \frac{1}{2}\cdot\frac{1}{2^z}\cdot\frac{|N_z|^2}{nm}\cdot\frac{(2^z)^2}{k_1} = \frac{|N_z|^2\cdot 2^z}{2nmk_1}.$$

Since $2^z \geqslant 1$,

$$\mathsf{sw}(f_{\vec{s}}) \geqslant \frac{1}{R}\cdot\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_1\text{-approval}}}) \geqslant \frac{|N_z|^2}{2R\cdot nm}\cdot\frac{1}{k_1}.$$

Case $2^z \in [k_r, k_{r+1}]$. Invoking Lemma 9 with above and $k \leftarrow k_r$, we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_r\text{-approval}}},\vec{u}'_z) \geqslant \frac{1}{2}\cdot\frac{1}{2^z}\cdot\frac{|N_z|^2}{nm}\cdot\frac{(\min\{2^z,k_r\})^2}{k_r} = \frac{|N_z|^2}{2nm}\cdot\frac{k_r}{2^z}.$$

By invoking Lemma 9 for $k \leftarrow k_{r+1}$, we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_{r+1}\text{-approval}}},\vec{u}'_z) \geqslant \frac{1}{2}\cdot\frac{1}{2^z}\cdot\frac{|N_z|^2}{nm}\cdot\frac{(\min\{2^z,k_{r+1}\})^2}{k_{r+1}} = \frac{|N_z|^2}{2nm}\cdot\frac{2^z}{k_{r+1}}.$$

Thus,

$$\begin{aligned}\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}},\vec{u}'_z) &\geqslant \frac{1}{R}\cdot\left(\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_r\text{-approval}}},\vec{u}'_z)+\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_{r+1}\text{-approval}}},\vec{u}'_z)\right)\\ &\geqslant \frac{1}{R}\cdot\frac{|N_z|^2}{2nm}\left(\frac{k_r}{2^z}+\frac{2^z}{k_{r+1}}\right) \overset{(1)}{\geqslant} \frac{|N_z|^2}{R\cdot nm}\cdot\sqrt{\frac{k_r}{k_{r+1}}},\end{aligned}$$

where (1) follows from the AM-GM inequality.

Case $2^z \in [k_R, m]$. Invoking Lemma 9 with above and $k \leftarrow k_R$, we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_1\text{-approval}}},\vec{u}'_z) \geqslant \frac{1}{2}\cdot\frac{1}{2^z}\cdot\frac{|N_z|^2}{nm}\cdot\frac{(\min\{2^z,k_R\})^2}{k_R} = \frac{|N_z|^2}{2nm}\cdot\frac{k_R}{2^z}.$$

Since $2^z \leqslant m$, we have

$$\mathsf{sw}(f_{\vec{s}}) \geqslant \frac{1}{R}\cdot\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_2\text{-approval}}}) \geqslant \frac{|N_z|^2}{2R\cdot nm}\cdot\frac{k_R}{m}.$$

For at least one value of $z \in [\log_2 m]$ we have $|N_z| \geqslant \frac{n}{\log m}$, by the pigeon-hole principle. Thus, we can take the minimum of the three cases above with $|N_z| \geqslant \frac{n}{\log m}$ to obtain a lower bound as follow

$$\begin{aligned}\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}) &\geqslant \frac{\left(\frac{n}{\log m}\right)^2}{2R\cdot nm}\min\left\{\frac{1}{k_1},\min_{r\in[R-1]}\left\{\sqrt{\frac{k_r}{k_{r+1}}}\right\},\frac{k_R}{m}\right\}\\ &= \frac{n}{2R\cdot m\log^2 m}\cdot\min\left\{\frac{1}{k_1},\sqrt{\frac{k_1}{k_2}},\sqrt{\frac{k_2}{k_3}},\ldots,\sqrt{\frac{k_{R-1}}{k_R}},\frac{k_R}{m}\right\}.\end{aligned}$$

□

A.4.2 Distortion Analysis

Theorem 15. *Fix any constant $\epsilon > 0$ and $k_1 < k_2 \ldots < k_R \in [(1-\epsilon)m]$. The distortion of the randomized approval mixture rule with $\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}$ scoring vector is*

$$O\left(R\log m \cdot \sqrt{\frac{n}{g}} \cdot \sqrt{\max\left\{k_1, \min\left(\frac{k_2}{k_1}, \frac{m}{(k_1)^2}\right), \ldots, \min\left(\frac{k_R}{k_{R-1}}, \frac{m}{(k_{R-1})^2}\right), \frac{m}{(k_R)^2}\right\}}\right),$$

where $g = \text{min-sw}(f^{\text{rand}}_{\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}})$.

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Let $a^* \in \arg\max_{a\in A} \mathsf{sw}(a, \vec{u})$ be an optimal alternative.

