Constrained Counting and Sampling: From Theory to Practice and Back

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National University of Singapore

SPIN 2019

Collaborators (30+ across 10 countries)

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Thanks to Joao Marques-Silva for slides on CDCL solving.

Model Checking of Software

- Ball and Rajamani; SPIN 2001: "safety properties of system software can be validated and invalidated using model checking....We model abstractions of C programs using boolean programs... [we use] Bebop, a tool for model checking boolean programs "
- Ball and Rajamani; SPIN 2000: "[In Bebop], we use Binary Decisions Diagrams (BDDs) to symbolically represent these summaries, which are binary relationships between sets of states."

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- Core underlying problem: **Boolean Satisfiability (SAT)**

Boolean Satisfiability (SAT); Given a Boolean expression, using "and" ($\wedge$) "or", ($\vee$) and "not" ($\neg$), *is there a satisfying solution* (an assignment of 0's and 1's to the variables that makes the expression equal 1)?

Example:

$$(\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3 \vee x_4) \wedge (x_3 \vee x_1 \vee x_4)$$

Solution: $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$

History:

- William Stanley Jevons, 1835-1882: "I have given much attention, therefore, to lessening both the manual and mental labour of the process, and I shall describe several devices which may be adopted for saving trouble and risk of mistake."
- Ernst Schröder, 1841-1902: "Getting a handle on the consequences of any premises, or at least the fastest method for obtaining these consequences, seems to me to be one of the noblest, if not the ultimate goal of mathematics and logic."

Complexity of Boolean Reasoning

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- Clay Institute, 2000: $1M Award!

Algorithmic Boolean Reasoning: Early History

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  $F = (r) \wedge (\bar{r} \vee s) \wedge (\bar{w} \vee a) \wedge (\bar{x} \vee \bar{a} \vee b)$

  What can we deduce?

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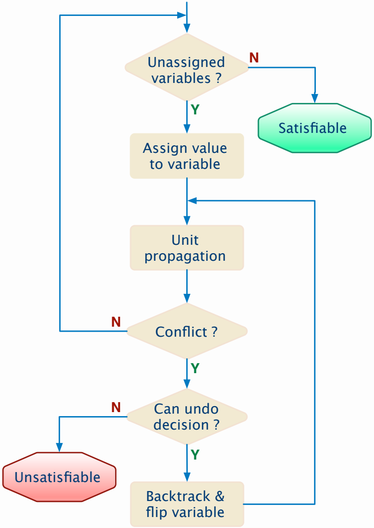
  What can we deduce?

  $s = 1$

The DPLL algorithm

[DL60,DLL62]



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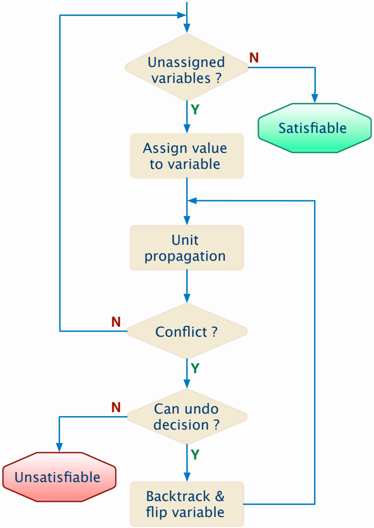

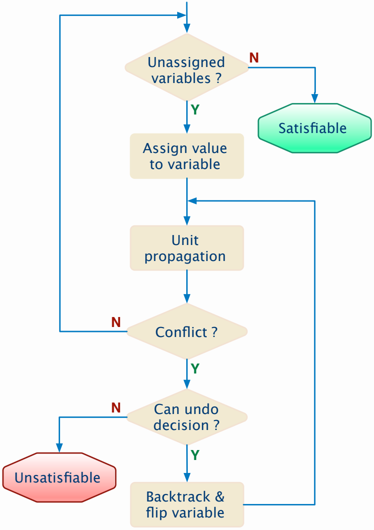

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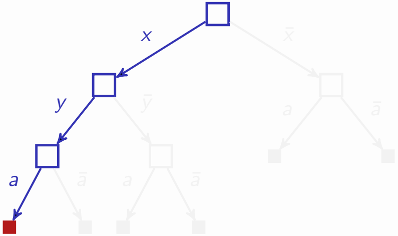
1	x	
2	y	
3	a	$\rightarrow b \rightarrow \bot$



