# Fair Division Under Boolean Valuations: Beyond Normalization

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**Abstract**

We study fair division of indivisible items when agents have arbitrary two-level preferences: the value of each agent for any set of items is Boolean, which need not be monotone or additive. Notably, we do not impose the standard assumption of *normalization*, i.e., different agents may value the empty set at different Boolean levels. Since the preferences are nonmonotone, *envy-freeness up to one item* (EF1) and *envy-freeness up to any item* (EFX) each admit several variants, depending on which items are tested for removal and whether they are removed from the envious agent's bundle or the envied agent's bundle. This paper investigates the existence of these variants of EF1 and EFX, on their own and together with economic efficiency, incentive compatibility, feasibility constraints, and lottery-based randomization. Our results highlight that the existence landscape depends crucially on the number of normalized agents, who value the empty bundle at the lower Boolean level.

Specifically, we uncover the conditions under which fine-grained variants of EF1, and the four variants of EFX, given by Bérczi et al. [8] exist. This includes the resolution of an open question posed by them: a variant of EFX, namely $\text{EFX}_0^+$, always exists for negative-Boolean valuations. For the setting where all agents value the empty bundle at zero, we show that fairness can be achieved along with other guarantees: there always exists a lottery over indivisible allocations that is ex-ante *envy-free* and ex-post *Pareto optimal* (PO) and EF1; additionally, there is a weakly group-strategyproof mechanism that outputs EF1 and PO allocations. Finally, we study a setting where agents have a matroid feasibility constraint on bundles (the same constraint applies to every agent); we characterize exactly when two agents are guaranteed a feasible EF1 allocation. Notably, the characterizing condition relates to the reconfiguration conjectures of White [19] and Gabow [13].

The authors used significant assistance from GPT-5.6-Sol for deriving theoretical results, verified any AI-generated proofs for correctness, and expanded on the exposition and simplified arguments, with the aid of GPT-5.6-Sol and Claude Opus 5.

# Contents

# 1 Introduction

Fair division studies how to allocate scarce resources — indivisible items, chores, time slots, inheritance, course seats — among agents with heterogeneous preferences, so that no agent feels that they were treated unfairly. Much of this theory, however, assumes that agents' preferences are *monotone* and *normalized*. A valuation is monotone nondecreasing if adding items always weakly increases utility, and monotone nonincreasing if adding items always weakly decreases utility. Normalization, which is an even more prevalent assumption, states that an agent's value for the empty set of items is zero. These assumptions primarily seem to serve the purpose of mathematical convenience. As one would expect, dropping monotonicity brings considerable nuance and complexity: well-known results for monotone nondecreasing valuations often fail outright or demand substantially new arguments for nonmonotone valuations [5; 12]. In a similar vein, normalization plays a central role in enabling fundamental fair-division results, such as Su's existence theorem for envy-free cake divisions [16].

However, in many real-world settings, agents' preferences obey neither monotonicity nor normalization. Having more of something is not always better: while taking two or three courses might benefit a student, taking a dozen courses can become burdensome and counterproductive. Similarly, advising a few students might make a researcher more productive overall, but can become a chore if the number of advisees increases beyond a threshold. Free disposal of items, which leads to monotone nondecreasing preferences,[1] is also unavailable in settings such as chore allocation. It is likewise natural for agents to assign a nonzero value to the empty bundle: a student might prefer taking no courses in a semester to taking an uninteresting course; if the items are perceived as chores, the empty bundle can be more valuable than nonempty bundles.

The existence of desirable allocations without any assumptions like normalization and monotonicity is therefore just as fundamental a question as its counterpart, yet this question remains underexplored. To isolate this question in its cleanest form, we study fair division of indivisible items under *Boolean valuations*: every agent $i$ has an arbitrary set valuation $v_i : 2^M \to \{0, 1\}$, with no monotonicity, additivity, or normalization assumption. This is the most general two-level preference over bundles one can model using set functions. At the ordinal level, it is simply a preference with two indifference classes, and the numerical values we assign are immaterial. Previous work [8; 11] has considered important subclasses of Boolean valuations in which every agent assigns the same value to the empty bundle, but not arbitrary mixed profiles containing agents of both types. Additionally, prior work focused primarily on fairness as the main objective; the interplay and tradeoffs between fairness and other desiderata remained underexplored.

The goal of this paper is to analyze the fairness guarantees that are possible for general Boolean valuations, with and without additional desiderata. Along with fairness, we consider economic efficiency, incentive compatibility for individuals and coalitions, structured feasibility constraints, and randomization over allocations.

As our results highlight, a single parameter turns out to govern much of the resulting landscape of attainable guarantees: the number of agents that place the empty bundle in the lower Boolean level. To intuitively understand the relevance of this parameter, consider an instance where only one agent values the empty bundle at zero, while the rest value the empty bundle at one. Here, giving all the items to the agent who values the empty bundle at zero results in an envy-free

[1] If items can be disposed of, then any new item that leads to a decrease in utility can be disposed at the agent's end, making preferences monotone nondecreasing.

allocation.

We will refer to the setting where $v_i(\varnothing) = 0$ for all agents, as the *normalized setting*. Otherwise, if $v_i(\varnothing) = 1$ for everyone, then a simple linear translation produces an equivalent valuation having range $\{-1, 0\}$ where the empty bundle is valued at zero; Bérczi et al. [8] refer to this as the *negative-Boolean* setting. The general setting we study allows different agents to place the empty bundle at possibly different Boolean levels.

### Our contributions

To the best of our knowledge, our paper is the first to consider fair division of indivisible items beyond the normalization assumption; dropping this assumption enables the model to capture subjectivity of agents' preferences regarding the value of the empty bundle. Our contributions span five related directions: we chart and strengthen the known existence landscape, resolve an open problem of Bérczi et al. [8], and establish positive and negative results on the interplay between fairness and other desiderata. In detail, our contributions are the following.

- *The EF1 landscape* (Sections 3 and 4). We begin by studying the variants of EF1 for nonmonotone valuations that are based on how the envy gets eliminated: removal from the envied bundle (EF1$^+$), from the envious agent's own bundle (EF1$_-$), or from either side (EF1$^+_-$); formal definitions appear in Section 2.2. Our starting point is a simple observation combining two known results: every Boolean profile, with no assumption on normalization, admits a complete EF1 allocation (Theorem 1). We then determine the tight conditions under which each stronger variant of EF1 exists, on its own and with Pareto optimality. Let $Z$ denote the set of agents that value the empty bundle at zero. Allocations satisfying EF1$^+$ are guaranteed to exist if and only if $Z \neq \varnothing$, i.e., as long as there is at least one agent that values the empty bundle at zero. On the other hand, allocations satisfying EF1$_-$ exist if and only if $|Z| \leqslant 1$. Conditions characterizing the existence of fair and efficient allocations are also uncovered: EF1$^+$ and PO allocations exist if and only if $Z \neq \emptyset$, and EF1$_-$ and PO allocations exist if and only if $|Z| = 1$. These characterizations appear in Theorem 2. We find it rather surprising that the characterizing conditions depend on a single parameter, $|Z|$, rather than on items being goods or chores.

- *The EFX landscape and an open question* (Section 5). We determine the universal-existence status of the four EFX variants (EFX$^+_-$, EFX$^0_-$, EFX$^+_0$, and EFX$^0_0$) defined by Bérczi et al. [8]; their definitions appear in Section 2.2, and Figure 1 maps their relationships with the three variants of EF1. The weakest notion, EFX$^+_-$, coincides with EF1 on the Boolean domain and hence always exists (Proposition 3); EFX$^0_-$ is guaranteed whenever $Z \neq \varnothing$, and this characterizing condition is tight because a profile with $Z = \varnothing$ may fail to admit EFX$^0_-$ allocations (Lemma 1 and Theorem 4); and the stronger notion, EFX$^0_0$, admits no universal guarantee [8]. For the fourth notion, EFX$^+_0$, we resolve an open problem, Question 16 posed by Bérczi et al. [8]: every instance with negative-Boolean valuations admits a complete EFX$^+_0$ allocation (Theorem 3), where the existence was previously known only for identical valuations [8, Theorem 9]. The proof, which is the technical highlight of the work, is based on a novel peel-and-match algorithm, in which candidate bundles are assigned to agents by a matching via Hall's theorem and, crucially, a failure of Hall's condition forms the basis of an induction argument rather than a dead end.

- *Fair, efficient, and weakly group-strategyproof mechanisms* (Section 6). We strengthen Bérczi et al.'s result on the existence of $\text{EFX}^0_-$ allocations [8, Theorem 7] for the normalized setting.[2] We give a simple weakly group-strategyproof mechanism that always outputs an $\text{EFX}^0_-$ and PO allocation. Given the reported valuations, the mechanism first maximizes social welfare (the sum of the values that the agents get in an allocation), subject to that, then minimizes the total number of items allocated to agents that receive a value of one, and finally uses a fixed tie-breaking rule.

- *Best-of-both-worlds guarantees* (Section 7). We show that if $Z \neq \emptyset$, i.e., at least one agent values the empty set at zero, then there exists a lottery over allocations that is ex-ante envy-free, and every allocation in its support is Pareto optimal, $\text{EF1}^+$, and $\text{EFX}^0_-$ (Theorem 6).[3]

- *Matroid constraints* (Section 8). We consider a fair division setting where allocated bundles must be independent sets of a matroid. We characterize, for two agents, exactly which matroids guarantee a feasible EF1 allocation for all Boolean profiles (Theorem 7). Notably, the characterizing condition (existence of a self-complementary component of the symmetric basis-pair exchange graph) connects the problem to the reconfiguration conjectures of White [19] and Gabow [13] in matroid theory. Known reconfiguration theorems yield feasible EF1 for strongly base-orderable, split, sparse paving, and regular matroids (Corollary 1). A partition matroid, by contrast, makes EF1 and Pareto optimality incompatible (Proposition 8).

## 2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of agents and $M = \{1, \ldots, m\}$ a finite set of indivisible items. Subsets $S \subseteq M$ are called *bundles*. A complete allocation $A = (A_1, \ldots, A_n)$ is a partition of $M$: $\bigcup_{i \in N} A_i = M$ and $A_i \cap A_j = \emptyset$ for all distinct $i, j \in N$. Each agent $i$ has an arbitrary valuation $v_i : 2^M \to \{0, 1\}$. Let $\mathcal{V}$ denote this domain, $\mathcal{V}_0 = \{v \in \mathcal{V} : v(\emptyset) = 0\}$, and $\mathcal{V}_1 = \{v \in \mathcal{V} : v(\emptyset) = 1\}$. For a given instance, we define $Z = \{i \in N : v_i(\emptyset) = 0\}$ and $E = N \setminus Z$ as the sets of agents that value the empty bundle at zero and at one, respectively.

An allocation $B$ *Pareto dominates* an allocation $A$ if $v_i(B_i) \geqslant v_i(A_i)$ for every agent $i$ and the inequality is strict for at least one agent. An allocation is *Pareto optimal* (PO) if no allocation Pareto dominates it.

Only the order of the two values matters for every fairness notion considered here.[4] Consequently, all results extend verbatim to arbitrary preferences with two indifference classes by mapping the lower class to zero and the higher class to one. The following perspective is often useful: for $v_i \in \mathcal{V}$, define the normalized translation $u_i(S) = v_i(S) - v_i(\emptyset)$. If $v_i \in \mathcal{V}_0$, then $u_i$ has range $\{0, 1\}$; if $v_i \in \mathcal{V}_1$, then $u_i$ has range $\{-1, 0\}$. Since the transformation is an agent-specific additive translation, it preserves every comparison between two bundles and hence the set of fair (EF1 or EFX) allocations remains unchanged.

[2] Recall, in the normalized setting every agent values the empty bundle at zero.

[3] This can be viewed as another extension of the existence of $\text{EFX}^0_-$ allocations [8, Theorem 7].

[4] The numerical values become relevant only for cardinal properties, such as proportionality or welfare approximation.

### 2.1 Three versions of EF1

We begin by defining variants of the envy-freeness up to one item (EF1) that we focus on. Note that, all these notions restore envy-freeness via a removal of an item from some agent's bundle.

**Definition 1** (EF1 variants)**.** We focus on the following three variants of EF1

- An allocation $A$ satisfies *goods-side envy-freeness up to one item* (EF$1^+$) if for all pairs of agents $i, j \in N$, either $v_i(A_i) \geqslant v_i(A_j)$, or there is an item $e \in A_j$ such that $v_i(A_i) \geqslant v_i(A_j \setminus \{e\})$.
- An allocation $A$ satisfies *chore-side envy-freeness up to one item* (EF$1_-$) if for all pairs of agents $i, j \in N$, either $v_i(A_i) \geqslant v_i(A_j)$, or there is an item $e \in A_i$ such that $v_i(A_i \setminus \{e\}) \geqslant v_i(A_j)$.
- An allocation $A$ is *two-sided envy-freeness up to one item* (EF$1^+_-$) if either removal is allowed, i.e., for every $i, j$, either $v_i(A_i) \geqslant v_i(A_j)$, or there is an $e \in A_i \cup A_j$ such that $v_i(A_i \setminus \{e\}) \geqslant v_i(A_j \setminus \{e\})$.

The superscript $+$ records removal from the envied bundle and the subscript $-$ removal from the envious agent's bundle, matching the EFX notation of Bérczi et al. Both EF$1^+$ and EF$1_-$ imply EF$1^+_-$. We use "EF1" without decoration for EF$1^+_-$, the removal-from-either-side notion standard for mixed manna.

For a bundle $S \subseteq M$, define its one-deletion lower star by

$$\underline{\partial}\, S = \{S\} \cup \{S \setminus \{e\} : e \in S\}.$$

For a Boolean valuation $v$, call $S$ *unsafe* if $v(T) = 0$ for every $T \in \underline{\partial}\, S$, and *robust* if $v(T) = 1$ for every $T \in \underline{\partial}\, S$. In particular, $\varnothing$ is unsafe for every $v \in \mathcal{V}_0$ and robust for every $v \in \mathcal{V}_1$.