Partition the agents based on their score to a^*, i.e., for $r \in [1, R]$ define $N_r = \{i \in N \mid \mathsf{rank}_i(a^*) \in [k_{r-1}, k_r]\}$ ($k_0 = 1$) and let $N_{R+1} = \{i \in N \mid \mathsf{rank}_i(a^*) \in [k_R, m]\}$ be the agents who give score of 0 to a^*. Furthermore,

$$\mathsf{sw}(a^*) = \sum_{r\in[R]} \mathsf{sw}_{N_r}(a^*) \leqslant R \cdot \max_{r\in[R]} \mathsf{sw}_{N_r}(a^*). \tag{9}$$

Suppose the maximum above is achieved at N_{r^*}. Next, we show upper bounds on distortion based on the value of r^*, and report the maximum of all as a distortion upper bound. Before doing so, to obtain an upper bound on the distortion, we round down agents utility to the nearest power of two, ignore utilities less than $\frac{1}{m^2}$ (replace with 0). Call the new utility profile $\vec{u}'$. Then,

$$\begin{aligned} &\mathsf{sw}(a^*, \vec{u}) \leqslant 2 \cdot \mathsf{sw}(a^*, \vec{u}') + \frac{n}{m^2} \ \text{ and } \ \mathsf{sw}(f(\vec{\sigma}), \vec{u}) \leqslant \mathsf{sw}(f(\vec{\sigma}), \vec{u}') \\ \Rightarrow \quad &\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \frac{n/m^2}{\mathsf{sw}(f(\vec{\sigma}), \vec{u})} + 2 \cdot \mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant 4 + 2 \cdot \mathsf{dist}(f(\vec{\sigma}), \vec{u}') \end{aligned}$$

where the last inequality is due to Lemma 7.

Strategy 1 (Welfare above a^*). Now, we subdivide each group N_r based on their utility for a^* and derive welfare lower bounds by invoking Lemma 9. Fix $r \in [2, R+1]$ and for $z \in \left[\lfloor \log_2 \frac{1}{k_{r-1}+1} \rfloor, \lfloor \log_2 \frac{1}{m^2} \rfloor\right]$, let $N_{r,z} = \{i \in N_r \mid u'_i(a^*) = 2^z\}$. Since for agents $i \in N_r$, $\mathsf{rank}_i(a^*) \geqslant k_{r-1} + 1$, $u'_i(a^*) \leqslant \frac{1}{k+1}$ (otherwise, the unit-sum assumption is violated since her total utility for her top $k_{r-1} + 1$ alternatives exceeds one). Now, for each subgroup, we invoke Lemma 9 with $\vec{s} \leftarrow \vec{s}_{k_{(r-1)}\text{-approval}}$, $T \leftarrow N_{r,z}$, $\vec{\sigma} \leftarrow \vec{\sigma}$, $\tau \leftarrow 2^z$, $\ell \leftarrow k_{r-1}$, $k \leftarrow k_{r-1}$, and we have

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}'_z) \geqslant 2^z \cdot \frac{|T|^2}{nm} \cdot \frac{(k_{r-1})^2}{k_{r-1}} = 2^z \cdot \frac{|N_{r,z}|^2 \cdot k_{r-1}}{nm} \tag{10}$$

Strategy 2 (Probability of a^*). Following strategy 2, for $r \in [1, R]$, we have

$$\Pr[a^* \in f(\vec{\sigma})] \geqslant \frac{1}{R} \cdot \frac{|N_r|}{nk_r} \quad \Rightarrow \quad \mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant R \cdot \frac{nk_r}{|N_r|}. \tag{11}$$

Strategy 3 (Absolute Welfare Guarantee). Following strategy 3 and by Lemma 17, we have

$$\mathsf{sw}(f) \geqslant \text{min-sw}_{n,m}(f) = g(n, m) \quad \Rightarrow \quad \mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant \frac{\mathsf{sw}(a^*)}{g(n, m)}. \tag{12}$$

We are ready to show distortion upper bounds based on the choice of r^*.

Case $r^* = 1$. In this case, we only apply strategies 2 and 3 to N_{r^*} (not the subgroups). By Equation (9) we have $\mathsf{sw}(a, \vec{u}) \leqslant R \cdot \mathsf{sw}_{N_1}(a, \vec{u})$. Furthermore, $\mathsf{sw}_{N_1}(a^*) \leqslant |N_1|$. Then, by Equations (11) and (12),

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant \min\left\{\frac{R \cdot |N_1|}{g(n, m)}, R \cdot \frac{nk_1}{|N_1|}\right\} \leqslant R \cdot \sqrt{\frac{nk_1}{g(n, m)}}. \tag{13}$$

Case $r^* \in [2, R]$. We use all the three strategies here. By the pigeonhole principle, there exists a $z^* \in [\log m^2]$ such that $\mathsf{sw}_{N_{r^*}}(a^*) \leqslant 2\log m \cdot \mathsf{sw}_{N_{r^*,z^*}}(a^*) \leqslant 2\log m \cdot 2^z \cdot |N_{r^*,z^*}|$. Thus, by Equation (10) in strategy 1,

