The Rise of Approximate Model Counting: A Child of SAT Revolution

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2^3 Years and Counting

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Special shout out to Mate Soos, maintainer of ApproxMC

Open source tool: `github.com/meelgroup/approxmc`

Boolean Satisfiability (SAT); Given a Boolean expression, using "and" ($\wedge$) "or", ($\vee$) and "not" ($\neg$), *is there a satisfying solution* (an assignment of 0's and 1's to the variables that makes the expression equal 1)?

Example:

$$(\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3 \vee x_4) \wedge (x_3 \vee x_1 \vee x_4)$$

Solution: $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$

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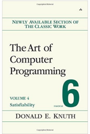

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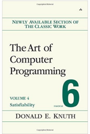

Industrial usage of SAT Solvers: Model Checking, Planning, Genome Rearrangement, Telecom Feature Subscription, Resource Constrained Scheduling, Noise Analysis, Games, $\cdots$

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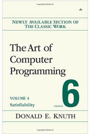

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Now that SAT is "easy", it is time to look beyond satisfiability

Counting

- Given
  - Boolean variables $X_1, X_2, \cdots X_n$
  - Formula F over $X_1, X_2, \cdots X_n$
- $\mathsf{Sol}(F) = \{$ solutions of F $\}$

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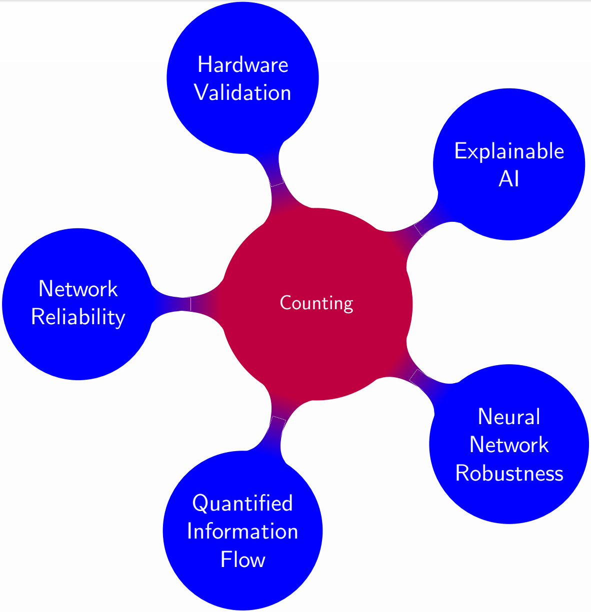
- Given $F := (X_1 \vee X_2)$
- $\mathsf{Sol}(F) = \{(0,1),(1,0),(1,1)\}$
- $|\mathsf{Sol}(F)| = 3$

Applications across Computer Science



Today's Menu

Testing of AI systems

Information Leakage

Network Reliability

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Information Leakage Model Counting

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Testing of AI systems

Information Leakage   Model Counting   Hashing Framework

Network Reliability

The Rise of Hashing-based Approach: Promise of Scalability and Guarantees
(S83,GSS06,GHSS07,CMV13b,EGSS13b,CMV14,CDR15,CMV16,ZCSE16,AD16 KM18,ATD18,SM19,ABM20,SGM20)

Testing of AI Systems

- Classical verification/testing setup for traditional systems
  - System captured as a model $M(\mathcal{I}, \mathcal{O})$ via logical constraints
  - Specification $\varphi(\mathcal{I}, \mathcal{O})$: relationship between input and output
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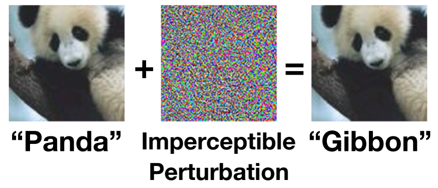

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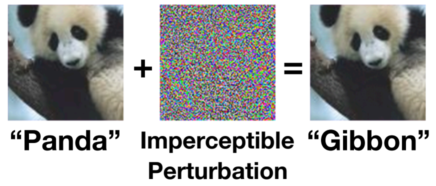


- Acceptable despite multiple executions with error
- Estimate the frequency of such behavior: Counting (BSSMS, 2019)

Quantitative Information Flow

Algorithm PC1 (SP, UI)

```
for i = 0; i <SP.length(); i++ do
    if SP[i] ≠ UI[i] then
        return "No"
    return "Yes"
end for
```

Algorithm PC2 (SP, UI)

```
match = "Yes"
for i = 0; i <SP.length(); i++ do
    if SP[i] ≠ UI[i] then
        match="No"
end for
return match
```

Quantification of Information Leakage between PC1 and PC2 via side-channels (such as time?)