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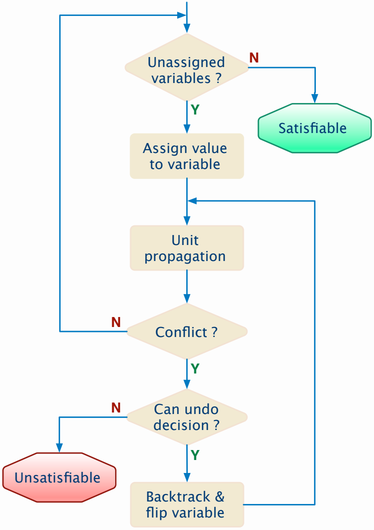

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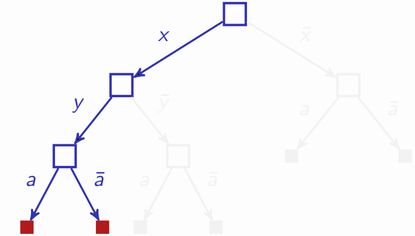

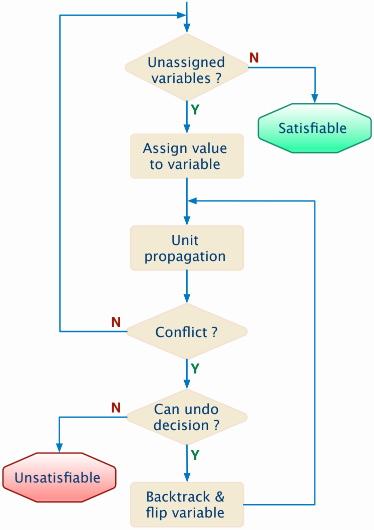

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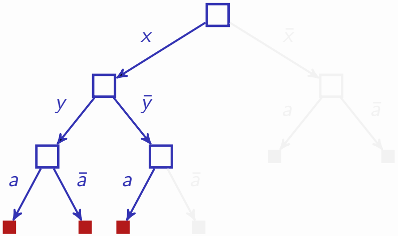

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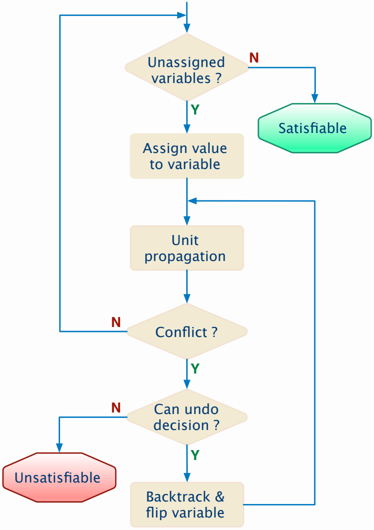

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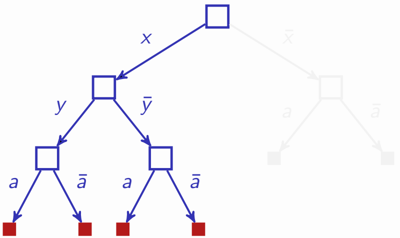

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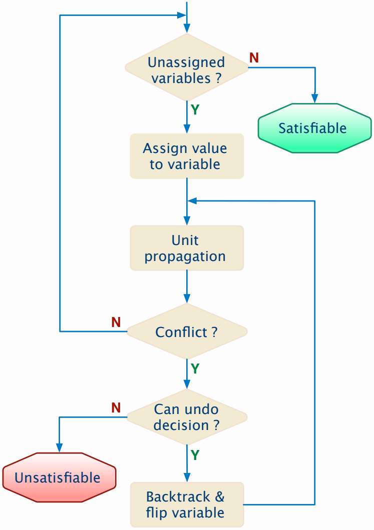