**Proposition 1** (Violation characterization)**.** *For every Boolean profile, allocation $A$, and ordered pair $i, j$, the pair $i, j$ violates EF1 if and only if $A_i$ is unsafe for $v_i$ and $A_j$ is robust for $v_i$.*

*Proof.* Under Boolean valuations, envy means $v_i(A_i) = 0 < 1 = v_i(A_j)$. Removing an item from $A_i$ fails to eliminate this envy for every possible removal exactly when $v_i(A_i \setminus \{e\}) = 0$ for every $e \in A_i$. Removing an item from $A_j$ fails for every possible removal exactly when $v_i(A_j \setminus \{e\}) = 1$ for every $e \in A_j$. These conditions say precisely that $A_i$ is unsafe and $A_j$ is robust. $\square$

### 2.2 Four nonmonotone EFX relaxations

Bérczi et al. [8] define four variants of envy-freeness up to any item (EFX) based on which items must be removed to restore envy-freeness. The following definitions will be useful for defining these notions. For an agent $i$ and a bundle $X \subseteq M$ define,

$$S_i^+(X) = \{e \in X : v_i(X) - v_i(X \setminus \{e\}) > 0\},$$
$$S_i^-(X) = \{e \in X : v_i(X) - v_i(X \setminus \{e\}) < 0\},$$
$$S_i^0(X) = \{e \in X : v_i(X) - v_i(X \setminus \{e\}) = 0\}.$$

We will describe the notions of EFX defined in Bérczi et al. using Table 1, where each row specifies a variant of EFX. An allocation $A$ satisfies a particular variant of EFX (corresponding to a row) if for all pairs of agents $i, j \in N$, either we have $v_i(A_i) \geqslant v_i(A_j)$, or two conditions hold: $(a)$

we have $v_i(A_i \setminus \{e\}) \geqslant v_i(A_j \setminus \{e\})$ *for all* $e \in T_i \cup T_j$ where $T_i \subseteq A_i$ and $T_j \subseteq A_j$ are the items we test from the envious and the envied agent's bundles respectively, and $(b)$ $T_i \cup T_j$ is non-empty. Table 1 below lists the set of items $T_i$ and $T_j$ that are tested for each of the four notions of EFX.

| Notion | $T_j$ (from envied agent's bundle) | $T_i$ (from envious agent's bundle) |
|---|---|---|
| $\mathrm{EFX}_0^0$ | $S_i^+(A_j) \cup S_i^0(A_j)$ | $S_i^-(A_i) \cup S_i^0(A_i)$ |
| $\mathrm{EFX}_-^0$ | $S_i^+(A_j) \cup S_i^0(A_j)$ | $S_i^-(A_i)$ |
| $\mathrm{EFX}_0^+$ | $S_i^+(A_j)$ | $S_i^-(A_i) \cup S_i^0(A_i)$ |
| $\mathrm{EFX}_-^+$ | $S_i^+(A_j)$ | $S_i^-(A_i)$ |

Table 1: The four nonmonotone EFX relaxations of Bérczi et al. [8]: for an envious ordered pair $i, j$, the items tested in the envied bundle $A_j$ and in the envious agent's own bundle $A_i$.

Since the tested set of every notion must be nonempty and each tested item eliminates the envy from its side, each of the four notions implies $\mathrm{EF1}_-^+$. The four notions are moreover partially ordered, as in Figure 1: for an envious pair, a tested item that does not change the bundle's value can never eliminate the envy from its side, so a notion that tests an $S^0$-set forces that set to be empty, and the tested sets of the stronger and the weaker notion then coincide. Hence $\mathrm{EFX}_0^0$ implies both $\mathrm{EFX}_-^0$ and $\mathrm{EFX}_0^+$, each of which implies $\mathrm{EFX}_-^+$. Two further implications hold under normalization: on $\mathcal{V}_0^n$ no agent envies an empty bundle, so $\mathrm{EFX}_-^0$ implies $\mathrm{EF1}^+$; and on $\mathcal{V}_1^n$ an envious agent never holds an empty bundle, so $\mathrm{EFX}_0^+$ implies $\mathrm{EF1}_-$. Neither implication holds in the mixed domain: a one-item instance with an empty *envied* bundle (respectively, an empty *envious* bundle) separates the notions. Figure 1 collects the seven fairness notions of this section, together with the universal-existence status that this paper establishes for each of them on the Boolean domain.

## Related work

Having stated the relevant definitions in the section above, we now mention some relevant work from the literature.

**Existence of EF1.** The existence of EF1 allocations and its variants is known for various valuation classes; Table 2 lists the known results and Appendix A lists the definitions of the concerned valuation classes. Envy-cycle elimination gives goods-side EF1 for monotone nondecreasing valuations [15]. EF1 also exists for additive mixed manna [1], for doubly monotone valuations [10], and for two agents with arbitrary set valuations [8]. Most directly relevant here, Bérczi et al. [8] prove $\mathrm{EFX}_-^0$, and hence EF1, for normalized Boolean valuations [8, Theorem 7]. They also prove the stronger $\mathrm{EFX}_0^+$ guarantee for identical negative-Boolean valuations [8, Theorem 9] and pose an open problem of whether the existence result holds for non-identical negative-Boolean valuations [8, Question 16]. Bhaskar et al. subsequently prove EF1 for non-identical negative-Boolean valuations and an arbitrary number of agents [11, Theorem 3], but the EFX question remained unresolved; we resolve it here.

**Dichotomous valuations.** A large body of work concerns valuations whose *marginals* are binary, rather than whose values are binary. For binary additive valuations, where every item is worth

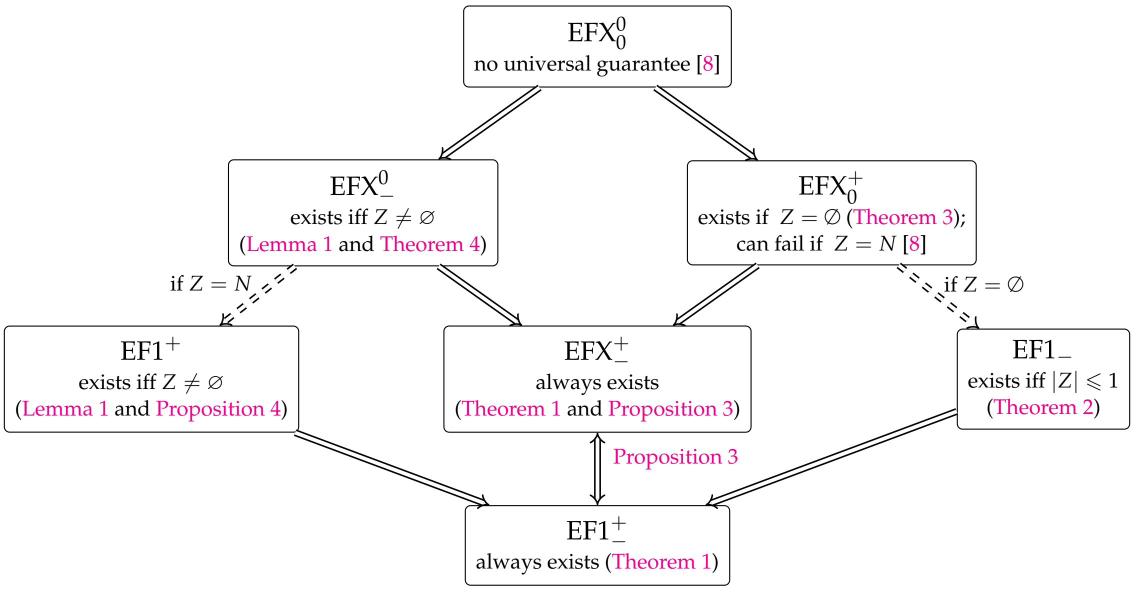


Figure 1: Implications among the three EF1 versions and the four EFX relaxations. Solid double arrows hold for arbitrary set valuations; the equivalence EFX$_-^+$ $\Leftrightarrow$ EF1$_-^+$ is specific to the Boolean domain (Proposition 3); and the dashed double arrows hold only on the indicated normalization endpoints, where the relevant empty bundle cannot occur. Each vertex records the existence status of its notion on arbitrary Boolean profiles, with the results establishing it.

zero or one and values add up, Halpern et al. [14] show that maximum Nash welfare with a fixed tie-breaking rule is simultaneously weakly group-strategyproof, EF1, and Pareto optimal. Babaioff et al. [2] extend the positive result to submodular valuations with binary marginals, also known as matroid-rank valuations: their deterministic truthful mechanism returns a Lorenz-dominating allocation, which is in particular EFX and of maximum Nash welfare, and randomizing over agent priorities makes it envy-free ex ante while retaining these properties ex post. Barman and Verma [3] prove that allocations that are both maximin fair and Pareto optimal exist for matroid-rank valuations, and subsequently [4] that the mechanism of Babaioff et al. is in fact weakly group-strategyproof for EF1. These papers call the property group strategyproofness and explicitly distinguish it from strong group strategyproofness. Viswanathan and Zick [18] give simpler and faster mechanisms that are strategyproof and output welfare-maximizing allocations for a wide range of fairness objectives [17]; see also Benabbou et al. [6] for fairness and efficiency in this class.

These classes and ours share the feature of being characterized by two levels, but they are not comparable. A valuation with binary marginals is monotone nondecreasing and its range is $\{0, 1, \ldots, m\}$, whereas a Boolean valuation has range $\{0, 1\}$ and is subject to no monotonicity, submodularity, or additivity requirement.

## 3 EF1 Existence for Unnormalized Boolean Valuations

We begin by settling the existence of EF1 allocations. The following theorem is a short synthesis of two established results. It nevertheless serves as the starting point for the paper: in Sections 4

| Domain | Guarantee | Source |
|---|---|---|
| Arbitrary valuations | $\text{EF1}^+_-$ | Open |
| Nonnegative (nonmonotone) valuations | $\text{EF1}^+_-$ | Open |
| Monotone nondecreasing set valuations | $\text{EF1}^+$ | Lipton et al. [15] |
| Additive mixed manna | $\text{EF1}^+_-$ | Aziz et al. [1] |
| Doubly monotone valuations | $\text{EF1}^+_-$ | Bhaskar et al. [10] |
| Two agents, arbitrary valuations | $\text{EF1}^+_-$ | Bérczi et al. [8] |
| Normalized Boolean valuations | $\text{EF1}^+_-$ | Bérczi et al. [8, Theorem 7] |
| Negative Boolean valuations | $\text{EF1}^+_-$ | Bhaskar et al. [11, Theorem 3] |

Table 2: EF1 and EFX existence results for different valuation classes.

and 5 we prove stronger, fine-grained existence guarantees, and Sections 6 and 7 combine fairness with economic efficiency and incentive compatibility.

**Theorem 1** (Universal Boolean EF1)**.** *Every instance with arbitrary valuations $v_i : 2^M \to \{0,1\}$ admits a complete EF1 allocation.*

*Proof.* Let $E = N \setminus Z$ be the agents that value the empty bundle at one. If $Z \neq \varnothing$, give the empty bundle to every agent in $E$, and allocate all items among the agents in $Z$ using the $\text{EFX}^0_-$ existence result of Bérczi et al. for heterogeneous normalized positive-Boolean valuations [8, Theorem 7]. This gives the required guarantee within $Z$. Every $i \in E$ receives value $v_i(\varnothing) = 1$, so $i$ envies no agent, and every $i \in Z$ assigns value $v_i(\varnothing) = 0$ to the bundle of each agent in $E$, which is no greater than $v_i(A_i)$. Hence every comparison involving $E$ is envy-free, and the full allocation is $\text{EFX}^0_-$, in particular EF1. If instead $Z = \varnothing$, then the transformation $u_i(S) = v_i(S) - 1$ results in a heterogeneous negative-Boolean instance, which admits a complete EF1 allocation by Bhaskar et al. [11, Theorem 3]; translation by a constant preserves every inequality in the definition of EF1, so the same allocation is EF1 for $v$. □

# 4 One-Sided EF1: A Fine-Grained Existence Analysis

Section 3 settles the existence of the two-sided notion, $\text{EF1}^+_-$. The one-sided notions behave very differently, with the size of the set $Z = \{i : v_i(\varnothing) = 0\}$ governing their existence exactly. We show that the guarantees for $\text{EF1}^+$ and $\text{EF1}_-$, both alone and in conjunction with Pareto optimality, are controlled by whether $|Z|$ is 0, 1, or at least 2. The resulting trichotomy is summarized in Theorem 2; the middle column $|Z| = 1$ is the unique regime in which a single allocation is simultaneously $\text{EF1}^+$, $\text{EF1}_-$, and Pareto optimal.

**Theorem 2.** *For Boolean profiles, universal existence of the one-sided notions is governed by* $|Z|$*:*

| | $\|Z\| = 0$ | $\|Z\| = 1$ | $\|Z\| \geqslant 2$ |
|---|---|---|---|
| $\mathrm{EF1}^{+}$ | ✗ | ✓ | ✓ |
| $\mathrm{EF1}_{-}$ | ✓ | ✓ | ✗ |
| $\mathrm{PO} \wedge \mathrm{EF1}^{+}$ | ✗ | ✓ | ✓ |
| $\mathrm{PO} \wedge \mathrm{EF1}_{-}$ | ✗ | ✓ | ✗ |

*where ✓ means every instance in that column admits such an allocation and ✗ means some instance admits none. Equivalently:* $\mathrm{EF1}^{+}$ *(with or without PO) is guaranteed iff* $Z \neq \varnothing$*;* $\mathrm{EF1}_{-}$ *is guaranteed iff* $|Z| \leqslant 1$*; and a Pareto-optimal* $\mathrm{EF1}_{-}$ *allocation is guaranteed iff* $|Z| = 1$*.*

This section is devoted to proving Theorem 2. To this end, we begin with a useful piece of notation: for every agent $i \in N$ and bundle $S \subseteq M$, let

$$v_i'(S) \;=\; \max_{T \in \underline{\partial} S} v_i(T),$$

so that $v_i'(S) = 0$ exactly when $S$ is unsafe for $v_i$, and $v_i'(S) = 1$ otherwise. For agents in $E = N \setminus Z$ (those with $v_i(\varnothing) = 1$) every bundle $S$ of size at most one has $v_i'(S) = 1$, since its lower star contains $\varnothing$. We begin by characterizing one-sided EF1 violations; the following proposition immediately follows from the relevant definitions on the Boolean domain.