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant 2R\log m \cdot \frac{2^z \cdot |N_{r^*,z^*}|}{2^z \cdot \frac{|N_{r^*,z^*}|^2 \cdot k_{(r^*-1)}}{nm}} = 2R\log m \cdot \frac{nm}{|N_{r^*,z^*}| \cdot k_{(r^*-1)}}$$

Combined with Equation (11) in strategy 2 we get

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant 2R\log m \cdot \min\left\{\frac{nm}{|N_{r^*,z^*}| \cdot k_{(r^*-1)}}, \frac{nk_{r^*}}{|N_{r^*,z^*}|}\right\}$$

Putting together with Equation (12) in strategy 3 we get

$$\begin{aligned}\mathsf{dist}(f(\vec{\sigma}), \vec{u}') &\leqslant \min\left\{2R\log m \cdot \frac{2^{z^*} \cdot |N_{r^*,z^*}|}{g(n,m)}, 2R\log m \cdot \min\left\{\frac{nm}{|N_{r^*,z^*}| \cdot k_{(r^*-1)}}, \frac{nk_{r^*}}{|N_{r^*,z^*}|}\right\}\right\}. \\ &\leqslant 2R\log m \cdot \sqrt{\frac{n}{g(n,m)} \cdot \frac{1}{k_{r-1}} \cdot \min\left\{\frac{m}{k_{r^*-1}}, k_{r^*}\right\}}\end{aligned}$$

Case $r^* = R + 1$. This case follows exactly like the previous case except that we cannot apply strategy 2, since probability of selection and the score of a^* from ranks below k_R is 0. Thus,

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant 2R\log m \cdot \sqrt{\frac{n}{g(n,m)} \cdot \frac{1}{k_{r-1}} \cdot \frac{m}{k_R}}.$$

Now, we report an upper bound on distortion by taking the maximum of all cases. Hence,

$$\begin{aligned}\mathsf{dist}(f(\vec{\sigma}), \vec{u}) &\leqslant 4 + 2 \cdot \mathsf{dist}(f(\vec{\sigma}), \vec{u}') \\ &\leqslant 4 + 4 \cdot R\log m \cdot \sqrt{\frac{n}{g(n,m)}} \cdot \\ &\qquad \sqrt{\max\left\{k_1, \min\left\{\frac{k_2}{k_1}, \frac{m}{(k_2)^2}\right\}, \ldots \min\left\{\frac{k_R}{k_{R-1}}, \frac{m}{(k_R)^2}\right\}, \frac{k_R}{m}\right\}}. \qquad \square\end{aligned}$$

Deriving Distortion of the Randomized t-Truncated Harmonic Rules. Next, we apply the result above to a natural extension of the harmonic scores, which was recently used by Gkatzelis et al. [26]. Define the *t-truncated harmonic* scoring vector as follows

$$\vec{s}_{t\text{-harmonic}} = (1, 1/2, \ldots, 1/t, 0, \ldots, 0),$$

for which the first t scores is equal to the $\vec{s}_{\mathsf{harmonic}}$ and the rest is 0. For simplicity, suppose t is a power of two. By rounding the scores, we have

$$\vec{s}'_{t\text{-harmonic}} = (1, 1/2, 1/4, 1/4, 1/8, \ldots, 1/t, 0, \ldots, 0),$$

Now, take the $\{1, 2, \ldots, \log_2 t\}$-mix-approval scoring vector. Since $s_i = \frac{1}{\log t}\sum_{j=\lceil \log i\rceil}^{\log t} \frac{1}{2^j}$ It holds that

$$s'_i \leqslant s_i \leqslant \log t \cdot s'_i,$$

by Lemma 1 we can approximate $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{t\text{-harmonic}}})$ up to $O(\log m)$ factor by analyzing the randomized $\{1, 2, \ldots, \log_2 t\}$-mix-approval rule. To analyze this rule, we utilize the bounds from Theorem 15 and Lemma 17. Note that $\frac{k_r}{k_{r-1}} = 2$ for all $r \in [2, R]$, $k_1 = 1$, and $k_R = t$. Then,

$$\begin{aligned}\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\{1,2,\ldots,\log_2 t\}\text{-mix-approval}}) &= \Omega\left(\frac{n}{m} \cdot \frac{1}{\log t \cdot \log^2 m} \cdot \min\left\{1, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \ldots, \sqrt{\frac{1}{2}}, \frac{t}{m}\right\}\right) \\ &= \Omega\left(\frac{n}{m} \cdot \frac{1}{\log t \cdot \log^2 m} \cdot \frac{t}{m}\right) = g\end{aligned}$$