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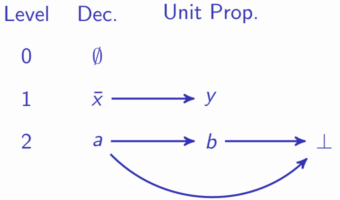

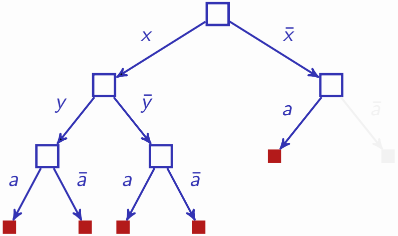

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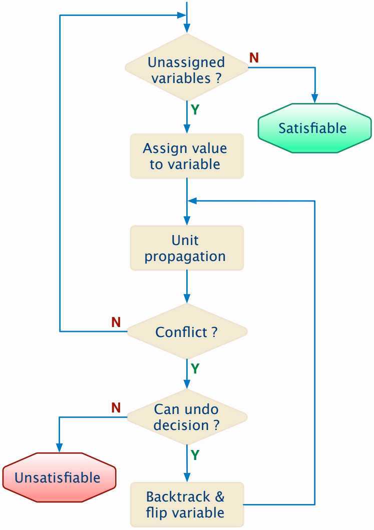

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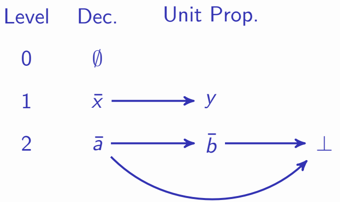
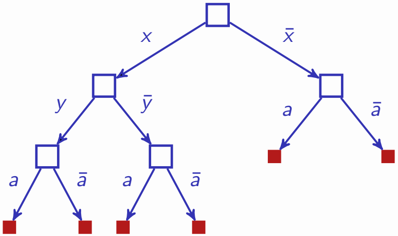
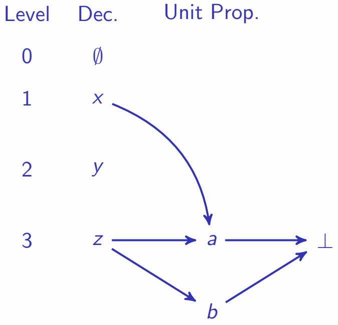




Clause learning

$(\bar{a} \vee \bar{b}) \wedge (\bar{z} \vee b) \wedge (\bar{x} \vee \bar{z} \vee a) \wedge (y \vee b)$



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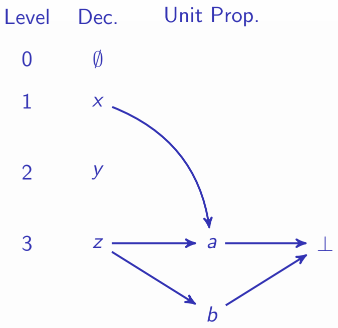

- Analyze conflict

[MSS96a,MSS96b,MSS96c,MSS96d,MSS99]

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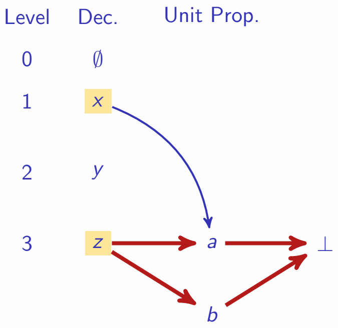

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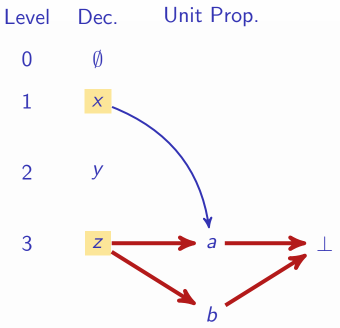