- Annotate every line of program with time taken and perform symbolic analysis
- Shannon Entropy $= \sum_t \Pr[\text{finishtime} = t] \log \frac{1}{\Pr[\text{time}=t]}$

(Bang et al., 2016)

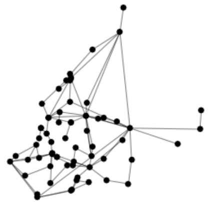
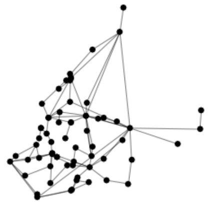


Figure: Plantersville, SC

- $G = (V, E)$; source node: s and terminal node t
- failure probability $g : E \to [0, 1]$
- Compute Pr[s and t are disconnected]?



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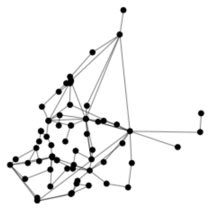

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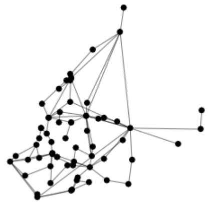
Reliability of Critical Infrastructure Networks



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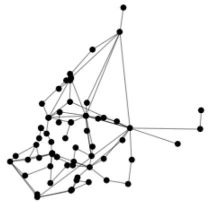

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(DMPV, AAAI 17, ICASP-13, RESS 2019)

Constrained Counting

Prior Work

Strong guarantees but poor scalability

- Exact counters (Birnbaum and Lozinskii 1999, Jr. and Schrag 1997, Sang et al. 2004, Thurley 2006)
- Hashing-based approach (Stockmeyer 1983, Jerrum Valiant and Vazirani 1986)

Weak guarantees but impressive scalability

- Bounding counters (Gomes et al. 2007,Kroc, Sabharwal, and Selman 2008, Gomes, Sabharwal, and Selman 2006, Kroc, Sabharwal, and Selman 2008)
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How to bridge this gap between theory and practice?

Obs 1 SAT Solver $\neq$ NP Oracle

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Obs 2 Memoryfulness

- Incremental Solving: Often easier to solve F followed by G if we G can be written as $G = F \wedge H$
- If $F \rightarrow C$ then $(F \wedge H) \implies C$

Counting in Bochum

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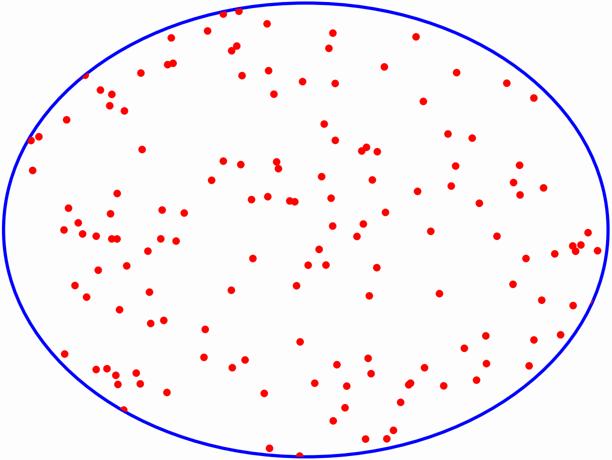
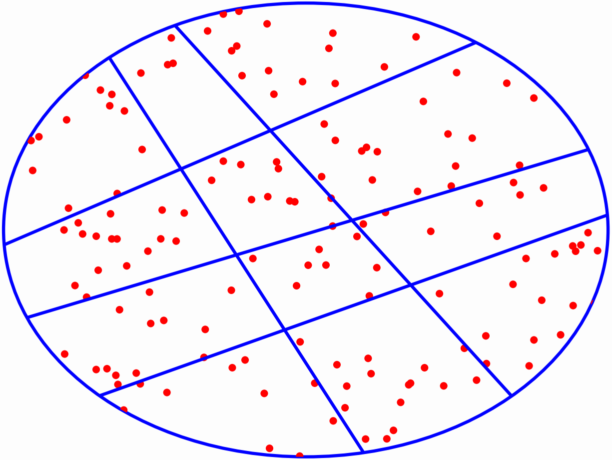
  - Potentially 2^n queries