**Proposition 2** (One-sided violations)**.** *Fix a Boolean profile, an allocation* $A$*, and an ordered pair* $i, j$ *with* $v_i(A_i) = 0$ *and* $v_i(A_j) = 1$*.*

(a) *The pair* $i, j$ *violates* $\mathrm{EF1}^{+}$ *iff* $A_j$ *is robust for* $v_i$*.*

(b) *The pair* $i, j$ *violates* $\mathrm{EF1}^{-}$ *iff* $A_i$ *is unsafe for* $v_i$*, i.e.* $v_i'(A_i) = 0$*.*

## 4.1 Two score-based allocation rules

We design two algorithms, one for each one-sided notion. Both are *score-based*: each of them maximizes a primary score over all complete allocations and then, among the allocations attaining that maximum, optimizes a secondary score. For any profile of Boolean valuation functions $p = (p_1, \ldots, p_n) \in (\mathcal{V}_0 \cup \mathcal{V}_1)^n$ and any allocation $A$, define the four scores

$$W_p(A) = \sum_i p_i(A_i), \quad C_p(A) = \sum_i p_i(A_i)\,|A_i|, \qquad W_p'(A) = \sum_i p_i'(A_i), \quad C_p'(A) = \sum_i p_i'(A_i)\,|A_i|,$$

where $p_i'(S) = \max_{T \in \underline{\partial} S} p_i(T)$ as above. When the profile under consideration $p$ is unambiguous, we drop the subscript and simply write $W$, $C$, $W'$, and $C'$. Let $\mathcal{A}$ denote the finite set of complete allocations.

**ALGORITHM 1:** GOODSOPT: the goods-side rule

**Input:** A Boolean profile $p$.
**Output:** A complete allocation.
$\mathcal{W} \leftarrow \arg\max_{A \in \mathcal{A}} W_p(A)$;
$\mathcal{O}^{+}(p) \leftarrow \arg\min_{A \in \mathcal{W}} C_p(A)$;
**return** *an arbitrary allocation in* $\mathcal{O}^{+}(p)$

**ALGORITHM 2:** CHORESOPT: the chores-side rule

**Input:** A Boolean profile $p$.
**Output:** A complete allocation.
$\mathcal{W} \leftarrow \arg\max_{A \in \mathcal{A}} W'_p(A)$;
$\mathcal{O}^-(p) \leftarrow \arg\max_{A \in \mathcal{W}} C'_p(A)$;
**return** *an arbitrary allocation in* $\mathcal{O}^-(p)$

Thus GOODSOPT (Algorithm 1) maximizes $W_p$ and, subject to that, *minimizes* $C_p$; CHORESOPT (Algorithm 2) maximizes $W'_p$ and, subject to that, *maximizes* $C'_p$. We write $A^+$ and $A^-$ for their outputs on $v$. All fairness and efficiency statements below hold for *every* allocation in the optimal sets $\mathcal{O}^+(v)$ and $\mathcal{O}^-(v)$, so the final arbitrary choice of the output allocation is immaterial.[5] Both algorithms optimize over all of $\mathcal{A}$: they serve the purpose of establishing existence and do not lead to polynomial-time algorithms.

Before analyzing these two rules, we show an equivalence that upgrades every EF1 guarantee to the weakest variant of EFX for free: on the Boolean domain, the weakest of the four EFX relaxations of Section 2.2 is not a strengthening of EF1 at all.

**Proposition 3.** *For Boolean valuations, an allocation is* $\mathrm{EFX}^+_-$ *if and only if it is EF1.*

*Proof.* Suppose first that $A$ is $\mathrm{EFX}^+_-$. For an envious pair $i, j$, the set $S_i^+(A_j) \cup S_i^-(A_i)$ is nonempty, and removal of any item in this set eliminates the envy from the corresponding side. Such an item is an EF1 witness.

Conversely, suppose $A$ is EF1 and $i$ envies $j$. Boolean valuations force $v_i(A_i) = 0$ and $v_i(A_j) = 1$. If an EF1 witness $e$ lies in $A_j$, then $v_i(A_j \setminus \{e\}) = 0$, so $e \in S_i^+(A_j)$. If it lies in $A_i$, then $v_i(A_i \setminus \{e\}) = 1$, so $e \in S_i^-(A_i)$. Thus the tested set for $\mathrm{EFX}^+_-$ is nonempty. Moreover, every item in $S_i^+(A_j)$ changes the Boolean value of $A_j$ from one to zero upon deletion, and every item in $S_i^-(A_i)$ changes the value of $A_i$ from zero to one upon deletion. Hence every tested item eliminates the envy, as required. □

In particular, Theorem 1 already yields a complete $\mathrm{EFX}^+_-$ allocation for every unnormalized Boolean instance, and each one-sided guarantee established below is automatically an $\mathrm{EFX}^+_-$ guarantee.

### 4.2 The goods side: $\mathrm{EF1}^+$

We now prove the fairness and efficiency guarantees of the allocations returned by Algorithm 1.

**Lemma 1.** *Every allocation $A$ returned by Algorithm 1 is Pareto optimal. If moreover $Z \neq \varnothing$, then $A$ is both* $\mathrm{EF1}^+$ *and* $\mathrm{EFX}^0_-$ *(which further imply EF1 and* $\mathrm{EFX}^+_-$*).*

*Proof.* Throughout, let $A$ be an allocation returned by Algorithm 1.

*Pareto optimality.* If $B$ Pareto dominates $A$, then $v_i(B_i) \geqslant v_i(A_i)$ for all $i$ with one strict inequality, so $W(B) > W(A)$, contradicting the definition of $\mathcal{O}^+(v)$.

*Fairness.* Suppose $Z \neq \varnothing$ and let $(a, j)$ be an envious ordered pair, so that

$$v_a(A_a) = 0 \qquad \text{and} \qquad v_a(A_j) = 1. \tag{1}$$

[5] Later, in Section 6, we show that GOODSOPT additionally underlies a weakly group-strategyproof and Pareto-optimal mechanism, and in Section 7 that the uniform lottery over $\mathcal{O}^+(v)$ is a best-of-both-worlds solution.

We establish two claims: (i) $v_a(A_j \setminus \{e\}) = 0$ for *every* $e \in A_j$, and (ii) $A_j \neq \varnothing$. Each claim, when violated, produces an allocation that Algorithm 1 would strictly prefer to $A$, that is, an allocation with larger $W$, or with the same $W$ and smaller $C$; the only exception is one configuration, which forces $Z = \varnothing$.

*Claim (i): deleting any item of the envied bundle eliminates the envy.* Suppose some $e \in A_j$ has $v_a(A_j \setminus \{e\}) = 1$. Reallocate $A'_a = A_j \setminus \{e\}$, $A'_j = A_a \cup \{e\}$, other bundles fixed. Then $v_a(A'_a) = 1$, so by Equation (1), $W(A') - W(A) = 1 + v_j(A_a \cup \{e\}) - v_j(A_j)$. If $v_j(A_j) = 0$, or if $v_j(A_j) = v_j(A_a \cup \{e\}) = 1$, the change in $W$ is positive, contradicting maximality of $W$ achieved by $A$. In the only other case, $v_j(A_j) = 1$ and $v_j(A_a \cup \{e\}) = 0$, welfare is unchanged while the joint contribution of $a, j$ to $C$ changes by $(|A_j| - 1) + 0 - (0 + |A_j|) = -1$, so $C$ strictly drops, again a contradiction.

*Claim (ii): the envied bundle is nonempty.* Suppose $A_j = \varnothing$. We first locate agent $a$ in $E$ with a nonempty bundle, then show that every other agent holds the empty bundle, and finally conclude that $Z = \varnothing$, contrary to the hypothesis.

From Equation (1), $v_a(\varnothing) = v_a(A_j) = 1$, so $a \in E$; and $v_a(A_a) = 0$ gives $A_a \neq \varnothing$.

Now let $k \neq a$. If $A_k \neq \varnothing$, dump $A_a$ onto $k$: $A'_a = \varnothing$, $A'_k = A_k \cup A_a$. As $a \in E$, $v_a(A'_a) = 1$, so $W(A') - W(A) = 1 + v_k(A_k \cup A_a) - v_k(A_k)$. This is positive unless $v_k(A_k) = 1$ and $v_k(A_k \cup A_a) = 0$, in which case welfare is unchanged and the joint contribution of $a, k$ to $C$ changes by $0 - |A_k| < 0$. Either way we contradict the optimality of $A$. Hence $A_k = \varnothing$, for all $k \neq a$.

Finally, fix any $k \neq a$, so that $A_k = \varnothing$ by the previous paragraph. If $v_k(\varnothing) = 0$, swap $a$ and $k$: $A'_a = \varnothing$, $A'_k = A_a$. Then $v_a(A'_a) = 1$ (up from 0) and $v_k(A_a) \geqslant 0 = v_k(\varnothing)$, so $W(A') - W(A) \geqslant 1 > 0$, a contradiction. Thus every $k \neq a$ lies in $E$, and with $a \in E$ this gives $Z = \varnothing$, contradicting the hypothesis. This proves Claim (ii).

By Claim (ii) the envied bundle $A_j$ is nonempty, and by Claim (i) any one of its items is an EF1$^+$ witness; hence $A$ is EF1$^+$. For EFX$^0_-$, the tested sets of the envious pair $(a, j)$ are $P_{aj} = S^+_a(A_j) \cup S^0_a(A_j)$ and $O_a = S^-_a(A_a)$. By Claim (i) every item of $A_j$ has deletion marginal $+1$ for $a$, so $P_{aj} = A_j$, which is nonempty by Claim (ii), and every $e \in P_{aj}$ eliminates the envy by Claim (i). Every $e \in O_a$ eliminates it automatically: $e \in S^-_a(A_a)$ means $v_a(A_a \setminus \{e\}) = 1 \geqslant v_a(A_j)$. Hence $A$ is EFX$^0_-$, and by Proposition 3 also EF1 and EFX$^+_-$. □

The role of the assumption $Z \neq \varnothing$ is to exclude an empty envied bundle, which the proof handles using one agent of $Z$. Note that Lemma 1 recovers the EFX$^0_-$ guarantee of Bérczi et al. [8, Theorem 7] for the normalized domain $\mathcal{V}^n_0$, and extends it to every profile with $Z \neq \varnothing$.

Next we show that the assumption $Z \neq \varnothing$ in Lemma 1 cannot be dropped: on the $Z = \varnothing$ domain, the EF1$^+$ guarantee fails not merely for the allocation returned by Algorithm 1 but for every allocation.

**Proposition 4.** *If $Z = \varnothing$, there are instances with no* EF1$^+$ *allocation.*

*Proof.* Take $n = 2$, $m = 1$, and $v_i(S) = \mathbf{1}[S = \varnothing]$ for both agents (so both lie in $E$). The single item goes to some agent $k$, giving $v_k(A_k) = 0$, while the other agent holds $\varnothing$, which is robust for $v_k$ and valued 1 by $v_k$. By Proposition 2(a), this ordered pair violates EF1$^+$, and this holds for every allocation. □

### 4.3 The chores side: EF1$_-$

Lemma 2 establishes the fairness guarantees of the allocations returned by Algorithm 2.

**Lemma 2.** *If $Z = \varnothing$, then every allocation $A$ returned by Algorithm 2 is* EF1$_-$*, and hence also* EFX$_-^+$.

*Proof.* Suppose the pair $(a, j)$ violates EF1$_-$ in allocation $A$. By Proposition 2(b), $v'_a(A_a) = 0$ and $v_a(A_j) = 1$. Since $a \in E$ and every bundle $S$ of size at most 1 has $v'_a(S) = 1$, the condition $v'_a(A_a) = 0$ forces $|A_a| \geqslant 2$.

Fix $c \in A_a$. Reallocate agent $a$'s bundle to $A_j \cup \{c\}$ and give the remainder to $j$:

$$A'_a = A_j \cup \{c\}, \qquad A'_j = A_a \setminus \{c\}, \qquad \text{other bundles fixed.}$$

Deleting $c$ from $A'_a$ leaves $A_j$, which $a$ values, so $v'_a(A'_a) = 1$; thus,

$$W'(A') - W'(A) = 1 + v'_j(A'_j) - v'_j(A_j).$$

If $v'_j(A_j) = 0$, or if $v'_j(A_j) = 1 = v'_j(A'_j)$, the increase in $W'$ is positive, contradicting maximality of $W'$. In the remaining case $v'_j(A_j) = 1$ and $v'_j(A'_j) = 0$, welfare $W'$ is unchanged and we compare $C'$, which $A$ maximizes. The joint contribution of $a, j$ to $C'$ changes by

$$\underbrace{|A_j| + 1}_{v'_a(A'_a)=1,\ |A'_a|=|A_j|+1} + \underbrace{0}_{v'_j(A'_j)=0} - \Big( \underbrace{0}_{v'_a(A_a)=0} + \underbrace{|A_j|}_{v'_j(A_j)=1} \Big) = 1,$$

so $C'$ strictly increases, again a contradiction. Hence no EF1$_-$ violation exists. Finally, EF1$_-$ implies EF1, which coincides with EFX$_-^+$ by Proposition 3. $\square$

*Remark* 1. Unlike its goods-side counterpart (Lemma 1), the guarantee of Lemma 2 does not upgrade to either of the two stronger EFX notions. Take $M = \{a, b, c\}$, let $v_2(S) = 1$ for all $S$, and let $v_1(S) = 1$ exactly for $S \in \{\varnothing, \{b\}, \{c\}\}$, so $Z = \varnothing$. The allocation $A = (\{a, b\}, \{c\})$ lies in $\mathcal{O}^-(v)$ (Algorithm 2): both agents have $v'_i(A_i) = 1$ (deleting $a$ from $\{a, b\}$ leaves $\{b\}$), so $W' = n$ and $C' = m$ are both maximal. Agent 1 envies agent 2, and the pair fails both notions. For EFX$_0^+$, the item $b$ has deletion marginal 0 in $A_1$, so it is tested, yet $v_1(\{a\}) = 0 < v_1(A_2)$. For EFX$_-^0$, the item $c$ has deletion marginal 0 in $A_2$, so it is tested, yet $v_1(A_1) = 0 < v_1(\varnothing)$. For EFX$_-^0$ this is no loss, since the notion can fail to exist when $Z = \varnothing$ (Theorem 4). But EFX$_0^+$ does exist when $Z = \varnothing$, as we establish in Theorem 3.