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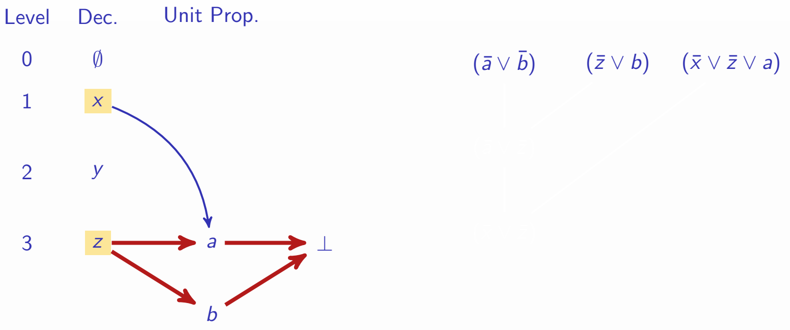

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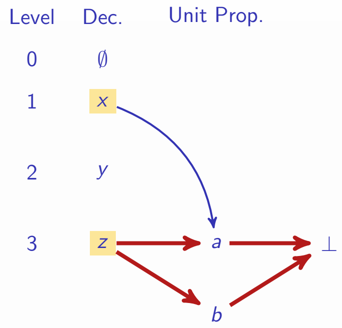

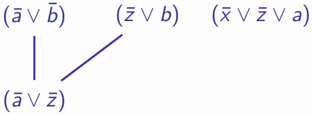

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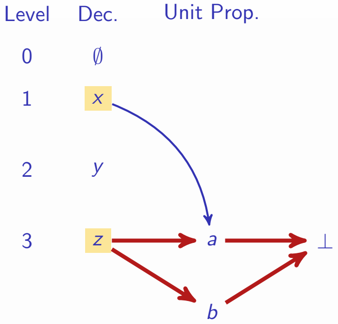

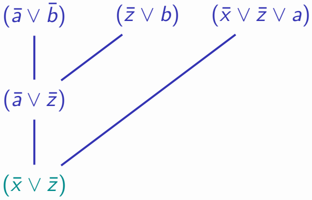

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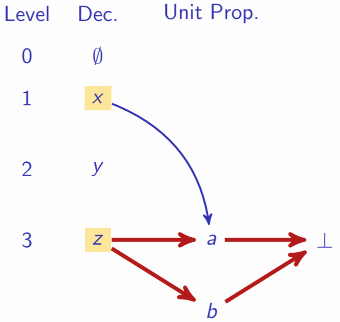

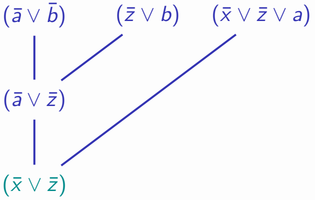

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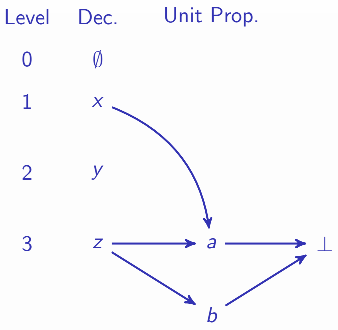
  - Create **new** clause: $(\bar{x} \vee \bar{z})$
- Can relate clause learning with resolution
  - Learned clauses result from (**selected**) resolution operations

Clause learning – after backtracking



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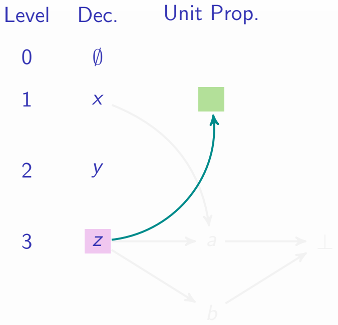