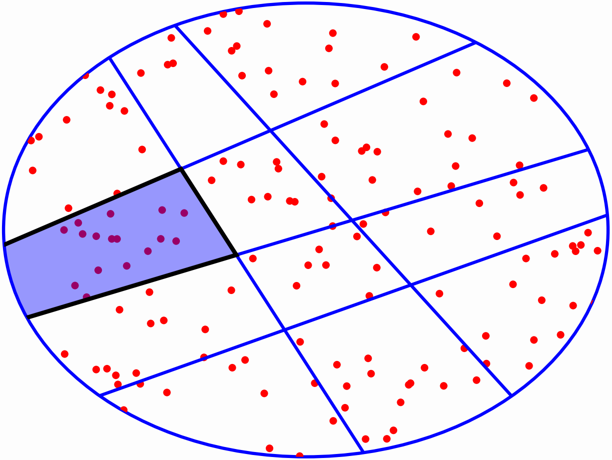
Can we do with lesser # of SAT queries – $\mathcal{O}(n)$ or $\mathcal{O}(\log n)$?

As Simple as Counting Dots





Pick a random cell



Estimate $=$ Number of solutions in a cell $\times$ Number of cells

Challenges

Challenge 1 How to partition into roughly equal small cells of solutions without knowing the distribution of solutions?

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Challenge 2 How many cells?

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- Designing function h : assignments $\rightarrow$ cells (hashing)
- Solutions in a cell α: $\mathsf{Sol}(F) \cap \{y \mid h(y) = \alpha\}$

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- Solutions in a cell α: $\mathsf{Sol}(F) \cap \{y \mid h(y) = \alpha\}$
- Deterministic h unlikely to work
- Choose h randomly from a large family H of hash functions

  $\boxed{\text{Universal Hashing (Carter and Wegman 1977)}}$

2-wise independent Hashing

- Let H be family of 2-wise independent hash functions mapping $\{0,1\}^n$ to $\{0,1\}^m$

$$\forall y_1, y_2 \in \{0,1\}^n, \alpha_1, \alpha_2 \in \{0,1\}^m, h \xleftarrow{R} H$$
$$\Pr[h(y_1) = \alpha_1] = \Pr[h(y_2) = \alpha_2] = \left(\frac{1}{2^m}\right)$$

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- The power of 2-wise independentity
  - Z be the number of solutions in a randomly chosen cell
  - $\mathsf{E}[Z] = \frac{|\mathsf{Sol}(F)|}{2^m}$
  - $\sigma^2[Z] \leq \mathsf{E}[Z]$

2-wise independent Hash Functions

- Variables: $X_1, X_2, \cdots X_n$
- To construct $h : \{0,1\}^n \rightarrow \{0,1\}^m$, choose m random XORs
- Pick every X_i with prob. $\frac{1}{2}$ and XOR them
  - $X_1 \oplus X_3 \oplus X_6 \cdots \oplus X_{n-2}$
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- Solutions in a cell: $F \wedge Q_1 \cdots \wedge Q_m$
- Performance of state of the art SAT solvers depends on the formulas (SAT Solvers != SAT oracles)

Improved 2-wise Independent Hash Functions

- Not all variables are required to specify solution space of F
  - $F := X_3 \iff (X_1 \vee X_2)$
  - X_1 and X_2 uniquely determines rest of the variables (i.e., X_3)
- Formally: if I is independent support, then $\forall \sigma_1, \sigma_2 \in \mathsf{Sol}(F)$, if σ_1 and σ_2 agree on I then $\sigma_1 = \sigma_2$
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- FP^{NP} procedure via reduction to Minimal Unsatisfiable Subset

Improved 2-wise Independent Hash Functions

- Not all variables are required to specify solution space of F
  - $F := X_3 \iff (X_1 \vee X_2)$
  - X_1 and X_2 uniquely determines rest of the variables (i.e., X_3)
- Formally: if I is independent support, then $\forall \sigma_1, \sigma_2 \in \mathsf{Sol}(F)$, if σ_1 and σ_2 agree on I then $\sigma_1 = \sigma_2$
  - $\{X_1, X_2\}$ is independent support but $\{X_1, X_3\}$ is not
- Random XORs need to be constructed only over I (CMV DAC14)
- Typically I is 1-2 orders of magnitude smaller than X
- Auxiliary variables introduced during encoding phase are *dependent* (Tseitin 1968)

Algorithmic procedure to determine I?