**Lemma 3.** *If $|Z| = 1$, there is a complete allocation that is Pareto optimal and envy-free; in particular it is both* EF1$^+$ *and* EF1$_-$.

*Proof.* Let $Z = \{z\}$ and give all items to $z$: $A_z = M$ and $A_i = \varnothing$ for $i \neq z$. Every $i \neq z$ lies in $E$ and holds $\varnothing$ at value 1, the maximum, so envies no one; and $z$ has $v_z(\varnothing) = 0 \leqslant v_z(M)$, so $z$ envies no one. Thus $A$ is envy-free. If $A$ is Pareto optimal, we are done. Otherwise some $B$ Pareto dominates $A$; every $i \neq z$ already attains value 1, so $v_i(B_i) = 1$ there, and the strict gain must come from $z$, giving $v_z(B_z) = 1$. Then all agents attain value 1 under $B$, so $B$ is envy-free, and it is Pareto optimal because no value can exceed 1. $\square$

*Remark* 2. When $|Z| = 1$, every allocation in $\mathcal{O}^+(v)$, the optimal set of Algorithm 1, is in fact envy-free. Let $A \in \mathcal{O}^+(v)$ and $Z = \{z\}$. Giving $M$ to $z$ and $\varnothing$ to every agent in $E$ yields welfare at least $n - 1$. If the maximum welfare is $n$, then $A$ gives every agent value one and is envy-free. Otherwise, the maximum welfare is $n - 1$, so $v_z(M) = 0$ and the benchmark allocation attains the pair $(n - 1, 0)$; the secondary objective then gives $C(A) = 0$, so every satisfied agent has an empty bundle. Because $v_z(\varnothing) = 0$, agent $z$ is the unique unsatisfied agent; hence every agent in $E$ is satisfied with the empty bundle and completeness forces $A_z = M$. Agent $z$ values her own bundle and every other bundle at zero, while every agent in $E$ already attains value one. Thus no agent envies another.

The next proposition identifies the domain where $\mathrm{EF1}_-$ fails to exist.

**Proposition 5.** *If $|Z| \geqslant 2$, there are instances with no* $\mathrm{EF1}_-$ *allocation.*

*Proof.* Let $z_1, z_2 \in Z$ be two agent that value the empty bundle at zero and take $m = 1$ such that $v_{z_1} = v_{z_2}$ equal to $\mathbf{1}[S = \{g\}]$, pick the valuation functions of the other agents arbitrarily. Under any allocation the single item is held by one agent, so at least one of $z_1, z_2$ holds $\varnothing$. This agent's envy towards the agent having the single item, cannot be eliminated via removal of an item from its own bundle (which is empty). Hence no allocation is $\mathrm{EF1}_-$. □

### 4.4 The trichotomy and its interaction with Pareto optimality

In this section, we prove Theorem 2. Before proving that, in the following proposition, we identify the condition where Pareto optimality is incompatible with two-sided EF1.

**Proposition 6.** *For every $n \geqslant 2$ and $m \geqslant 2$, there is a profile in $\mathcal{V}_1^n$ at which no allocation is both Pareto optimal and EF1.*

*Proof.* Let every agent have the identical valuation $v(S) = \mathbf{1}[S = \varnothing]$. We first characterize the Pareto-optimal allocations. If two agents hold nonempty bundles, moving one entire bundle to the other agent makes its former owner empty and therefore strictly better off, while the recipient remains at value zero and every other agent is unaffected. Thus a Pareto-optimal allocation has exactly one nonempty bundle, which by completeness equals $M$. Conversely, such an allocation is Pareto optimal: improving the unique holder requires making her bundle empty, which forces some currently empty agent to receive an item and fall from value one to zero.

Fix a Pareto-optimal allocation, and let $k$ hold $M$. Since $m \geqslant 2$, the bundle $M$ and every one-item deletion from it are nonempty, so $M$ is unsafe for $v_k$. Every other agent holds $\varnothing$, which is robust for $v_k$. Proposition 1 therefore gives an EF1 violation from $k$ to every other agent. □

Finally, we combine the lemmas and propositions presented above to prove Theorem 2.

### 4.5 Proof of Theorem 2

*Proof.* Each entry of the table is established by one of the preceding results.

$\mathrm{EF1}^+$ *and* $\mathrm{PO} \wedge \mathrm{EF1}^+$. If $Z \neq \varnothing$, then Lemma 1 provides an allocation that is simultaneously Pareto optimal and $\mathrm{EF1}^+$; this establishes both rows in the columns $|Z| = 1$ and $|Z| \geqslant 2$. If $Z = \varnothing$, then Proposition 4 exhibits an instance admitting no $\mathrm{EF1}^+$, and therefore no Pareto-optimal $\mathrm{EF1}^+$ allocation; this establishes both rows in the column $|Z| = 0$.

$\mathrm{EF1}_-$. If $|Z| = 0$, then Lemma 2 shows the existence of an $\mathrm{EF1}_-$ allocation; if $|Z| = 1$, then Lemma 3 provides an $\mathrm{EF1}_-$ (indeed envy-free) allocation. If $|Z| \geqslant 2$, then Proposition 5 exhibits an instance admitting no $\mathrm{EF1}_-$ allocation.

$\mathrm{PO} \wedge \mathrm{EF1}_-$. If $|Z| = 1$, then the allocation of Lemma 3 is Pareto optimal and envy-free, hence $\mathrm{EF1}_-$. If $|Z| = 0$, then Proposition 6 shows that on $\mathcal{V}_1^n$ no Pareto-optimal allocation is even $\mathrm{EF1}_-^+$; since $\mathrm{EF1}_-$ implies $\mathrm{EF1}_-^+$, no Pareto-optimal allocation is $\mathrm{EF1}_-$ either. If $|Z| \geqslant 2$, then $\mathrm{EF1}_-$ itself may not exist by Proposition 5. □

The middle column of the table in Theorem 2 is the sweet spot. When exactly one agent values the empty bundle at zero, the allocation of Lemma 3 is at once Pareto optimal, $\mathrm{EF1}^{+}$, and $\mathrm{EF1}_{-}$. The two one-sided guarantees therefore overlap only at $|Z| = 1$, while $\mathrm{EF1}^{+}_{-}$ (Section 3) is the fairness notion that exists across all three columns.

# 5 The EFX Landscape

| Notion | Status on arbitrary Boolean profiles | Source of the conclusion |
|---|---|---|
| $\mathrm{EFX}^{+}_{-}$ | Always exists | Theorem 1 and Proposition 3 |
| $\mathrm{EFX}^{0}_{-}$ | Exists whenever $Z \neq \emptyset$, but not universally | Lemma 1 and Theorem 4 |
| $\mathrm{EFX}^{+}_{0}$ | Exists if $Z = \emptyset$, but not universally | Theorem 3; counterexample by Bérczi et al. [8] |
| $\mathrm{EFX}^{0}_{0}$ | No universal guarantee | Counterexample by Bérczi et al. [8] |

Table 3: The EFX landscape on arbitrary Boolean profiles.

This section studies the existence of the four variants of EFX for Boolean valuations. In particular, we show that $\mathrm{EFX}^{+}_{0}$ exists for negative-Boolean valuations, and show that $\mathrm{EFX}^{0}_{-}$ exists if and only if $Z$ is nonempty. The weakest notion is already resolved: by Proposition 3, $\mathrm{EFX}^{+}_{-}$ coincides with EF1, so Theorem 1 yields a complete $\mathrm{EFX}^{+}_{-}$ allocation for every Boolean instance. Moreover, Lemma 1 already provides a complete $\mathrm{EFX}^{0}_{-}$ allocation whenever $Z \neq \emptyset$, via Algorithm 1.

## 5.1 $\mathrm{EFX}^{+}_{0}$ for negative-Boolean valuations

Bérczi et al. prove $\mathrm{EFX}^{+}_{0}$ existence for identical negative-Boolean valuations [8, Theorem 9] and explicitly pose as an open question whether the assumption of identical valuations can be dropped [8, Question 16]. The question remained open: Bhaskar et al. later obtained the weaker EF1 guarantee for the negative-Boolean valuations [11, Theorem 3], but not $\mathrm{EFX}^{+}_{0}$. Theorem 3 below answers the question affirmatively.

The algorithm of Bérczi et al. repairs a violation by transferring an item from the envious agent to the envied agent. Under nonidentical valuations, however, a repair for one agent can immediately create the reverse violation for the other. Our proof proceeds differently. We fix one reference agent and split the ground set into blocks that are minimal for that agent, in the sense that deleting any single item from a block raises its value; a block can then be given safely to any agent who values it at one, so we look for a matching that assigns every block to such an agent. If the matching exists, the resulting allocation is already $\mathrm{EFX}^{+}_{0}$. If it does not exist, then Hall's condition fails, and the set of blocks witnessing the failure has a smaller neighbourhood than itself; these blocks can be distributed among the reference agent and the agents in that neighbourhood, and every remaining agent values each of these blocks at zero. The remaining agents and items therefore form a strictly smaller instance whose solution can be combined with the partial allocation without creating any new violation, which is what drives the induction.

We keep the convention of the rest of the paper: the valuations are Boolean, and the domain of this subsection is $\mathcal{V}_1^n$, that is, $v_i(\emptyset) = 1$ for every agent, or equivalently $Z = \emptyset$. This is exactly the heterogeneous negative-Boolean domain of Bérczi et al. after the translation $u_i = v_i - 1$, which sends our values 1 and 0 to their values 0 and $-1$; as noted in Section 2, such a translation preserves every bundle comparison and every deletion marginal, and hence every notion of Section 2.2.

For $v_i \in \mathcal{V}_1$, call a bundle $S$ *deletion-secure for agent $i$* if either $v_i(S) = 1$, or $v_i(S) = 0$ and $v_i(S \setminus \{e\}) = 1$ for every $e \in S$. Thus every value-one bundle is deletion-secure, as is every zero-valued bundle whose every one-item deletion has value one; in particular, every robust bundle is deletion-secure. The following lemma characterizes $\mathrm{EFX}_0^+$ on this domain in terms of deletion-secure bundles.

**Lemma 4.** *An allocation $A$ of an instance with $v_i \in \mathcal{V}_1$ for every agent $i$ is $\mathrm{EFX}_0^+$ if and only if, for every agent $i$, either $A_i$ is deletion-secure for $i$, or $v_i(A_j) = 0$ for every $j \neq i$.*

*Proof.* Fix an agent $i$. If $v_i(A_i) = 1$, then $i$ envies no agent, and $A_i$ is deletion-secure for $i$; both sides of the stated equivalence hold for $i$. So assume $v_i(A_i) = 0$. If $v_i(A_j) = 0$ for every $j \neq i$, then again $i$ envies no agent and contributes no violation, matching the second alternative.

It remains to treat the case $v_i(A_i) = 0$ and $v_i(A_j) = 1$ for some $j \neq i$, that is, $i$ envies $j$. For every $e \in A_i$, the deletion marginal $v_i(A_i) - v_i(A_i \setminus \{e\})$ is 0 or $-1$, so $A_i = S_i^-(A_i) \cup S_i^0(A_i)$ and every item of $A_i$ is tested by $\mathrm{EFX}_0^+$; such an item eliminates the envy precisely when $v_i(A_i \setminus \{e\}) = 1$. Moreover, $A_i \neq \varnothing$, because $v_i(\varnothing) = 1 \neq 0 = v_i(A_i)$, so the tested set is nonempty. Finally, every $e \in S_i^+(A_j)$ satisfies $v_i(A_j \setminus \{e\}) = 0 = v_i(A_i)$, so every tested item of $A_j$ eliminates the envy automatically. Consequently the pair $(i, j)$ satisfies the requirement of $\mathrm{EFX}_0^+$ if and only if $v_i(A_i \setminus \{e\}) = 1$ for every $e \in A_i$, which, since $v_i(A_i) = 0$, is exactly the statement that $A_i$ is deletion-secure for $i$. As this condition does not depend on $j$, the agent $i$ contributes no violation if and only if $A_i$ is deletion-secure for $i$, or $i$ envies nobody, i.e. $v_i(A_j) = 0$ for every $j \neq i$. □

**Theorem 3.** *Every instance with $v_i \in \mathcal{V}_1$ for every agent $i$, that is, every instance with negative-Boolean valuations, admits a complete $\mathrm{EFX}_0^+$ allocation. Such an allocation can be computed in polynomial time in the value-oracle model.*[6]

*Proof.* We argue by induction on the number of agents $n$. If $n = 1$, give all items to the single agent; there is no second agent, so the allocation is trivially $\mathrm{EFX}_0^+$. Let $n \geqslant 2$ and fix an agent $d \in N$, called the *reference agent*.

We first cut the ground set into blocks that are minimal for the reference agent, in the sense that deleting any single item from a block raises its value for $d$. Formally, we construct pairwise disjoint nonempty blocks $C_1, \ldots, C_k \subseteq M$ and residual sets $M = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_k$ for a suitable $k$, using an iterative procedure defined below.

We start with $R_0 = M$. For $t = 1, 2, \ldots$: if $R_{t-1}$ is deletion-secure for $d$, stop and set $k = t - 1$. Otherwise $R_{t-1}$ is not deletion-secure for $d$, so $v_d(R_{t-1}) = 0$ and there is an item $e \in R_{t-1}$ with $v_d(R_{t-1} \setminus \{e\}) = 0$. Starting from $S = R_{t-1} \setminus \{e\}$, repeatedly replace $S$ by $S \setminus \{f\}$ for some $f \in S$ with $v_d(S \setminus \{f\}) = 0$, for as long as such an item $f$ exists. The loop ends at a set $C_t \subseteq R_{t-1} \setminus \{e\}$ with

$$v_d(C_t) = 0 \qquad \text{and} \qquad v_d(C_t \setminus \{f\}) = 1 \quad \text{for every } f \in C_t,$$

so that $C_t$ is deletion-secure for $d$; note that $C_t \neq \varnothing$, since $v_d(\varnothing) = 1 \neq 0$. Set $R_t = R_{t-1} \setminus C_t$ and continue.