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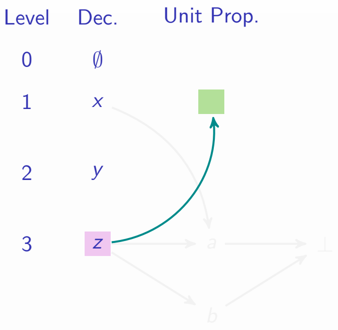

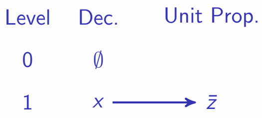

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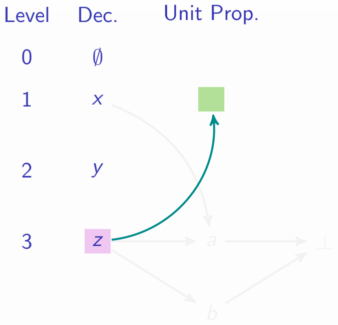
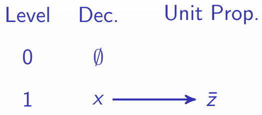




- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1
- Learned clauses are asserting [MSS96a,MSS99]
- Backtracking differs from plain DPLL:
  - Always bactrack after a conflict [ZMMM01]

The Modern CDCL SAT Solvers

- Clause learning & non-chronological backtracking [MSS96a,MSS99,BS97,Z97]

- Search restarts [GSK98,BMS00,H07,B08]

- Lazy data structures

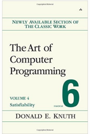
- Conflict-guided branching

- ...

The Modern CDCL SAT Solvers

- Clause learning & non-chronological backtracking [MSS96a,MSS99,BS97,Z97]
  - Exploit UIPs [MSS96a,SSS12]
  - Minimize learned clauses [SB09,VG09]
  - Opportunistically delete clauses [MSS96a,MSS99,GN02]
- Search restarts [GSK98,BMS00,H07,B08]
- Lazy data structures
  - Watched literals [MMZZM01]
- Conflict-guided branching
  - Lightweight branching heuristics [MMZZM01]
  - Phase saving [S00,PD07]
- ...

Modern SAT solvers are able to deal routinely with practical problems that involve millions of variables, although such problems were regarded as hopeless just a few years ago. (Donald Knuth, 2016)



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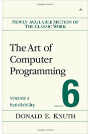

Industrial usage of SAT Solvers: Model Checking, Planning, Genome Rearrangement, Telecom Feature Subscription, Resource Constrained Scheduling, Noise Analysis, Games, $\cdots$

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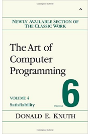

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Now that SAT is "easy", it is time to look beyond satisfiability

Constrained Counting and Sampling

- Given
  - Boolean variables $X_1, X_2, \cdots X_n$
  - Formula F over $X_1, X_2, \cdots X_n$

- $\mathsf{Sol}(F) = \{$ solutions of F $\}$

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- Constrained Counting: Determine $|\mathsf{Sol}(F)|$
- Constrained Sampling: Randomly sample from $\mathsf{Sol}(F)$ such that $\Pr[\text{y is sampled}] = \frac{1}{|\mathsf{Sol}(F)|}$

Constrained Counting and Sampling

- Given
  - Boolean variables $X_1, X_2, \cdots X_n$
  - Formula F over $X_1, X_2, \cdots X_n$
  - Weight Function W: $\{0, 1\}^n \mapsto [0, 1]$
- $\mathsf{Sol}(F) = \{$ solutions of F $\}$
- $W(F) = \Sigma_{y \in \mathsf{Sol}(F)} W(y)$
- Constrained Counting: Determine $W(F)$
- Constrained Sampling: Randomly sample from $\mathsf{Sol}(F)$ such that $\Pr[\text{y is sampled}] = \frac{W(y)}{W(F)}$

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- Given
  - $F := (X_1 \vee X_2)$
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Constrained Counting and Sampling