- FP^{NP} procedure via reduction to Minimal Unsatisfiable Subset
- Two orders of magnitude runtime improvement

(IMMV; CP15, Constraints16)

The Hope of Short XORs

- If we pick every variable X_i with probability p .
  - Expected Size of each XOR: np
  - $\mathsf{E}[Z_m] = \frac{|\mathsf{Sol}(F)|}{2^m}$
  - $\sigma^2[Z_m] \leq \mathsf{E}[Z_m] + \sum_{\sigma_1 \in \mathsf{Sol}(F)} \sum_{\substack{\sigma_2 \in \mathsf{Sol}(F) \\ w = d(\sigma_1, \sigma_2)}} r(w, m)$
    - where, $r(w, m) = \left(\left(\frac{1}{2} + \frac{(1-2p)^w}{2} \right)^m - \frac{1}{2^m} \right)$
  - For $p = \frac{1}{2}$, we have $\frac{\sigma^2[Z_m]}{\mathsf{E}[Z_m]} \leq 1$
- Earlier Attempts (GSS07,EGSS14,ZCSE16,AD17,ATD18)
  - $\sum_{\sigma_1 \in \mathsf{Sol}(F)} \sum_{\substack{\sigma_2 \in \mathsf{Sol}(F) \\ w = d(\sigma_1, \sigma_2)}} r(w, m) \leq \sum_{\sigma_1 \in \mathsf{Sol}(F)} \sum_{w=0}^{n} \binom{n}{w} r(w, m)$
  - $\binom{n}{w}$ grows very fast with n, so could not upper bound $\frac{\sigma^2[Z_m]}{\mathsf{E}[Z_m]}$
  - The weak bounds lead to significant slowdown: typically $100\times$ to $1000\times$ factor of slowdown! (ATD18,ABM20)

The Power of Isoperimetric Inequalities

- $\sum_{\sigma_1 \in \mathrm{Sol}(F)} \sum_{\substack{\sigma_2 \in \mathrm{Sol}(F) \\ w = d(\sigma_1, \sigma_2)}} r(w, m) = \sum_{w=0}^{n} C_F(w) r(w, m)$
- $C_F(w) = |\{\sigma_1, \sigma_2 \in \mathrm{Sol}(F) \mid d(\sigma_1, \sigma_2) = w\}|$

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- Isoperimetric Inequalities! (Rashtchian and Raynaud 2019)

Lemma

$$\sum_{w=0}^{n} C_F(w) r(w, m) \leq \sum_{w=0}^{n} \binom{8e\sqrt{n \cdot \ell}}{w} r(w, m) \ \textit{where}\ \ell = \log |\mathrm{Sol}(F)|$$

  - $\frac{\binom{n}{w}}{\binom{8e\sqrt{n \cdot \ell}}{w}} \approx \left(\frac{n}{\ell}\right)^{\frac{w}{2}}$

From Linear to Logarithmic Size XORs

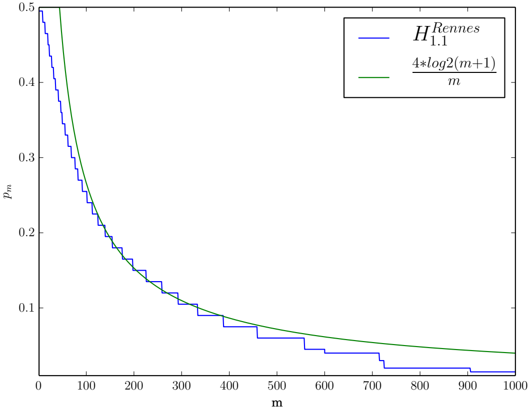
Theorem (MA, LICS-20)

For all q, k, $|\mathsf{Sol}(F)| \leq k \cdot 2^m$, $p = \mathcal{O}(\frac{\log m}{m})$ *we have*

$$\frac{\sigma^2[Z_m]}{\mathsf{E}[Z_m]} \leq q (a\ constant)$$

Recall, average size of XORs: $n \cdot p$
Improvement of p *from* $\frac{m/2}{m}$ *to* $\frac{\log m}{m}$

Sparse Hash Functions



$H_{1.1}^{Rennes}$: Sparse hash functions that guarantee $q = 1.1$

Handling CNF+XOR Formulas

- CNF + Sparse XORs are still CNF+XOR formulas.
- Translating XORs to CNF and performing CDCL is not sufficient

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- CNF + Sparse XORs are still CNF+XOR formulas.
- Translating XORs to CNF and performing CDCL is not sufficient
  - XORs can be solved by Gaussian elimination
- CryptoMiniSAT: Solver designed to perform CDCL and Gaussian Elimination in tandem (SNC09)
- BIRD2 (Blast, Inprocess, Recover, Detach, and Destroy): Tighter integration (SM19, SGM20)

Challenges

Challenge 1 How to partition into roughly equal small cells of solutions without knowing the distribution of solutions?