[6] In the value-oracle model, we can query an agent $i$ with a bundle $S \subseteq M$ to get $v_i(S)$ in $O(1)$ time.

Each block is nonempty, so each iteration strictly shrinks the residue and the loop terminates after $k \leqslant m$ iterations. On termination, $M$ is the disjoint union of $C_1, \dots, C_k$ and $R_k$, we have $v_d(C_t) = 0$ for every $t \leqslant k$, and $R_k$ is deletion-secure for $d$.

We next record which agents can safely receive which blocks. Put $p = \min\{k, n-1\}$, and let $H$ be the bipartite graph with parts $\mathcal{L} = \{C_1, \dots, C_p\}$ and $N \setminus \{d\}$, where a block $C_t$ and an agent $i$ are adjacent exactly when $v_i(C_t) = 1$. Whether $H$ admits a matching that saturates $\mathcal{L}$ decides how we proceed, and we treat the two possibilities in turn.

Suppose first that some matching of $H$ saturates $\mathcal{L}$. Fix such a matching, give each block $C_t \in \mathcal{L}$ to the agent matched to it, give $R_p$ to $d$, and give the empty bundle to every remaining agent. The bundles $C_1, \dots, C_p, R_p$ are pairwise disjoint with union $M$, so the allocation is complete. Every agent matched to a block $C_t$ has $v_i(C_t) = 1$, and every agent receiving $\varnothing$ has $v_i(\varnothing) = 1$; in both cases the bundle is deletion-secure for its owner. For the reference agent there are two possibilities. If $p = k$, then $d$ receives $R_k$, which is deletion-secure for $d$ by the construction above. If $p = n - 1 < k$, then every agent other than $d$ is matched to a block, and $v_d(C_t) = 0$ for every block, so $v_d(A_j) = 0$ for every $j \neq d$ and the second alternative of Lemma 4 applies to $d$ even when $R_p$ is not deletion-secure. In either case Lemma 4 shows that the allocation is $\mathrm{EFX}_0^+$.

It remains to treat the case in which no matching of $H$ saturates $\mathcal{L}$. Here the blocks that witness the failure themselves split the instance into a part we can allocate at once and a strictly smaller part that we solve recursively. Fix a maximum matching $F$ of $H$ and a block $C^\star \in \mathcal{L}$ that $F$ leaves unmatched. Call a path of $H$ *$F$-alternating from $C^\star$* if it starts at $C^\star$, its first edge lies outside $F$, and its edges thereafter lie alternately in $F$ and outside $F$; the one-vertex path $C^\star$ is $F$-alternating from $C^\star$ as well. Let

$$
\begin{aligned}
X &= \{C \in \mathcal{L} : \text{some } F\text{-alternating path from } C^\star \text{ ends at } C\}, \\
Q &= \{i \in N \setminus \{d\} : \text{some } F\text{-alternating path from } C^\star \text{ ends at } i\},
\end{aligned}
$$

so that $C^\star \in X$. Every agent of $Q$ is matched by $F$: an $F$-alternating path from $C^\star$ ending at an unmatched agent would be an augmenting path, contradicting the maximality of $F$.

We claim that $F$ matches the agents of $Q$ with the blocks of $X \setminus \{C^\star\}$, one to one. Let $i \in Q$, and let $C$ be the block matched to $i$ by $F$, which exists by the previous paragraph. Appending the edge $\{i, C\} \in F$ to an $F$-alternating path from $C^\star$ ending at $i$ gives an $F$-alternating path from $C^\star$ ending at $C$, so $C \in X$; and $C \neq C^\star$ because $C^\star$ is unmatched. This assigns to every $i \in Q$ a block of $X \setminus \{C^\star\}$, and distinct agents receive distinct blocks because $F$ is a matching. Conversely, let $C \in X \setminus \{C^\star\}$ and consider an $F$-alternating path from $C^\star$ ending at $C$; since the path has at least one edge and alternates starting outside $F$, its last edge lies in $F$ and joins $C$ to an agent $i$, which lies in $Q$ because the path truncated before $C$ ends at $i$. Hence every block of $X \setminus \{C^\star\}$ is assigned to some agent of $Q$, and the assignment is a bijection; in particular $|Q| = |X| - 1$.

We claim next that the set of agents adjacent in $H$ to at least one block of $X$ is exactly $Q$. Every agent of $Q$ is adjacent to the block matched to it, which lies in $X$, so one inclusion is clear. For the other, let $C \in X$ and let $i$ be an agent adjacent to $C$. If $\{i, C\} \notin F$, take an $F$-alternating path from $C^\star$ ending at $C$; its last edge lies in $F$, or the path is the one-vertex path $C^\star$, so appending $\{C, i\}$ preserves the alternation and shows $i \in Q$. If $\{i, C\} \in F$, then $C \neq C^\star$, and by the bijection of the previous paragraph the block $C$ is matched to an agent of $Q$, which must be $i$ because $F$ is a matching. In both cases $i \in Q$.

Now allocate as follows. Give $C^\star$ to $d$, and give every block $C \in X \setminus \{C^\star\}$ to the agent of $Q$

matched to it by $F$; by the bijection above, each agent of $Q$ receives exactly one block. Write

$$P = Q \cup \{d\}, \qquad M_P = \bigcup_{C \in X} C$$

for the set of agents served so far and the set of items they receive. Every agent of $P$ holds a bundle that is deletion-secure for her: the bundle $C^\star$ is a peeled block and hence deletion-secure for $d$, and each $i \in Q$ receives a block $C$ with $v_i(C) = 1$.

Consider the residual instance with agent set $N' = N \setminus P$ and item set $M' = M \setminus M_P$, in which every agent $i \in N'$ keeps the restriction of $v_i$ to the subsets of $M'$; this restriction still assigns the value 1 to $\varnothing$, so the residual instance again has all valuations in $\mathcal{V}_1$. Since $|Q| + 1 = |X| \leqslant p \leqslant n - 1$, we have $|N'| = n - |X| \geqslant 1$, and $|N'| < n$ because $d \in P$. By the induction hypothesis, the residual instance admits a complete $\mathrm{EFX}_0^+$ allocation $(A_i)_{i \in N'}$ of $M'$. Let $A$ be the allocation of $M$ among $N$ that gives every agent of $P$ the bundle assigned above and every agent of $N'$ her bundle in the residual allocation. Since $M_P$ and $M'$ partition $M$, the allocation $A$ is complete.

We verify the condition of Lemma 4 for every agent of $N$, which proves that $A$ is $\mathrm{EFX}_0^+$. For $i \in P$, the bundle $A_i$ is deletion-secure for $i$, as observed above. Let now $i \in N'$. Every bundle held by an agent of $P$ is a block of $X$, and $i \notin Q$, so $i$ is adjacent in $H$ to no block of $X$ by the previous claim; that is,

$$v_i(A_j) = 0 \qquad \text{for every } j \in P. \tag{2}$$

If $A_i$ is deletion-secure for $i$, the first alternative of the characterization holds for $i$ and we are done. Otherwise $A_i$ is not deletion-secure for $i$; applying Lemma 4 to the residual allocation, which is $\mathrm{EFX}_0^+$ by the induction hypothesis, yields $v_i(A_j) = 0$ for every $j \in N' \setminus \{i\}$, and together with Equation (2) this gives $v_i(A_j) = 0$ for every $j \in N \setminus \{i\}$, which is the second alternative. Note that the correctness of the induction depends upon Equation (2): the bundles allocated to $P$ are worth zero to every agent of $N'$, so no agent of the residual instance envies an agent of $P$, and the recursive allocation can be computed without any reference to the bundles already assigned.

*Running time.* Deciding whether a set is deletion-secure for $d$ takes at most $m + 1$ value queries, and each block is produced by at most $m$ deletions, so producing all the blocks uses $O(m^3)$ value queries in total. The graph $H$ is built with at most $nm$ queries. A maximum matching of $H$, and the vertex sets $X$ and $Q$, are computable in polynomial time by standard algorithms for maximum matching. Every recursive call removes at least one agent, namely $d$, so there are at most $n$ calls. Hence the algorithm runs in polynomial time. □

### 5.2 Tightly characterizing the existence of $\mathrm{EFX}_-^0$

**Theorem 4.** *Every instance with Boolean valuations where $Z = \{i : v_i(\varnothing) = 0\} \neq \varnothing$ admits a complete $\mathrm{EFX}_-^0$ allocation. This is best possible as a guarantee at the level of valuation classes: there is an instance with $Z = \varnothing$ that admits no $\mathrm{EFX}_-^0$ allocation.*

*Proof.* Existence of $\mathrm{EFX}_-^0$ when $Z \neq \varnothing$ follows from Lemma 1. For the negative result, take two agents, three items, and the identical valuation $v(S) = 1$ if and only if $|S| \leqslant 1$. Every allocation has a bundle-size split of $(0, 3)$ or $(1, 2)$, up to exchanging the agents.

If the bundle sizes are $(0, 3)$, the three-item holder has value zero and envies the empty bundle, which has value one. The envied bundle has no item, and deletion of any item from the three-item bundle leaves value zero. Thus the tested set $S_i^+(A_j) \cup S_i^0(A_j) \cup S_i^-(A_i)$ is empty, violating $\mathrm{EFX}_-^0$.

If the bundle sizes are $(1, 2)$, the two-item holder has value zero and envies the singleton, which has value one. The singleton item has zero deletion marginal because deleting it leaves the empty bundle, also of value one. It is therefore tested under $\mathrm{EFX}^0_-$, but its deletion does not eliminate the envy. Hence neither bundle-size pattern is $\mathrm{EFX}^0_-$. $\square$

### 5.3 The remaining variants of EFX

No universal guarantee on arbitrary Boolean profiles is possible for either $\mathrm{EFX}^+_0$ or $\mathrm{EFX}^0_0$. Bérczi et al. give a two-agent, three-item identical positive-Boolean valuation $v(S) = \mathbf{1}[|S| \geqslant 2]$ with no $\mathrm{EFX}^+_0$ allocation, and a two-agent, two-item identical additive Boolean instance with singleton values one and zero with no $\mathrm{EFX}^0_0$ allocation [8]. The first counterexample complements Theorem 3: $\mathrm{EFX}^+_0$ can fail when $v_i(\varnothing) = 0$ for every agent, but always exists when $v_i(\varnothing) = 1$ for every agent. Table 3 summarizes the resulting landscape.

## 6 Combining Fairness and Efficiency with Incentives

In the setting where every agent values the empty bundle at zero, Bérczi et al. [8] showed that $\mathrm{EFX}^0_-$ allocations always exist. We strengthen this by designing a deterministic weakly group-strategyproof mechanism that always outputs PO and $\mathrm{EFX}^0_-$ allocations.

The mechanism of this section is simply Algorithm 1 with a tie-breaking rule. Recall from Algorithm 1 the scores $W_p(A) = \sum_{i \in N} p_i(A_i)$ and $C_p(A) = \sum_{i \in N} p_i(A_i)|A_i|$ of an allocation $A \in \mathcal{A}$ for any profile of Boolean valuation functions $p = (p_1, \ldots, p_n) \in (\mathcal{V}_0 \cup \mathcal{V}_1)^n$. These scores are, respectively, the number of agents reporting value one for their bundles and the total size of those agents' bundles. Fix a strict total order $\rhd$ on the set of all allocations, independent of all reports; this tie-breaking device, which plays no role in the existence results of Section 4, is what turns Algorithm 1 into a deterministic mechanism. The mechanism $\mathcal{M}_\rhd$, stated as Algorithm 3, runs GOODSOPT on the reported profile $p$ and resolves its final choice by $\rhd$: it maximizes $W_p(A)$, minimizes $C_p(A)$ among the maximizers, and returns the allocation that is maximal according to $\rhd$ among the allocations that remain. In particular, $\mathcal{M}_\rhd(p) \in \mathcal{O}^+(p)$.

**ALGORITHM 3:** The mechanism $\mathcal{M}_\rhd$

**Input:** A reported Boolean profile $p = (p_1, \ldots, p_n)$; a strict total order $\rhd$ on $\mathcal{A}$, fixed independently of $p$.
**Output:** A complete allocation.
$\mathcal{W} \leftarrow \arg\max_{A \in \mathcal{A}} W_p(A)$;
$\mathcal{O}^+(p) \leftarrow \arg\min_{A \in \mathcal{W}} C_p(A)$;
**return** *the $\rhd$-maximal allocation in $\mathcal{O}^+(p)$*

We show that this mechanism resists not only unilateral deviations by agents but also coordinated misreporting attempts by a group of agents.

**Definition 2** (Weak group strategyproofness)**.** A mechanism $f$ is *weakly group-strategyproof* if no coalition can misreport so that all of its members strictly benefit: for every nonempty coalition $K \subseteq N$, every true profile $v \in \mathcal{V}^n$, and every joint misreport $\widehat{v}_K = (\widehat{v}_i)_{i \in K}$, some member $i \in K$ has

$$v_i\big(f_i(\widehat{v}_K, v_{-K})\big) \ \leqslant \ v_i\big(f_i(v)\big).$$

Taking $|K| = 1$ recovers strategyproofness.

**Theorem 5.** *If the agents are allowed to report any valuations from $\mathcal{V}^n$, the mechanism $\mathcal{M}_\triangleright$ is weakly group-strategyproof and returns a Pareto-optimal allocation. If the reports lie in $\mathcal{V}_0^n$, its output additionally satisfies the fairness guarantees* EF1$^+$ *and* EFX$^0_-$, *and hence* EFX$^+_-$, *with respect to the reported profile.*

*Proof.* Fix a reported profile $p$, and write $A = \mathcal{M}_\triangleright(p)$.