Furthermore, $\mathsf{dist}(f^{\mathsf{rand}}_{\{1,2,\ldots,\log_2 t\}\text{-mix-approval}})$ is at most

$$O\left(\log t \cdot \log m \cdot \sqrt{\frac{n}{g}} \cdot \sqrt{\max\left\{1, \min\left(2, m\right), \min\left(2, \frac{m}{2^2}\right), \min\left(2, \frac{m}{4^2}\right), \ldots, \min\left(2, \frac{m}{t^2}\right), \frac{m}{t^2}\right\}}\right)$$
$$= O\left(\log t \cdot \log m \cdot \sqrt{\frac{n}{g}} \cdot \max\left\{1, \sqrt{\frac{m}{t^2}}\right\}\right)$$
$$= O(\mathrm{polylog}(t,m) \cdot \sqrt{\frac{m^2}{t}} \cdot \max\left\{1, \sqrt{\frac{m}{t^2}}\right\})$$
$$= \begin{cases} O(\frac{m\sqrt{m}}{t\sqrt{t}} \cdot \mathrm{polylog}(t,m)) & \text{if } t \leqslant \sqrt{m}, \\ O(\frac{m}{\sqrt{t}} \cdot \mathrm{polylog}(t,m)) & \text{if } t \in [\sqrt{m}, m]. \end{cases}$$

B Random k-Committee Member Rules

B.1 Lower Bound

We begin by proving a lower bound on the distortion of *any* randomized rule with a support size of at most k, including, but not limited to, the ones that pick the winner uniformly at random from this support.

Theorem 16. *For $k \in [m/3]$, any randomized voting rule that has a support size of at most k incurs a distortion of at least $\Omega(m^2/k^2)$.*

Proof. Consider voting rule f, which has a support of size at most k. We create a preference profile $\vec{\sigma}$ and argue that the distortion of any such voting rule on this preference profile is at least $m^2/6k^2$. Partition the alternatives into two sets, A_1 with size $m-k$, and A_2 with size k. In $\vec{\sigma}$ each member of A_1 appears as the top choice of $n/(m-k)$ agents, and each member of A_2 appears as the second choice of n/k agents, in a way that each pair appears in the top-2 positions of $n/(k(m-k))$ agents. If the rule gives positive probability to all members of A_2, then consider the utility profile where each agent has utility 1 for her top choice and 0 for the rest. The social welfare of all alternatives with positive probability is zero, hence the distortion of this voting rule is unbounded. Now we can assume that there exists $a^* \in A_2$, that has probability zero in the output. Let $A_1^+ \subseteq A_1$ be the set of alternatives in A_1 that get positive probability in $f(\vec{\sigma})$, and $A_1^- = A_1 \setminus A_1^+$. Consider the utility profile agents that have members of A_1^+ as their top choice has the utility of $1/m$ for all the alternatives, agents that have members of A_1^- as their top choice and a^* as their second choice have utility $1/2$ for their top 2 alternatives and 0 for the rest, and other agents have utility 1 for their top choice and 0 for the rest. In this utility profile, we have

$$\mathsf{sw}(a^*) \geqslant \frac{1}{2} \cdot \frac{|A_1^-|n}{k(m-k)} \geqslant \frac{(m-2k)n}{2k(m-k)} \geqslant \frac{n}{4k},$$

and for $a' \in A_1^+ \cup A_2 \setminus \{a^*\}$:

$$\mathsf{sw}(a') \leqslant \frac{1}{m} \cdot \frac{|A_1^+|n}{m-k} \leqslant \frac{3k}{2m^2}.$$

That gives a bound on the social welfare of any alternative with positive probability and implies that $\mathbb{E}_{a\sim f(\vec{\sigma})}[\mathsf{sw}(c)] \leqslant \frac{3k}{2m^2}$ which means $\mathsf{dist}(f(\vec{\sigma}), \vec{\sigma}) \geqslant \frac{n/4k}{3k/2m^2} = \frac{m^2}{6k^2}$. □

We are now ready to prove the desired lower bound for random k-committee member rules.

Theorem 3. *For $k \in [m]$, the random k-committee member rule incurs $\Omega(\max(k, m^2/k^2))$ distortion. This lower bound is at least $\Omega(m^{2/3})$ for all k.*

Proof. First, consider the preference profile where all the agents have the same ranking with a^* as their top choice. Now consider the utility profile where each agent has utility 1 for his top choice and zero for the rest. In this utility profile, if the selected committee does not include a^* then the distortion is unbounded, and if a^* is part of the committee then the distortion is k. On the other hand, by we have the lower bound of $\Omega(m^2/k^2)$ which gives us the desired bound of $\Omega\left(\max\left(k, \frac{m^2}{k^2}\right)\right)$. □

ALGORITHM 1: Top-Biased Stable k-Committee

Input: Preference profile $\vec{\sigma}$, Committee size k
Output: Shortlisted Committee of size k
$A_{\mathsf{stable}} \leftarrow$ an approximately stable committee of size $k/3$
$A_{\mathsf{plu}} \leftarrow$ top $k/3$ alternatives with the highest plurality score
$N_1 \leftarrow$ voters whose top vote is among A_{plu}
$A_{\mathsf{greedy}} \leftarrow \emptyset$
for $i \in N_1$ **do** $\mathsf{hits}(i) = 1$
for $t = 1$ *to* $k/3$ **do**
  $\overline{A} \leftarrow A \setminus (A_{\mathsf{plu}} \cup A_{\mathsf{greedy}} \cup A_{\mathsf{stable}})$
  for $i \in N_1$ **do**
    $S_i \leftarrow$ top $m/(\mathsf{hits}(i) + 1)$ alternatives of i among $\overline{A}$
  $a^* \leftarrow \arg\max_{a \in \overline{A}} |\{i \in N_1 \mid a \in S_i\}|$
  $A_{\mathsf{greedy}} \leftarrow A_{\mathsf{greedy}} \cup \{a^*\}$
  for $i \in N_1$ *and* $a^* \in S_i$ **do**
    $\mathsf{hits}(i) \leftarrow \mathsf{hits}(i) + 1$
return $A_{\mathsf{plu}} \cup A_{\mathsf{greedy}} \cup A_{\mathsf{stable}}$