- Independent Support-based XORs
- Specialized CNF Solvers

Challenge 2 How many cells?

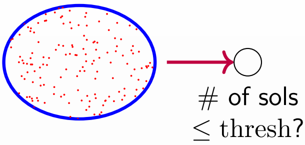
Challenge 2: How many cells?

- A cell is small if it has $\approx \mathrm{thresh} = 5(1 + \frac{1}{\varepsilon})^2$ solutions
- We want to partition into 2^{m^*} cells such that $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\mathrm{thresh}}$

Challenge 2: How many cells?

- A cell is small if it has $\approx \mathrm{thresh} = 5(1+\frac{1}{\varepsilon})^2$ solutions
- We want to partition into 2^{m^*} cells such that $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\mathrm{thresh}}$
  - Check for every $m = 0, 1, \cdots n$ if the number of solutions $\leq \mathrm{thresh}$

ApproxMC



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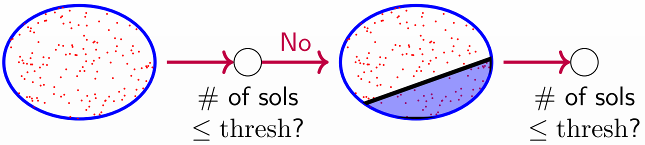

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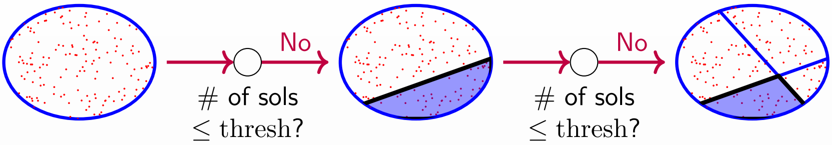

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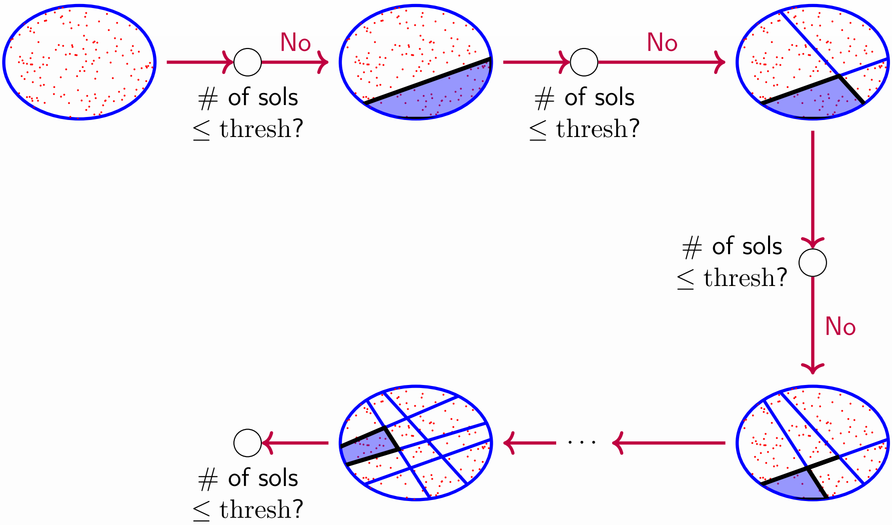

No
# of sols
$\leq$ thresh?
No
# of sols
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No
...
# of sols
$\leq$ thresh?