*Pareto optimality and fairness.* Since $A \in \mathcal{O}^+(p)$ is returned by Algorithm 1, both claims are exactly Lemma 1 applied to the profile $p$: Pareto optimality holds unconditionally, and EF1$^+$ together with EFX$^0_-$ (hence EFX$^+_-$, by Proposition 3) holds since $Z(p) = \{i : p_i(\varnothing) = 0\} = N$.

*Weak group strategyproofness.* Fix a nonempty coalition $K \subseteq N$, a true profile $v \in \mathcal{V}^n$, and misreports $\widehat{v}_i \in \mathcal{V}$ for $i \in K$; let $\widehat{v}$ denote the profile that agrees with the misreports on $K$ and with $v$ elsewhere. Let $X = \mathcal{M}_\triangleright(v)$ and $Y = \mathcal{M}_\triangleright(\widehat{v})$. A profitable joint deviation would require every member to gain: $v_i(X_i) = 0$ and $v_i(Y_i) = 1$ for all $i \in K$; indeed this implies that $Y \neq X$. Put $w^* = W_v(X)$ and $c^* = C_v(X)$.

For any allocation $B$, changing only the coalition's reports changes its scores by

$$W_{\widehat{v}}(B) - W_v(B) = \sum_{i \in K} \big(\widehat{v}_i(B_i) - v_i(B_i)\big), \qquad C_{\widehat{v}}(B) - C_v(B) = \sum_{i \in K} \big(\widehat{v}_i(B_i) - v_i(B_i)\big)|B_i|;$$

in particular, if $\widehat{v}_i(B_i) = v_i(B_i)$ for every $i \in K$, then $B$ has the same pair of scores under both profiles. Since $v_i(Y_i) = 1$ for every member, each term $\widehat{v}_i(Y_i) - v_i(Y_i)$ is nonpositive, whence

$$W_{\widehat{v}}(Y) = W_v(Y) + \sum_{i \in K} \big(\widehat{v}_i(Y_i) - 1\big) \leqslant W_v(Y) \leqslant w^*,$$

with equality throughout only if $\widehat{v}_i(Y_i) = 1$ for every $i \in K$ and $W_v(Y) = w^*$.

If $\widehat{v}_i(X_i) = 1$ for some member $i$, then, since $v_j(X_j) = 0$ for every $j \in K$, the welfare of $X$ can only increase: $W_{\widehat{v}}(X) \geqslant w^* + 1$. By the preceding bound, $Y$ cannot be selected. Hence $\widehat{v}_i(X_i) = 0 = v_i(X_i)$ for every member, and $X$ has the same pair of scores $(w^*, c^*)$ under both profiles.

For $Y$ to defeat $X$ under $\widehat{v}$, it cannot have strictly larger welfare by the preceding bound. If it has welfare $w^*$, the bound is tight, which forces $\widehat{v}_i(Y_i) = v_i(Y_i) = 1$ for every $i \in K$ and $W_v(Y) = w^*$. The scores of $Y$ are therefore also unchanged by the deviation. It cannot have $C_{\widehat{v}}(Y) < c^*$, because then $C_v(Y) < c^*$ would contradict the optimality of $X$. If the two score pairs are equal, the fixed order $\triangleright$ selects $X$ at the truthful profile and therefore must rank $X$ above $Y$ at the deviating profile as well. Thus $Y$ could not have been chosen by $\mathcal{M}_\triangleright$ on the profile $\widehat{v}$, a contradiction. □

We call the property above weak group strategyproofness: it rules out joint deviations that strictly benefit every coalition member. Strong group strategyproofness instead rules out a joint deviation under which every coalition member is weakly better off and at least one is strictly better off. Prior work shows that strong group strategyproofness is incompatible with Pareto optimality even for unit-demand dichotomous valuations [2; 4], which form a subclass of $\mathcal{V}_0$. Hence no deterministic mechanism on the normalized Boolean domain can satisfy both strong group strategyproofness and Pareto optimality, even without imposing a fairness requirement.

**Example 1** (The score mechanism is not strongly group-strategyproof)**.** Let $N = \{1, 2, 3\}$, $M = \{a, b, c, d\}$, $B_2 = \{a, b\}$, and $B_3 = \{b, c\}$. Let $v_1(S) = 0$ for every $S \subseteq M$, and, for each $i \in \{2, 3\}$, let $v_i(S) = 1$ if and only if $S = B_i$. At the truthful profile, the maximum welfare is one because $B_2 \cap B_3 \neq \varnothing$, and every welfare maximizer has secondary score two. Let $X = \mathcal{M}_\triangleright(v)$, let $w \in \{2, 3\}$

be the unique agent with $X_w = B_w$, let $\ell$ be the other agent in $\{2,3\}$, and put $T = M \setminus B_\ell$. Consider the coalition $K = \{1, \ell\}$. Agent 1 reports $\widehat{v}_1(S) = 1$ if and only if $S = T$, agent $\ell$ reports $\widehat{v}_\ell(S) = 1$ if and only if $S \in \{B_\ell, M\}$, and agent $w$ reports truthfully. The bundles $T$ and $B_\ell$ partition $M$, while $B_w$ intersects both of them. Therefore the unique reported-welfare-two allocation gives $T$ to agent 1, $B_\ell$ to agent $\ell$, and $\varnothing$ to agent $w$; no allocation has reported welfare three. The mechanism selects this allocation regardless of its secondary score or the fixed tie-breaking rule. Under the true valuations, agent 1's value remains zero, while agent $\ell$'s value increases from zero to one. Thus every coalition member is weakly better off and one is strictly better off, so $\mathcal{M}_\triangleright$ is not strongly group-strategyproof.

## 7 Best of Both Worlds: Adding Ex-Ante Envy-Freeness

In this section we show that randomization strengthens fairness without sacrificing any of the guarantees obtained so far: whenever some agent values the empty bundle at zero, drawing an allocation uniformly at random from the optimal set of Algorithm 1 yields a lottery that is envy-free in expectation, while every allocation it may output is Pareto optimal, EF1$^+$, and EFX$^0_-$.

The condition $Z \neq \varnothing$ is tight as a class-wide guarantee. Indeed, for every $n, m \geqslant 2$, Proposition 6 gives a profile with $Z = \varnothing$ at which no allocation is simultaneously Pareto optimal and EF1; hence no lottery can have support consisting of Pareto-optimal EF1$^+$ allocations, regardless of its ex-ante properties. In that counterexample, the uniform lottery over the Pareto-optimal allocations is ex-ante envy-free by symmetry, so it is precisely the ex-post fairness requirement that fails.

Throughout this section we assume $Z \neq \varnothing$. For an allocation $A$ and an ordered pair $i \neq j$ define the *envy differential*

$$d_A(i,j) \;=\; v_i(A_i) - v_i(A_j) \;\in\; \{-1, 0, +1\},$$

and say that *i envies j* in $A$ if $d_A(i,j) = -1$.

**Definition 3** (ex-ante EF; BoBW lottery)**.** A lottery (probability distribution) $p$ over allocations is *ex-ante envy-free* if

$$\mathbb{E}_{A\sim p}\big[v_i(A_i)\big] \;\geqslant\; \mathbb{E}_{A\sim p}\big[v_i(A_j)\big] \qquad \text{for every ordered pair } i \neq j,$$

equivalently $\mathbb{E}_{A\sim p}[d_A(i,j)] \geqslant 0$ for all $i \neq j$. A *BoBW lottery* is a lottery that is ex-ante EF and whose support consists of allocations that are Pareto optimal and EF1$^+$.

The lottery is built directly on GoodsOpt (Algorithm 1): instead of selecting a single allocation from the optimal set $\mathcal{O}^+(v)$, we randomize uniformly over the whole set. Recall the scores of Section 4.1 on the true profile: $W(A) = \sum_{i\in N} v_i(A_i)$ is the utilitarian welfare, and $C(A) = \sum_{i\in N} v_i(A_i)|A_i|$ is the number of items the agents having utility 1 hold. Let $(W^*, C^*)$ be the optimal score pair, so that

$$\mathcal{O}^+(v) = \big\{A \in \mathcal{A} :\; W(A) = W^* \text{ and } C(A) = C^*\big\}$$

is the family of allocations computed by GoodsOpt (Algorithm 1). Indeed, $\mathcal{O}^+(v) \neq \varnothing$.

**Definition 4** (the uniform optimizer lottery)**.** The lottery $p^{\mathrm{unif}}$ is the uniform distribution over $\mathcal{O}^+(v)$: it draws each $A \in \mathcal{O}^+(v)$ with probability $1/|\mathcal{O}^+(v)|$.

**Theorem 6.** *Let $v_1, \ldots, v_n \colon 2^M \to \{0,1\}$ be arbitrary Boolean valuations with $Z = \{i : v_i(\varnothing) = 0\} \neq \varnothing$. Then $p^{\mathrm{unif}}$ (Definition 4) is a BoBW lottery: it is ex-ante envy-free, and every allocation in its support is Pareto optimal,* EF1$^+$*, and* EFX$^0_-$.

Since the support of the lottery satisfies $\mathrm{supp}(p^{\mathrm{unif}}) = \mathcal{O}^+(v)$, the ex-post guarantees of PO, EF1$^+$, and EFX$^0_-$ directly follow from Lemma 1. The remainder of this section proves the ex-ante part.

For $i \neq j$, the *swap operator* $T_{ij}$ maps an allocation $A$ to the allocation $A' = T_{ij}(A)$ defined by

$$A'_i = A_j, \qquad A'_j = A_i, \qquad A'_k = A_k \quad (k \notin \{i,j\}).$$

Note that $T_{ij}$ is an involution ($T_{ij}$ is its own inverse) on the set of all allocations, i.e., $T_{ij}(T_{ij}(A)) = A$.

**Lemma 5** (swap closure)**.** *Let $A \in \mathcal{O}^+(v)$ and suppose $d_A(i,j) = -1$. Then $A' = T_{ij}(A) \in \mathcal{O}^+(v)$ and $d_{A'}(i,j) = +1$.*

*Proof.* Since the valuations are Boolean, $d_A(i,j) = -1$ means $v_i(A_i) = 0, v_i(A_j) = 1$.

First, we will show that the swap preserves welfare and pins down $j$'s values. Only agents $i$ and $j$ change bundles, so

$$\begin{aligned} W(A') - W(A) &= \big(v_i(A'_i) + v_j(A'_j)\big) - \big(v_i(A_i) + v_j(A_j)\big) \\ &= \big(v_i(A_j) + v_j(A_i)\big) - \big(v_i(A_i) + v_j(A_j)\big) \\ &= 1 + v_j(A_i) - v_j(A_j). \end{aligned}$$

Since $A$ attains the maximum welfare $W^*$, we must have $W(A') - W(A) \leqslant 0$, i.e. $v_j(A_j) - v_j(A_i) \geqslant 1$. As both values lie in $\{0,1\}$, this forces $v_j(A_j) = 1, v_j(A_i) = 0$, and consequently $W(A') = W(A) = W^*$, so $A'$ attains the maximum welfare.

Now, we will prove that the swap preserves the secondary cost. By the above argument,

$$v_i(A'_i) = v_i(A_j) = 1, \qquad v_j(A'_j) = v_j(A_i) = 0,$$

so among $\{i,j\}$ exactly $j$ is satisfied in $A$ and exactly $i$ is satisfied in $A'$. Every agent $k \notin \{i,j\}$ keeps her bundle, hence her satisfaction status and her contribution to $C$ are unchanged. Therefore

$$C(A') - C(A) = |A'_i| - |A_j| = |A_j| - |A_j| = 0,$$

so $C(A') = C(A) = C^*$.

Hence, $A'$ attains the optimal score pair $(W^*, C^*)$, and therefore $A' \in \mathcal{O}^+(v)$. Finally,

$$d_{A'}(i,j) = v_i(A'_i) - v_i(A'_j) = v_i(A_j) - v_i(A_i) = 1 - 0 = +1. \qquad \square$$

**Proposition 7.** *The lottery $p^{\mathrm{unif}}$ is ex-ante envy-free.*

*Proof.* Fix an ordered pair $i \neq j$ and partition $\mathcal{O}^+(v)$ according to the value of $d_A(i,j)$:

$$S_d = \{A \in \mathcal{O}^+(v) : d_A(i,j) = d\} \qquad \text{for } d \in \{-1, 0, +1\}.$$

By Lemma 5, the swap operator $T_{ij}$ maps $S_{-1}$ into $S_{+1}$. Since $T_{ij}$ is an involution on the set of all allocations, it is injective, and therefore

$$|S_{-1}| \leqslant |S_{+1}|.$$

Consequently,

$$\mathbb{E}_{A\sim p^{\text{unif}}}\left[v_i(A_i) - v_i(A_j)\right] = \frac{1}{|\mathcal{O}^+(v)|}\sum_{A\in\mathcal{O}^+(v)} d_A(i,j) = \frac{|S_{+1}| - |S_{-1}|}{|\mathcal{O}^+(v)|} \geqslant 0.$$

As the ordered pair $(i,j)$ was arbitrary, $\mathbb{E}_{A\sim p^{\text{unif}}}[v_i(A_i)] \geqslant \mathbb{E}_{A\sim p^{\text{unif}}}[v_i(A_j)]$ for all $i \neq j$; that is, $p^{\text{unif}}$ is ex-ante envy-free. □

*Proof of Theorem 6.* The support of $p^{\text{unif}}$ is $\mathcal{O}^+(v)$, and by Lemma 1 every $A \in \mathcal{O}^+(v)$ is Pareto optimal, EF1$^+$, and EFX$^0_-$; this establishes the ex-post guarantee. Proposition 7 establishes ex-ante envy-freeness. Hence $p^{\text{unif}}$ is a BoBW lottery. □

# 8 Matroid Constraints

Until now every partition of the items has been admissible. We now require the bundles to be independent in a matroid, which models settings such as course allocation with per-category quotas. For two agents we determine exactly which matroids admit a feasible EF1 allocation for every Boolean profile; the answer concerns the way in which one pair of complementary independent sets can be turned into another by exchanging single items, and through this route the fairness question meets two conjectures of matroid theory that have been open for decades.