B.2 Upper Bound

Our algorithm uses the notion of *Approximately Stable Committees* introduced by Jiang et al. [40] and used by Ebadian et al. [19] in the design of the *Stable Committee Rule*.

Definition 17 (Approximately Stable Committee[40])**.** *A committee $X \subseteq A$ of size k is α-stable w.r.t. preference profile $\vec{\sigma}$ if for any candidate $a \in A$ we have $|i \in N \colon a \succ_i X| \leqslant \alpha \cdot \frac{n}{k}$, where $a \succ_i X$ means that voter i prefers a to every member of X.*

Theorem 18 ([40])**.** *Given any preference profile and $k \in [m]$, a 16-stable committee of size k exists.*

Algorithm 1 starts with selecting an approximately stable committee of size $k/3$ and $k/3$ alternatives with the highest plurality scores. Then, to gain more social welfare from the alternatives N_1 whom top alternative is selected by the algorithm, it proceeds as follows. Initialize $\mathsf{hits}(i)$ to one. The point of the hits count is that we can ensure a welfare of $\frac{\mathsf{hits}(i)}{m}$ by when the procedure ends. Initially, we can ensure a welfare of $\frac{1}{m}$ for all agents in N_1. Next, the sets S_i are set to be the top $m/2$ alternatives of agents in N_1. We will show that picking any of these alternatives will guarantee a welfare of at least $\frac{2}{m}$ for a user. The algorithm makes a greedy choice a^* that hits the highest number of agents and adds that alternative to the selected committee and increases the hit number of the agents hit by a^*. After the second hit, it updates the sets S_i to be the top $m/3$ remaining alternatives of the hit agent. Then $m/4$ after the thirds, and so forth. The algorithm selects the remaining $k/3$ alternatives of the committee as described.

B.2.1 Absolute Welfare Lower Bound

For the absolute welfare lower bound analysis, we will show that for each voter $i \in N_1$ we can guarantee a utility of at least $\mathsf{hits}(i)/m$. First, we present a helpful technical lemma.

Lemma 18. *Consider agent $i \in N$ with preference profile σ_i and utility function u_i, and a set $A' \subseteq A$ of alternatives. If A' includes the top choice of i, and for any $2 \leqslant \ell \leqslant t$ at least $t - \ell + 2$ members of A' appear in the top m/ℓ choices of i, then the total utility of i for members of A' is at least t/m, i.e. $\sum_{a \in A'} u_i(a) \geqslant t/m$.*

Proof. We can write u_i as a weighted sum of m dichotomous utility functions, i.e. $u_i = \sum_{j=1}^{m} \alpha_j \mathbb{1}_{i,j}$, where $\sum_{j=1}^{m} \alpha_j = 1$. For $j \in [m]$ we define

$$g(j) = \begin{cases} t - \lceil m/j \rceil + 2 & j \geqslant m/t, \\ 1 & o.w., \end{cases}$$

as a lower bound on the number of members of A' that appear in the top j positions of this agent's preference ranking. Each of these alternatives gets $1/j$ utility in $\mathbb{1}_{i,j}$. If we sum it up for all values of

j, we have

$$u_i(a) \geqslant \sum_{j \geqslant \mathsf{rank}_i(a)} \alpha_j/j \implies \sum_{a \in A'} u_i(a) \geqslant \sum_{j=1}^{m} \alpha_j \frac{g(j)}{j}.$$

For $j < m/t$, $g(j)/j \geqslant t/m$, and after that $\frac{t-\lceil m/j \rceil+2}{j}$ is increasing up to $j = {}^{2m}/_{t+2}$, decreasing afterwards, and is minimized at $j = m$. That means $g(j)/j$ is always greater than t/m, which implies

$$\sum_{a \in A'} u_i(a) \geqslant \sum_{j=1}^{m} \alpha_j \frac{g(j)}{j} \geqslant \frac{t}{m} \sum_{j=1}^{m} \alpha_j = \frac{t}{m}.$$ □

Theorem 19. *For $k \in [m]$, there exists a deterministic k-committee selection voting rule f_k^*, that for any pair of $\vec{\sigma}, \vec{u}$ guarantees $\sum_{a \in f_k^*(\vec{\sigma})} \mathsf{sw}(a, \vec{u}) \geqslant \frac{nk\sqrt{k}}{6m^2}$.*

Proof. Let $\widehat{A}$ be the set of alternatives returned by Algorithm 1. First, we show by Lemma 18 that for all agents $i \in N_1$, $\sum_{a \in \widehat{A}} u_i(a) \geqslant \mathsf{hits}(i)/m$. That is because A_{plu} includes the top choice of members of N_1. In addition by changing S_i in line 9 of the Algorithm 1, we make sure that the i-th hit in an agent's preference ranking is among his m/i top alternatives. This means that at the end if there are t hits in an agent's ranking, then for any $2 \leqslant \ell \leqslant t$ at least $t - \ell + 2$ members of $A_{\mathsf{greedy}} \cup A_{\mathsf{plu}}$ appear in that agent's top m/ℓ positions.