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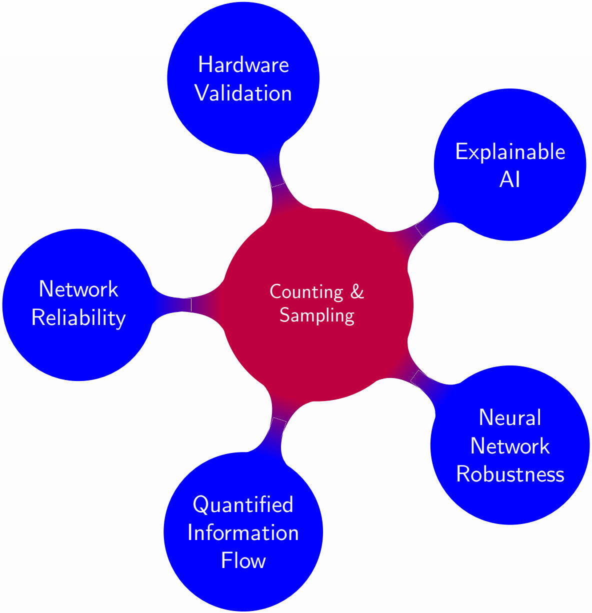
Applications across Computer Science



Today's Menu

Testing of AI systems

Network Reliability

Hardware Validation

Today's Menu

Testing of AI systems

Network Reliability   Constrained Counting

Hardware Validation

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Network Reliability   Constrained Counting   Hashing Framework

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Testing of AI Systems

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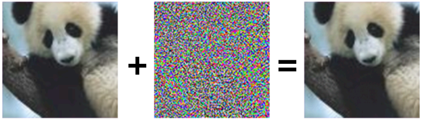

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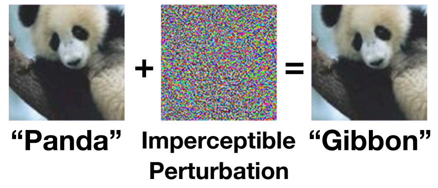

- Acceptable despite multiple executions with error: From satisfiability to counting

(BSSMS, 2019)

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Can we predict likelihood of a region facing blackout?

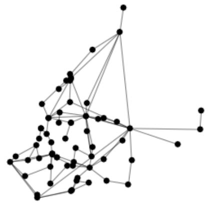

Figure: Plantersville, SC

- $G = (V, E)$; source node: s and terminal node t
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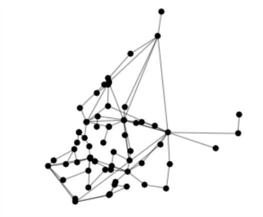

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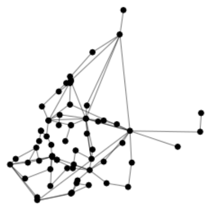
Reliability of Critical Infrastructure Networks



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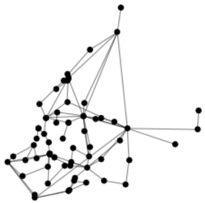

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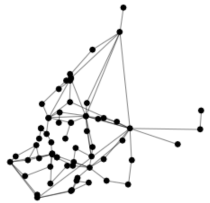

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(DMPV, AAAI 17, ICASP-13, RESS 2019)

Prior Work

Strong guarantees but poor scalability

- Exact counters (Birnbaum and Lozinskii 1999, Jr. and Schrag 1997, Sang et al. 2004, Thurley 2006)
- Hashing-based approach (Stockmeyer 1983, Jerrum Valiant and Vazirani 1986)

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- Bounding counters (Gomes et al. 2007,Kroc, Sabharwal, and Selman 2008, Gomes, Sabharwal, and Selman 2006, Kroc, Sabharwal, and Selman 2008)
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How to bridge this gap between theory and practice?