## 8.1 Matroid preliminaries

Let $\mathcal{K} = (M, \mathcal{I})$ be the matroid. That is, $M$ is the ground set of items and $\mathcal{I} \subseteq 2^M$ is the family of *independent sets*, satisfying three axioms, $(i)$ the non-emptiness axiom which requires $\varnothing \in \mathcal{I}$, $(ii)$ the hereditary axiom which states that if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$, and $(iii)$ the augmentation axiom, with the requirement that if $X, Y \in \mathcal{I}$ with $|X| < |Y|$, then some $e \in Y \setminus X$ satisfies $X \cup \{e\} \in \mathcal{I}$.

A *basis* is an inclusion-maximal independent set. The augmentation axiom implies that all bases have the same cardinality, called the *rank* of the matroid. In a uniform matroid of rank $r$, a set is independent exactly when it has cardinality at most $r$. A partition matroid partitions $M$ into blocks $P_q$ with capacities $r_q$, and a set $S$ is independent exactly when $|S \cap P_q| \leqslant r_q$ for every block $P_q$.

We assume that there is a common matroid $\mathcal{K} = (M, \mathcal{I})$ for every agent. An allocation $A$ is *feasible* if $A_i \in \mathcal{I}$ for every $i$; the allocation continues to satisfy $\cup_i A_i = M$ and $A_i \cap A_j = \varnothing$ for all distinct $i, j \in N$. We assume that at least one complete feasible allocation exists.

Fairness is defined exactly as in Section 2: a complete feasible allocation $A$ is EF1 if, for every ordered pair of agents $i, j$, either $v_i(A_i) \geqslant v_i(A_j)$, or some item $e \in A_i \cup A_j$ satisfies $v_i(A_i \setminus \{e\}) \geqslant v_i(A_j \setminus \{e\})$. The constraint thus restricts which allocations are admissible, not which comparisons the definition may make; and since the independent sets of a matroid are closed under taking subsets, the bundles obtained by deleting a single item in the fairness test are themselves independent.

In the following section, we uncover the necessary and sufficient condition that the feasibility matroid $\mathcal{K}$ must satisfy to enable the existence of feasible EF1 allocations.

### 8.2 A characterization for two agents

Let $\mathcal{F}_{\mathcal{K}}$ be the set of independent sets of $\mathcal{K}$ whose complement sets are also independent sets, formally,

$$\mathcal{F}_{\mathcal{K}} = \{B \subseteq M : B \in \mathcal{I} \text{ and } M \setminus B \in \mathcal{I}\}.$$

Thus every $B \in \mathcal{F}_{\mathcal{K}}$ represents the ordered feasible allocation $(B, M \setminus B)$. Define a graph $H_{\mathcal{K}}$ on vertex set $\mathcal{F}_{\mathcal{K}}$, joining distinct vertices $B, B'$ whenever

$$\underline{\partial}\, B \cap \underline{\partial}\, B' \neq \varnothing \qquad \text{or} \qquad \underline{\partial}(M \setminus B) \cap \underline{\partial}(M \setminus B') \neq \varnothing.$$

Define the *complementation map*

$$\tau : \mathcal{F}_{\mathcal{K}} \to \mathcal{F}_{\mathcal{K}}, \qquad \tau(B) = M \setminus B.$$

It is well defined, because the two conditions defining $\mathcal{F}_{\mathcal{K}}$ are exchanged when $B$ is replaced by $M \setminus B$, and it satisfies $\tau(\tau(B)) = B$. Furthermore, $\tau$ is an automorphism of $H_{\mathcal{K}}$: replacing $B$ by $M \setminus B$ and $B'$ by $M \setminus B'$ simultaneously exchanges the two conditions of the adjacency rule above, so $B$ and $B'$ are adjacent if and only if $\tau(B)$ and $\tau(B')$ are. Consequently $\tau$ maps every connected component of $H_{\mathcal{K}}$ onto a connected component. For a component $K \subseteq \mathcal{F}_{\mathcal{K}}$ of $H_{\mathcal{K}}$, we write $\tau(K) = \{\tau(B) : B \in K\}$, and we call $K$ *self-complementary* if $\tau(K) = K$.

**Theorem 7.** *An instance with two agents and a common matroid feasibility constraint represented by $\mathcal{K}$ admits a complete feasible EF1 allocation if and only if $H_{\mathcal{K}}$ has a connected component $K$ satisfying $\tau(K) = K$.*

*Proof.* For sufficiency, let $K$ be a self-complementary component and fix $B \in K$. Since $\tau(K) = K$, the vertex $\tau(B)$ also lies in $K$, so the connectedness of $K$ provides a path $B = X_0, X_1, \ldots, X_\ell = \tau(B)$ in $H_{\mathcal{K}}$. Call a vertex $X \in \mathcal{F}_{\mathcal{K}}$ *agreeable for $v_1$* if agent 1 is the source of no EF1 violation when she receives $X$ and the other agent receives $M \setminus X$, and also when she receives $M \setminus X$ and the other agent receives $X$. We claim that the path contains an agreeable vertex.

Recall from Section 2 that a bundle $S$ is *unsafe* for $v_1$ if $v_1(T) = 0$ for every $T \in \underline{\partial}\, S$, and *robust* for $v_1$ if $v_1(T) = 1$ for every $T \in \underline{\partial}\, S$. By Proposition 1, agent 1 holding $S$ while the other agent holds $M \setminus S$ is an EF1 violation exactly when $S$ is unsafe and $M \setminus S$ is robust for $v_1$.

Assume, for contradiction, that no vertex of the path is agreeable. Consider a vertex $X_q$ along the path. Exactly one of the two ways of assigning the pair $\{X_q, M \setminus X_q\}$ to the two agents makes agent 1 the source of a violation: at least one by the assumption, and not both, because a bundle cannot be at once unsafe and robust for $v_1$. We may therefore label $X_q$ by 0 if $X_q$ is unsafe and $M \setminus X_q$ is robust for $v_1$, and by 1 in the opposite case.

We now show that adjacent vertices of the path carry the same label, this will eventually lead to a contradiction. Indeed, suppose $X$ has label 0 and an adjacent vertex $Y$ has label 1. Then every set in $\underline{\partial}\, X$ has value zero for $v_1$ and every set in $\underline{\partial}\, Y$ has value one, so $\underline{\partial}\, X \cap \underline{\partial}\, Y = \varnothing$; likewise every set in $\underline{\partial}(M \setminus X)$ has value one and every set in $\underline{\partial}(M \setminus Y)$ has value zero, so $\underline{\partial}(M \setminus X) \cap \underline{\partial}(M \setminus Y) = \varnothing$. Both conditions in the definition of $H_{\mathcal{K}}$ fail, so $X$ and $Y$ are not adjacent. Hence, by contradiction, the label is a fixed constant along the path.

On the other hand, interchanging $X$ and $M \setminus X$ interchanges the two cases of the labelling, so $\tau(X)$ carries the label opposite to that of $X$; in particular the endpoints $X_0 = B$ and $X_\ell = \tau(B)$ carry opposite labels. This contradiction implies that there must be at least one agreeable vertex on the path.[7]

Fix an agreeable vertex $X$ of the path. Both $X$ and $M \setminus X$ are independent, because $X \in \mathcal{F}_{\mathcal{K}}$. Offer the two bundles $X$ and $M \setminus X$ to agent 2, let her keep one that she weakly prefers, and give the other to agent 1. Agent 2 does not envy agent 1, and agent 1 is the source of no violation, whichever of the two bundles she receives, because $X$ is agreeable for $v_1$. The result is a complete feasible EF1 allocation for the profile $(v_1, v_2)$.

For necessity, suppose no connected component maps onto itself under $\tau$. The components then occur in disjoint pairs $K, \tau(K)$. Choose a coloring $c$ of the components by zero and one such that $c(\tau(K)) = 1 - c(K)$. Define a Boolean valuation $v$ as follows. If $S \in \underline{\partial}\, B$ for some $B \in \mathcal{F}_{\mathcal{K}}$, set $v(S) = c(K_B)$, where $K_B$ is the component containing $B$; assign value zero to all remaining sets. This is well defined: if $\underline{\partial}\, B \cap \underline{\partial}\, B' \neq \varnothing$, then either $B = B'$ or $B, B'$ are adjacent, and hence belong to the same component.

Fix $B \in \mathcal{F}_{\mathcal{K}}$. Every set in $\underline{\partial}\, B$ has value $c(K_B)$, while every set in $\underline{\partial}(M \setminus B)$ has value $1 - c(K_B)$. If $c(K_B) = 0$, then $B$ is unsafe and $M \setminus B$ robust; if $c(K_B) = 1$, the reverse holds. Consequently, under the identical profile $(v, v)$, every feasible allocation has one agent holding an unsafe bundle and the other holding a robust bundle. Proposition 1 shows that no feasible allocation is EF1, completing the proof. □

### 8.3 Basis-pair reconfiguration and positive classes

We now specialize Theorem 7 to the case in which the ground set is exactly the union of two bases. We show that the graph $H_{\mathcal{K}}$ then records single exchanges of items between a pair of complementary bases, and we explain how the self-complementary-component condition relates to two long-standing conjectures on such exchanges; this yields feasible EF1 allocations for several classes of matroids.

Suppose that $\mathcal{K}$ has rank $r$, that $|M| = 2r$, and that $M$ can be partitioned into two independent sets; we refer to this as the *base-partition case*. Each part of such a partition has size at most $r$ and the two sizes add up to $2r$, so both parts have size exactly $r$ and are therefore bases. Hence $\mathcal{F}_{\mathcal{K}}$ consists precisely of those bases whose complement is again a basis, and a vertex $B$ of $H_{\mathcal{K}}$ encodes the ordered pair of complementary bases $(B, M \setminus B)$, with $\tau$ interchanging the two entries of the pair.

Recall that a *symmetric exchange* [19] is a procedure that transforms a pair of disjoint bases $(B, M \setminus B)$ into $(B - x + y,\ (M \setminus B) - y + x)$, where $x \in B$ and $y \in M \setminus B$ are such that both new sets are again bases.[8] In the base-partition case, adjacency of two vertices in $H_{\mathcal{K}}$ exactly corresponds to the existence of a symmetric exchange between the two pairs corresponding to each vertex. Indeed, if $B' = B - x + y$ arises from $B$ by a symmetric exchange, then $B \setminus \{x\} = B' \setminus \{y\}$ lies in $\underline{\partial}\, B \cap \underline{\partial}\, B'$, so $B$ and $B'$ are adjacent. Conversely, let $B \neq B'$ be adjacent, and suppose first that $\underline{\partial}\, B \cap \underline{\partial}\, B' \neq \varnothing$. A common member of the two lower stars has size $r$ or $r - 1$; it cannot have size $r$, since that would force $B = B'$, so it is of the form $B \setminus \{x\} = B' \setminus \{y\}$ with $x \in B$ and $y \in B'$, whence $B' = B - x + y$. Both complements are bases because $B' \in \mathcal{F}_{\mathcal{K}}$, so this is a symmetric

[7] The logical structure of this argument deserves a remark, since the labelling above is available only under the supposition that no vertex of the path is agreeable. What the argument establishes is that *at least one* vertex of the path is agreeable; it does not exhibit that vertex, and in particular it does not show that the chosen starting vertex $B$ is agreeable. If $B$ happens to be agreeable, then the path is not needed at all.

[8] For simplicity we denote $X \cup \{g\}$ and $X \setminus \{g\}$ using $X + g$ and $X - g$ respectively.

exchange. If instead $\underline{\partial}(M \setminus B) \cap \underline{\partial}(M \setminus B') \neq \varnothing$, the same argument applied to the complements gives $M \setminus B' = (M \setminus B) - y + x$, which is again a symmetric exchange. Thus $H_{\mathcal{K}}$ is the graph whose vertices are the pairs of complementary bases and whose edges join two pairs that differ by one symmetric exchange.

White [19] conjectured that any two pairs of bases with the same union can be transformed into one another by a sequence of symmetric exchanges, and Gabow [13] conjectured the stronger statement that such exchanges can be carried out in a serial fashion. These conjectures relate to the condition of Theorem 7 as follows. In the base-partition case, the pairs $(B, M \setminus B)$ and $(M \setminus B, B)$ have the same union, namely $M$; White's conjecture applied to them asserts that they are connected by symmetric exchanges, that is, by the previous paragraph, that $B$ and $\tau(B)$ lie in the same connected component of $H_{\mathcal{K}}$. This is precisely the statement that the component containing $B$ is self-complementary, which is what Theorem 7 requires. Note that only one vertex $B$ and its complement are needed, so the condition of Theorem 7 is weaker than the conjectures.

The conjectures are known to hold for several classes of matroids, which the previous paragraph converts into fairness guarantees. Bérczi and Schwarcz prove the required exchange property for split matroids, a class that includes sparse paving matroids [7], and Bérczi, Mátravölgyi, and Schwarcz prove it for regular matroids [9]. For a strongly base-orderable matroid the property is immediate: there is a bijection $\phi : B \to M \setminus B$ such that exchanging the elements of any subset of the pairs $\{x, \phi(x)\}$ leaves both sets bases, so performing these exchanges one pair at a time traces a path in $H_{\mathcal{K}}$ from $B$ to $\tau(B)$. Combining these facts with Theorem 7 gives the following.

**Corollary 1.** *In a two-agent base-partition instance, arbitrary Boolean valuations admit a feasible EF1 allocation whenever the matroid is strongly base-orderable, split, sparse paving, or regular.*

The exact condition in Theorem 7 is weaker than global basis-pair connectivity: it requires only one self-complementary component. Consequently, the theorem identifies the precise reconfiguration property needed by arbitrary Boolean valuations without assuming a stronger open conjecture. The following corollary follows by padding a feasible instance to the base-partition case and using the fact that uniform matroids and partition matroids are strongly base-orderable.

**Corollary 2.** *For two agents, arbitrary Boolean valuations admit a complete feasible EF1 allocation under every uniform-matroid or partition-matroid constraint for which a complete feasible allocation exists.*

*Proof.* For a uniform matroid of rank $r$, the existence of a complete feasible allocation implies $|M| \leqslant 2r$. Add $2r - |M|$ dummy items; the enlarged rank-$r$ uniform matroid has a ground set of size $2r$ and can therefore be partitioned into two bases. For a partition matroid with blocks $P_q$ and capacities $r_q$, the existence of a complete feasible allocation implies $|P_q| \leqslant 2r_q$ for every block. Add $2r_q - |P_q|$ dummy items to block $P_q$. Each enlarged block then has size $2r_q$, so splitting every block into two sets of size $r_q$ partitions the enlarged ground set into two bases. Thus, in either case, the enlarged instance is a base-partition instance of a strongly base-orderable matroid.