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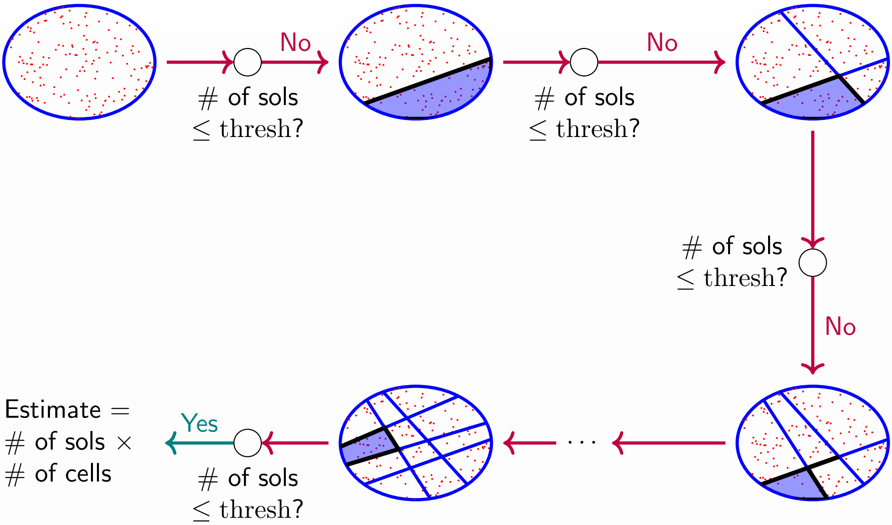
No
# of sols
≤ thresh?
No
# of sols
≤ thresh?
# of sols
≤ thresh?
No
Estimate =
# of sols ×
# of cells
Yes
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· · ·

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  - Query n: Is $\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_n) \leq \mathrm{thresh}$
- Stop at the first m where Query m returns YES and return estimate as $\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_m) \times 2^m$
- Observation: $\#(F \wedge Q_1 \cdots \wedge Q_i \wedge Q_{i+1}) \leq \#(F \wedge Q_1 \cdots \wedge Q_i)$
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  - Challenge Query i and Query j are not independent
  - Independence crucial to analysis (Stockmeyer 1983, $\cdots$)
  - Key Insight: The probability of making a bad choice of Q_i is very small for $i \ll m^*$

(CMV, IJCAI16)

Taming the Curse of Dependence

Let $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\text{thresh}}$ $(m^* = \log(\frac{|\mathsf{Sol}(F)|}{\text{thresh}}))$

Lemma (1)

ApproxMC terminates with $m \in \{m^* - 1, m^*\}$ with probability $\geq$ 0.8

Lemma (2)

For $m \in \{m^* - 1, m^*\}$, estimate obtained from a randomly picked cell lies within a tolerance of ε of $|\mathsf{Sol}(F)|$ with probability $\geq$ 0.8

Repeat $\mathcal{O}(\log(1/\delta))$ times and return the median

ApproxMC

Theorem (Correctness)

$$\Pr\left[\frac{|\mathrm{Sol}(F)|}{1+\varepsilon} \leq \mathit{ApproxMC}(F, \varepsilon, \delta) \leq |\mathrm{Sol}(F)|(1+\varepsilon)\right] \geq 1-\delta$$

Theorem (Complexity)

$\mathit{ApproxMC}(F, \varepsilon, \delta)$ makes $\mathcal{O}(\frac{\log n \log(\frac{1}{\delta})}{\varepsilon^2})$ calls to SAT oracle.

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Theorem (FPRAS for DNF; (MSV, FSTTCS 17; CP 18, IJCAI-19))

If F is a DNF formula, then ApproxMC is FPRAS – different from the Monte-Carlo based FPRAS for DNF (Karp, Luby 1983)

Reliability of Critical Infrastructure Networks

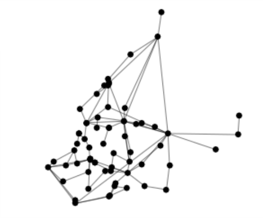
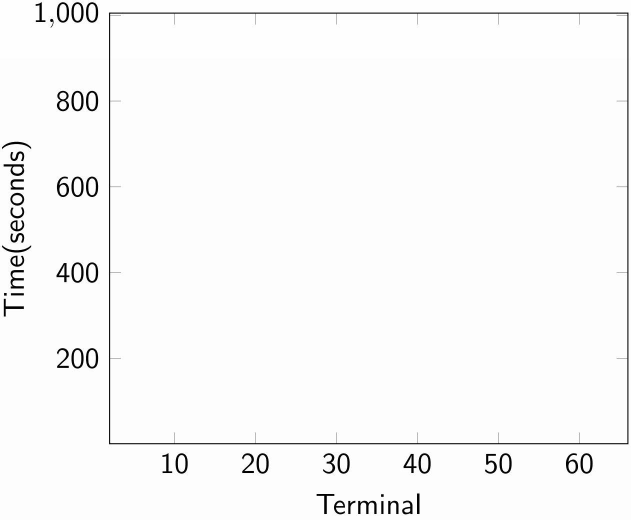


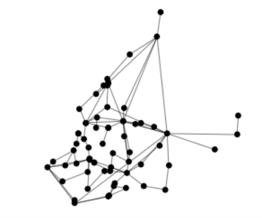
Figure: Plantersville, SC

- $G = (V, E)$; source node: s
- Compute Pr[t is disconnected]?