Consequently, we have

$$\sum_{a \in \widehat{A}} \mathsf{sw}(a) \geqslant \sum_{i \in N_1} \frac{\mathsf{hits}(i)}{m}. \tag{14}$$

Lower bound on the total number of hits. Next, we show $\sum_{i \in N_1} \mathsf{hits}(i) \geqslant |N_1| \cdot \sqrt{k}$. Let $a_1, a_2, \ldots, a_{k/3}$ be the sequence of alternatives greedily picked in the algorithm (lines 6-13). For $t \in [k/3]$, let $\mathsf{hits}^t(i)$ be the number of hits of voter i at the beginning of iteration t and h_t be the number of voters that were hit by a_t during the t-th iteration. Let $h_{\min} = \min_{t \in [k/3]} h_t$. Indeed we have

$$\sum_{i \in N_1} \mathsf{hits}(i) = \sum_{t \in [k/3]} h_t \geqslant h_{\min} \cdot k/3. \tag{15}$$

Moreover, since at iteration t we pick the candidate which hits the highest number of agents, h_t is at least as much as the average total $|S_i|$'s, i.e.

$$h_t \geqslant \frac{1}{m} \cdot \sum_{i \in N_1} |S_i| \geqslant \frac{1}{m} \cdot \sum_{i \in N_1} \frac{m}{\mathsf{hits}^t(i) + 1} \geqslant \frac{|N_1|^2}{\sum_{i \in N_1} \mathsf{hits}^t(i) + |N_1|},$$

where the last transition follows from the AM-HM inequality. Since the RHS is minimized at time $t = k/3$ (the sum in the denominator is non-decreasing), and by Equation (15), we have

$$\frac{3}{k} \cdot \sum_{i \in N_1} \mathsf{hits}(i) \geqslant h_{\min} \geqslant \frac{|N_1|^2}{\sum_{i \in N_1} \mathsf{hits}(i) + |N_1|}$$

Denote $\alpha = \sum_{i \in N_1} \mathsf{hits}(i)$. Then, we have $\alpha \cdot (\alpha + |N_1|) \geqslant \frac{k}{3}|N_1|^2$, which holds only if

$$\sum_{i \in N_1} \mathsf{hits}(i) = \alpha \geqslant \frac{1}{2} \cdot |N_1| \left(\sqrt{1 + 4k/3} - 1 \right) \geqslant |N_1| \sqrt{k/3}, \tag{16}$$

where the last transition holds for $k \geqslant 1$.

Deriving the bound. Moreover, $|N_1| \geqslant nk/3m$, since the $k/3$ alternatives with the highest number of top votes must have a total of at least $k/3m$ fraction of the n top votes. This observation combined with Equations (14) and (16) yields

$$\sum_{a \in \widehat{A}} \mathsf{sw}(a) \geqslant \frac{1}{3\sqrt{3}} \cdot \frac{nk\sqrt{k}}{m^2}.$$ □

B.2.2 Distortion Analysis

The goal of this section is to prove the following theorem.

Theorem 4. *There is a polynomial-time computable random k-committee member rule with distortion $O(\max\{k, m^2/(k\sqrt{k})\})$. This is minimized at $k = m^{4/5}$, where the bound becomes $O(m^{4/5})$.*

Proof. We select $\frac{k}{3}$ members of the committee using $f^*_{\frac{k}{3}}$ from Theorem 19 rule and for the rest, we select a 16-stable committee of size $\frac{k}{3}$.

For a preference profile $\vec{\sigma}$ and utility profile $\vec{u}$, let a^* be the optimal alternative, and $X \subseteq A$ be a committee of size k selected by our rule. By Theorem 18 we know that $|i \in N\colon a^* \succ_i X| \leqslant \frac{48n}{k}$. That means for at least $n - \frac{48n}{k}$ agents, at least one member of the selected committee gets as much utility as a^*. The maximum utility that a^* can get from the rest of the agents is $\frac{48n}{k}$. That means:

$$\mathsf{sw}(a^*, \vec{u}) \leqslant \sum_{a \in X} \mathsf{sw}(a, \vec{u}) + \frac{48n}{k}.$$