Extend each valuation by setting $\widetilde{v}_i(S) = v_i(S \cap M)$, and apply Corollary 1 to obtain a feasible EF1 allocation $\widetilde{A}$ of the enlarged instance. Set $A_i = \widetilde{A}_i \cap M$ for each agent $i$. The resulting allocation $A$ is complete and feasible for the original instance. Fix an ordered pair of agents $i, j$. If $\widetilde{v}_i(\widetilde{A}_i) \geqslant \widetilde{v}_i(\widetilde{A}_j)$, then $v_i(A_i) \geqslant v_i(A_j)$. Otherwise, let $e \in \widetilde{A}_i \cup \widetilde{A}_j$ be an EF1 witness. If $e \in M$, then the same item witnesses EF1 for $A$. If $e$ is a dummy item, then $\widetilde{v}_i(\widetilde{A}_i \setminus \{e\}) \geqslant \widetilde{v}_i(\widetilde{A}_j \setminus \{e\})$ reduces to $v_i(A_i) \geqslant v_i(A_j)$, so no witness is needed. Hence $A$ is EF1. $\square$

The question for arbitrary Boolean valuations under matroid constraints is therefore already tied, in the two-agent case, to a longstanding basis-pair reconfiguration problem.

### 8.4 EF1 and Pareto optimality can conflict under a partition matroid

We show that EF1 and PO are incompatible under partition-matroid constraints. Pareto optimality in this section is relative to the family of complete feasible allocations.

**Proposition 8.** *There is a two-agent instance with identical normalized Boolean valuations and a partition-matroid constraint in which feasible EF1 allocations and feasible Pareto-optimal allocations both exist, but no feasible allocation satisfies both properties.*

*Proof.* Let $M = \{a, b, c, d\}$. The partition matroid has two capacity-one blocks, $\{a, c\}$ and $\{b, d\}$. Let both agents have the normalized valuation

$$v(S) = 1 \iff S \in \{\{c\}, \{d\}, \{c, d\}\}.$$

There are four ordered complete feasible allocations. The allocations with bundle pair $\{a, b\}, \{c, d\}$, in either orientation, have utility vectors $(0, 1)$ and $(1, 0)$. They are the only Pareto-optimal allocations: keeping the agent who receives value one at value one forces her to retain $\{c, d\}$. Yet $\{a, b\}$ is unsafe and $\{c, d\}$ is robust, so each such allocation violates EF1 by Proposition 1.

The remaining two allocations have bundle pairs $\{a, d\}, \{b, c\}$, in either orientation. Both agents receive value zero, so these allocations are envy-free and hence EF1. Each is Pareto dominated by orienting $\{c, d\}$ toward one agent and $\{a, b\}$ toward the other. Thus no feasible allocation is both EF1 and Pareto optimal. □

## 9 Discussion

As our results highlight, the value of the empty bundle is more than a normalization convention: it dictates the guarantees we can achieve. An item may raise one bundle from zero to one and lower another from one to zero, so calling it a good or a chore is generally meaningless; in contrast, the value agents assign to $\varnothing$ dictates which one-sided version of EF1 we can achieve. The sharp transition at $|Z| = 1$ makes the point especially vivid.

Our mechanism-design results leave two distinct questions open. On $\mathcal{V}_0^n$, Theorem 5 gives a weakly group-strategyproof and Pareto-optimal mechanism whose output is $\mathrm{EFX}_{-}^{0}$. At the opposite endpoint $Z = \varnothing$, Proposition 6 shows that, for every $n, m \geqslant 2$, there exists a profile at which no allocation is simultaneously Pareto optimal and EF1; hence no mechanism can guarantee both properties throughout $\mathcal{V}_1^n$. This does not rule out a strategyproof EF1 mechanism without Pareto optimality on $\mathcal{V}_1^n$, and it does not classify mixed profiles with $0 < |Z| < n$.

**Open Question 1:** Can a strategyproof rule guarantee EF1 on $\mathcal{V}_1^n$ for arbitrary $n$ and $m$? On which Boolean profiles can strategyproofness, EF1, and Pareto optimality be achieved simultaneously?

For two agents we obtain a complete answer for the fairness achievable under matroid feasibility constraints: a feasible EF1 allocation is guaranteed for every Boolean profile exactly when the exchange graph $H_{\mathcal{K}}$ has a self-complementary component (Theorem 7). This condition is weaker than the exchange property conjectured by White and by Gabow, and it already holds for several classes of matroids, which is how Corollary 1 is obtained. Extending the characterization to more agents seems to require a different object, since a graph records only exchanges between two bundles whereas $n$ bundles may be permuted in many ways at once; agent-specific matroids are further out of reach, as there is then no single exchange graph shared across agents.

**Open Question 2:** Which matroids guarantee a feasible EF1 allocation for every Boolean profile and every number of agents? In the two-agent case, must the basis-pair exchange graph contain a self-complementary component?

The natural next domain is preferences with three indifference classes rather than two. There a single deletion can move a bundle by more than one level, so the distinction between unsafe and robust bundles that drives all of our arguments no longer captures the effect of removing an item, and it is not clear which of our results survive.

## AI Disclosure

Many of the mathematical proofs and counterexamples were derived by OpenAI Codex (GPT-5.6-Sol at Max effort) based on explicit questions, research directions, literature connections, proof and search strategies, and inspirations supplied by the authors. The authors have verified all mathematical details, presented simplified arguments and expositions, often with the aid of GPT-5.6-Sol and Claude Opus 5. The authors retain full responsibility for all the content.

## References


[1] Haris Aziz, Ioannis Caragiannis, Ayumi Igarashi, and Toby Walsh. Fair allocation of indivisible goods and chores. *Autonomous Agents and Multi-Agent Systems*, 36(1):3, 2022. doi: 10.1007/s10458-021-09532-8.

[2] Moshe Babaioff, Tomer Ezra, and Uriel Feige. Fair and truthful mechanisms for dichotomous valuations. *Proceedings of the AAAI Conference on Artificial Intelligence*, 35(6):5119–5126, 2021. doi: 10.1609/aaai.v35i6.16647.

[3] Siddharth Barman and Paritosh Verma. Existence and computation of maximin fair allocations under matroid-rank valuations. In *Proceedings of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 169–177, 2021.

[4] Siddharth Barman and Paritosh Verma. Truthful and fair mechanisms for matroid-rank valuations. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*, pages 4801–4808, 2022.

[5] Siddharth Barman and Paritosh Verma. Fair division beyond monotone valuations with applications to equitable graph partitioning. In *Proceedings of the 2026 Annual ACM–SIAM Symposium on Discrete Algorithms (SODA)*, pages 6613–6641, 2026. doi: 10.1137/1.9781611978971.236.

[6] Nawal Benabbou, Mithun Chakraborty, Ayumi Igarashi, and Yair Zick. Finding fair and efficient allocations for matroid rank valuations. *ACM Transactions on Economics and Computation*, 9(4):21:1–21:41, 2021.

[7] Kristóf Bérczi and Tamás Schwarcz. Exchange distance of basis pairs in split matroids. *SIAM Journal on Discrete Mathematics*, 38(1):132–147, 2024. doi: 10.1137/23M1565115.

[8] Kristóf Bérczi, Erika R. Bérczi-Kovács, Endre Boros, Fekadu Tolessa Gedefa, Naoyuki Kamiyama, Telikepalli Kavitha, Yusuke Kobayashi, and Kazuhisa Makino. Envy-free relaxations for goods, chores, and mixed items. *Theoretical Computer Science*, 1002:114596, 2024. doi: 10.1016/j.tcs.2024.114596.

[9] Kristóf Bérczi, Bence Mátravölgyi, and Tamás Schwarcz. Reconfiguration of basis pairs in regular matroids. *Journal of Combinatorial Theory, Series B*, 177:105–142, 2026. doi: 10.1016/j.jctb.2025.10.009. Preliminary version in *Proceedings of the 56th Annual ACM Symposium on Theory of Computing (STOC)*, pages 1653–1664, 2024.

[10] Umang Bhaskar, A. R. Sricharan, and Rohit Vaish. On approximate envy-freeness for indivisible chores and mixed resources. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX-/RANDOM)*, volume 207 of *LIPIcs*, pages 1:1–1:23. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. doi: 10.4230/LIPIcs.APPROX/RANDOM.2021.1.

[11] Umang Bhaskar, Gunjan Kumar, Yeshwant Pandit, and Rakshitha. Towards envy-freeness relaxations for general nonmonotone valuations. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 298–306. International Foundation for Autonomous Agents and Multiagent Systems, 2025. Extended version available as arXiv:2411.19881.

[12] Vittorio Bilò, Martin Loebl, and Cosimo Vinci. Approximately envy-free and equitable allocations of indivisible items for non-monotone valuations. *Proceedings of the AAAI Conference on Artificial Intelligence*, 40(20):16700–16708, 2026. doi: 10.1609/aaai.v40i20.38712.

[13] Harold N. Gabow. Decomposing symmetric exchanges in matroid bases. *Mathematical Programming*, 10(1):271–276, 1976. doi: 10.1007/BF01580677.

[14] Daniel Halpern, Ariel D. Procaccia, Alexandros Psomas, and Nisarg Shah. Fair division with binary valuations: One rule to rule them all. In *Web and Internet Economics — 16th International Conference (WINE)*, volume 12495 of *Lecture Notes in Computer Science*, pages 370–383. Springer, 2020. doi: 10.1007/978-3-030-64946-3_26.

[15] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (EC)*, pages 125–131. ACM, 2004. doi: 10.1145/988772.988792.

[16] Francis Edward Su. Rental harmony: Sperner's lemma in fair division. *The American Mathematical Monthly*, 106(10): 930–942, 1999. doi: 10.1080/00029890.1999.12005142.

[17] Vignesh Viswanathan and Yair Zick. A general framework for fair allocation under matroid rank valuations. In *Proceedings of the 24th ACM Conference on Economics and Computation (EC)*, pages 1129–1152, 2023. doi: 10.1145/3580507.3597675.

[18] Vignesh Viswanathan and Yair Zick. Yankee swap: A fast and simple fair allocation mechanism for matroid rank valuations. In *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 179–187, 2023.

[19] Neil L. White. A unique exchange property for bases. *Linear Algebra and its Applications*, 31:81–91, 1980. doi: 10.1016/0024-3795(80)90209-8.


# A Valuation classes

Several valuation classes appear in the discussion of related work; we collect their definitions here. Throughout this subsection, $v : 2^M \to \mathbb{R}$ denotes the valuation of a single agent, normalized so that $v(\varnothing) = 0$. For a bundle $S \subseteq M$ and an item $e \notin S$, the *marginal value* of $e$ with respect to $S$ is

$$v(e \mid S) \;=\; v(S \cup \{e\}) - v(S),$$

and $v$ is *submodular* if marginal values never increase as the bundle grows, that is, if $v(e \mid S) \geqslant v(e \mid T)$ whenever $S \subseteq T \subseteq M \setminus \{e\}$.

We first record the classes in which the value of a bundle is the sum of the values of its items.

- *Additive*: $v(S) = \sum_{e \in S} v(\{e\})$ for every bundle $S$, where $v(\{e\}) \geqslant 0$ for every item $e$.
- *Binary additive*: additive, with the further requirement that $v(\{e\}) \in \{0, 1\}$ for every item $e$. Each item is thus either wanted or unwanted, and the value of a bundle counts the wanted items that it contains.
- *Additive mixed manna*: $v(S) = \sum_{e \in S} v(\{e\})$ for every bundle $S$, where $v(\{e\})$ is now an arbitrary real number. An item $e$ is a *good* if $v(\{e\}) > 0$, a *chore* if $v(\{e\}) < 0$, and irrelevant to the agent if $v(\{e\}) = 0$.

The next two classes keep the requirement that every marginal value be zero or one, but drop additivity.

- *Dichotomous*: $v(e \mid S) \in \{0, 1\}$ for every bundle $S$ and every item $e \notin S$. Such a valuation is monotone nondecreasing, and its range is contained in $\{0, 1, \ldots, m\}$.
- *Matroid rank*, also called *binary submodular*: dichotomous and submodular. Equivalently, $v$ is the rank function of a matroid on $M$, so that $v(S)$ is the size of a largest independent subset of $S$. Every binary additive valuation is of this form.

The remaining classes are defined by the signs of the marginal values alone, and impose no structure beyond them.

- *Monotone nondecreasing*: $v(S) \leqslant v(T)$ whenever $S \subseteq T$; equivalently, $v(e \mid S) \geqslant 0$ for every bundle $S$ and every item $e \notin S$, so that every item is a good.
- *Doubly monotone*: the item set splits into two parts, $M^+$ and $M^-$, such that $v(e \mid S) \geqslant 0$ for every $e \in M^+$ and $v(e \mid S) \leqslant 0$ for every $e \in M^-$, for every bundle $S$. Every item is therefore a good or a chore for the agent, but the split may differ from agent to agent, and the value of a bundle need not be the sum of the values of its items.
- *Nonnegative*: $v(S) \geqslant 0$ for every bundle $S$, with no monotonicity requirement.
- *Arbitrary set valuations*: no requirement beyond $v$ being a function on $2^M$.

The Boolean valuations studied in this paper are nonnegative, but they are subject to no other requirement, and in particular they are incomparable with the classes based on binary marginals: a Boolean valuation has range $\{0, 1\}$ and need not be monotone nondecreasing, whereas a dichotomous valuation is monotone nondecreasing and may take any value in $\{0, 1, \ldots, m\}$. The two classes intersect precisely in the monotone nondecreasing valuations of $\mathcal{V}_0$.

Finally, a profile $(v_1, \ldots, v_n)$ is *identical* if all agents share the same valuation. We use *heterogeneous* to mean *not necessarily identical*: no restriction is imposed on how the agents' valuations relate to one another.