Timeout = 1000 seconds



(DMPV, AAAI17)



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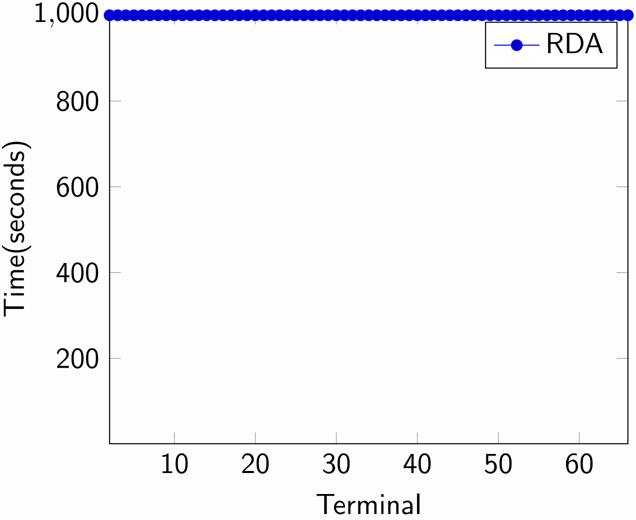

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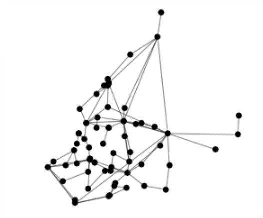

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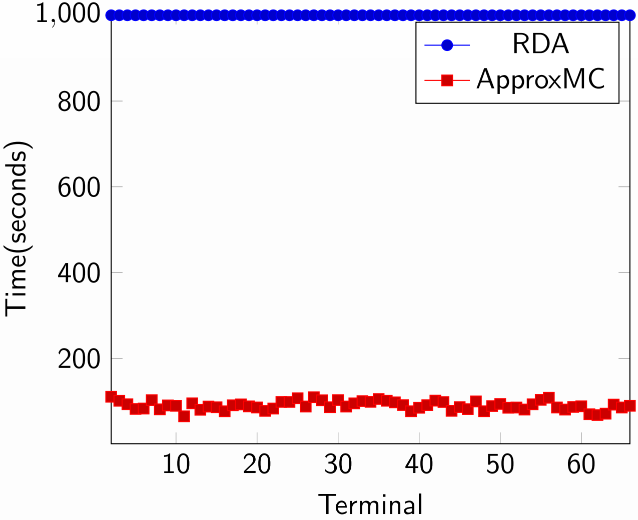

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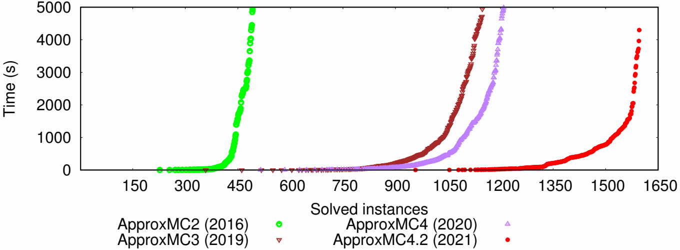
Improvements Over the Years



Challenge Problems

Enabling “Counting Revolution”

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Neural Network Robustness Handle 1000 neurons per layer

Civil Engineering Reliability for Los Angeles Transmission Grid

Security Leakage Measurement for C++ program with 1K lines

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Technical Directions

- Beyond SAT: Satisfiability Modulo Theory
- Formula-Dependent Sparser XOR constraints
- Tighter integration between solvers and algorithms

Enabling "Counting Revolution"

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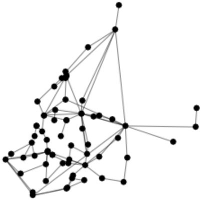
Questions?

Reliability of Critical Infrastructure Networks



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- $G = (V, E)$; source node: s and terminal node t
- failure probability $g : E \to [0, 1]$
- Compute Pr[s and t are disconnected]?