Let $\mathbb{U}[X]$ be a uniform distribution over the members of X, we have

$$\begin{aligned}
\mathsf{dist}(\mathbb{U}[X], \vec{u}) &= \frac{\mathsf{sw}(a^*, \vec{u})}{\frac{1}{|X|}\sum_{a \in X} \mathsf{sw}(a, \vec{u})} \\
&= \frac{2k \cdot \mathsf{sw}(a^*, \vec{u})}{2\sum_{a \in X} \mathsf{sw}(a, \vec{u})} \\
&\leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\max(0, \mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}) + \frac{n\frac{k}{3}\sqrt{\frac{k}{3}}}{6m^2}} && \text{(by Theorem 19)} \\
&\leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\max(0, \mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}) + \frac{nk\sqrt{k}}{32m^2}}.
\end{aligned}$$

Now we consider two cases, first if $\mathsf{sw}(a^*, \vec{u}) \geqslant \frac{96n}{k}$. In this case we have

$$\begin{aligned}
\mathsf{dist}(\mathbb{U}[X], \vec{u}) &\leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\max(0, \mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}) + \frac{nk\sqrt{k}}{32m^2}} \leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}} \\
&\leqslant \frac{4k\mathsf{sw}(a^*, \vec{u})}{\mathsf{sw}(a^*, \vec{u})} = 4k = O\left(k\right).
\end{aligned}$$

Then we consider the other case where $\mathsf{sw}(a^*, \vec{u}) < \frac{96n}{k}$. Here we have

$$\begin{aligned}
\mathsf{dist}(\mathbb{U}[X], \vec{u}) &\leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\max(0, \mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}) + \frac{nk\sqrt{k}}{32m^2}} \leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\frac{nk\sqrt{k}}{32m^2}} \\
&\leqslant \frac{2k\frac{96n}{k}}{\frac{nk\sqrt{k}}{32m^2}} = O\left(\frac{m^2}{k\sqrt{k}}\right).
\end{aligned}$$

These two bounds together give us the desired lower bound of $O(\max\{k, m^2/(k\sqrt{k})\})$. $\square$

C Additional Experimental Results

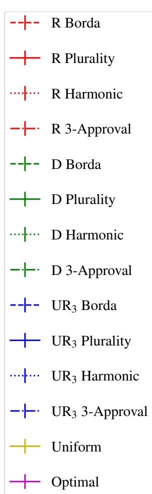
In this section, we provide some complementary results that give a better perspective on the empirical efficiency of the rules we study. The setup of these experiments is the same as described in Section 5.

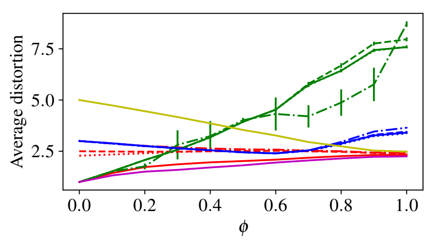
Results. Figures 2a to 2c show the average distortion of different rules for different values of ϕ with $m \in \{5, 25, 50\}$, respectively. For $m = 5$, we find that randomized plurality is better than every other rule, regardless of the value of ϕ. But as m grows larger, we can see that random committee member rules begin to perform almost as well as randomized positional scoring rules for $\phi \leqslant 0.5$. In addition, as we have seen in Section 5, when ϕ grows large (moving towards the impartial culture), randomized positional scoring rules outperform deterministic and random committee member rules as well as the uniform benchmark. Overall, these plots reinforce the claim we made in Section 5 that there almost always seems to be an explainable randomized rule that achieves better efficiency than deterministic rules.

Figure 2d is the counterpart of Figure 1d presented in Section 5, where instead of plotting the (average) best value of k against ϕ, we plot the (average) distortion achieved at the said best value of k against ϕ, for various random committee member rules. In a sense, this shows the limit of how well each random committee member rule can perform, when paired with its corresponding optimal k. As we can see, all the four rules we consider achieve approximately the same average distortion under these optimized conditions, which increases almost linearly with ϕ.

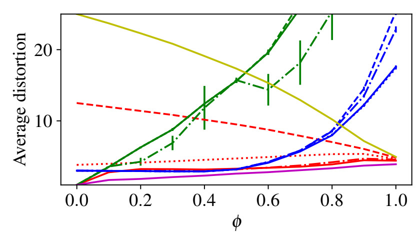
Figure 3 shows the value of k that yields the minimum distortion for different scoring vectors as a function of m, averaged over 100 runs and fixing the value of $\phi \in \{0.1, 0.5, 1\}$. Recall that theoretically, the value of k that optimizes the worst-case distortion is between $\Omega(m^{2/3})$ and $O(m^{4/5})$. For optimizing the average distortion, it turns out that the best k is close to 1 when ϕ is small, but as ϕ grows the best k gets closer to m. The growth as a function of m is highly sublinear for small ϕ, but almost linear for large ϕ.