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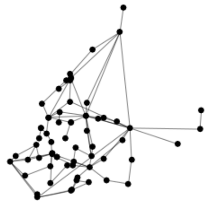

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- $W(\pi) = \Pr(\pi)$

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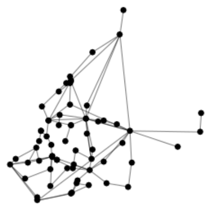

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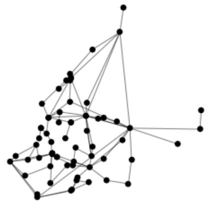

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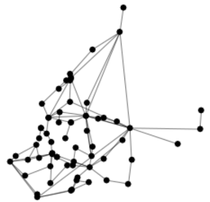

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Constrained Counting

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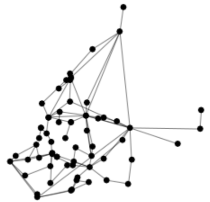
(DMPV, AAAI 17, ICASP-13, RESS 2019)

Reliability of Critical Infrastructure Networks



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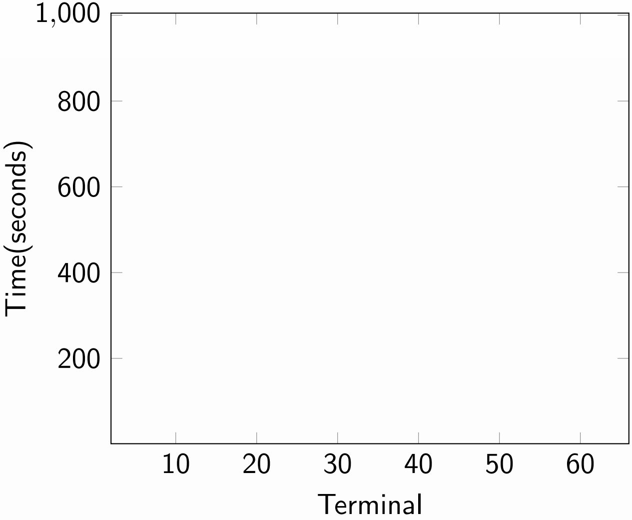
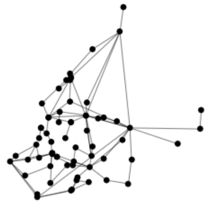


(DMPV, AAAI17)



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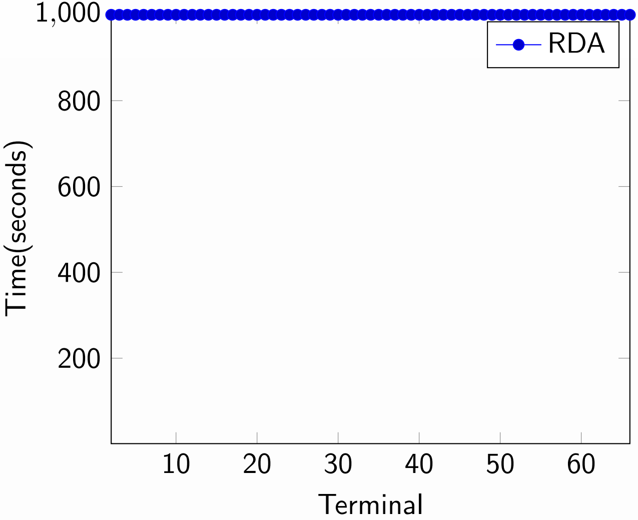

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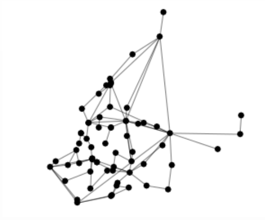

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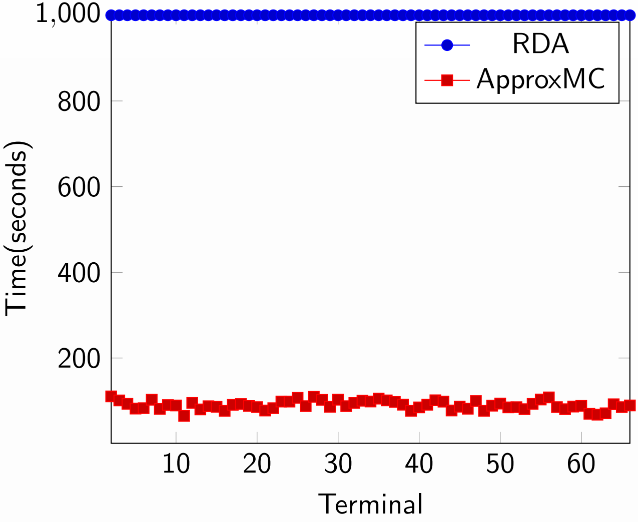

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