We have also presented the (average) distortion achieved at these best values of k. Once again, we can notice that while the average distortion certainly grows with m, the different random committee member rules we consider perform about the same when paired with their optimal k. This average distortion is small when ϕ is small, and is approximately $\sqrt{m}$ for $\phi = 1$. An interesting observation is that for relatively small values of ϕ (i.e., $\phi = 0.1$ and $\phi = 0.5$), the distortion is very similar to the (average) best value of k. This indicates that for these values of ϕ, it may be the case that the optimal alternative is almost always part of the committee, for all four of the employed voting rules. Further, since k is small, not much preference information is observed. Hence, the worst case social welfare of any other alternatives included in the committee can be very small, bringing the distortion of choosing a random committee member close to the inverse of the probability of choosing the optimal alternative from the committee, which is k.

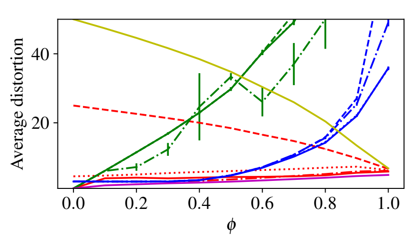




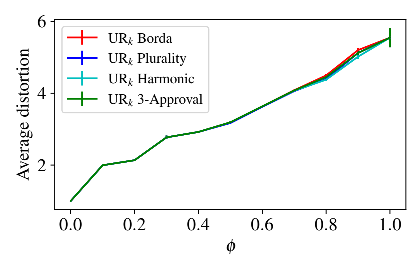
(a) Average distortion with $m = 5$.



(b) Average distortion with $m = 25$.

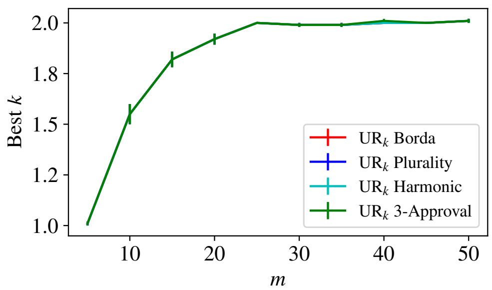


(c) Average distortion with $m = 50$.

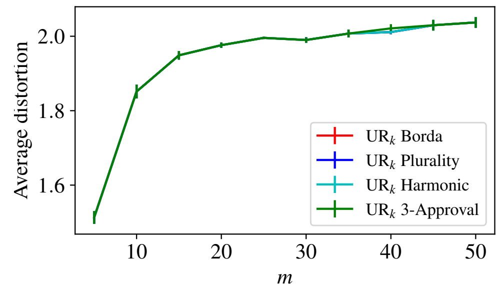


(d) Average distortion with best k based on ϕ, for $m = 25$.

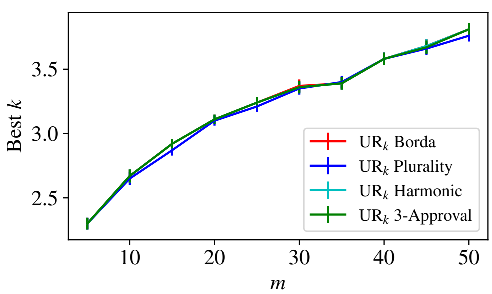
Figure 2: All figures show results averaged over 150 runs along with the standard error. Figures 2a to 2c share the legend on the left.



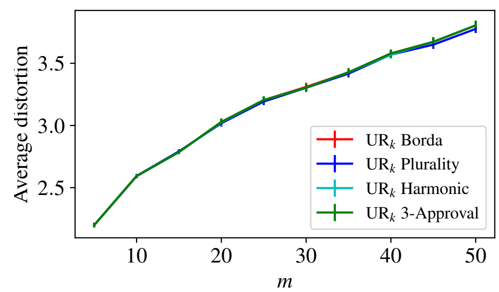
(a) Best value of k based on m, for $\phi = 0.1$.



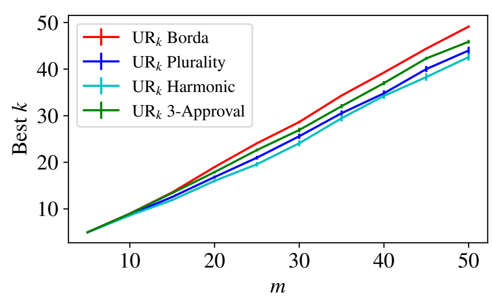
(b) Average distortion with best k based on m, for $\phi = 0.1$.



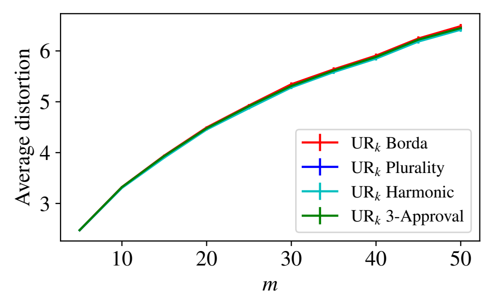
(c) Best value of k based on m, for $\phi = 0.5$.



(d) Average distortion with best k based on m, for $\phi = 0.5$.



(e) Best value of k based on m, for $\phi = 1$.



(f) Average distortion with best k based on m, for $\phi = 1$.

Figure 3: All figures show results averaged over 100 runs along with the standard error.