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- ApproxCount$(F, W, \varepsilon, \delta)$: Compute C such that

$$\Pr[\frac{W(F)}{1+\varepsilon} \leq C \leq W(F)(1+\varepsilon)] \geq 1-\delta$$

From Weighted to Unweighted Counting

Boolean Formula F and weight function $W : \{0,1\}^n \to \mathbb{Q}^{\geq 0}$

Boolean Formula F'

$$W(F) = c(W) \times |\mathsf{Sol}(F')|$$

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How do we estimate $|\mathsf{Sol}(F')|$?

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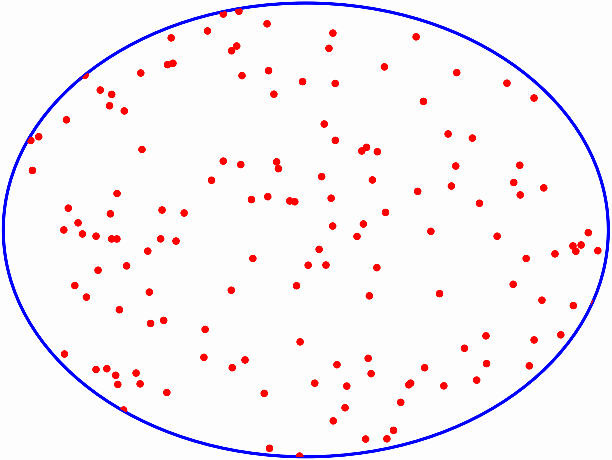
  - Potentially 2^n queries

Can we do with lesser # of SAT queries – $\mathcal{O}(n)$ or $\mathcal{O}(\log n)$?

As Simple as Counting Dots



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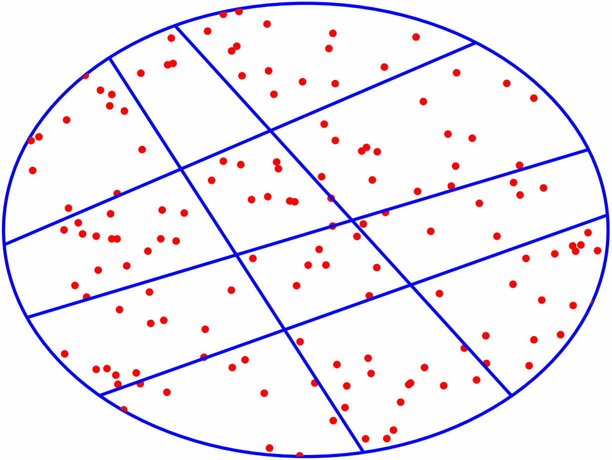
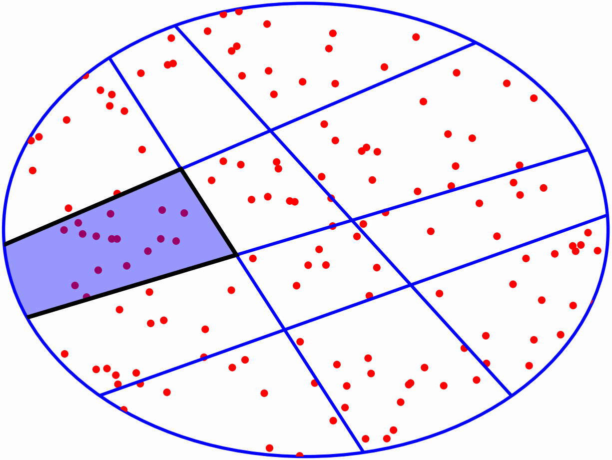


Pick a random cell



Estimate = Number of solutions in a cell × Number of cells

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- Deterministic h unlikely to work
- Choose h randomly from a large family H of hash functions

Universal Hashing (Carter and Wegman 1977)

2-Universal Hashing

- Let H be family of 2-universal hash functions mapping $\{0,1\}^n$ to $\{0,1\}^m$

$$\forall y_1, y_2 \in \{0,1\}^n, \alpha_1, \alpha_2 \in \{0,1\}^m, h \xleftarrow{R} H$$
$$\Pr[h(y_1) = \alpha_1] = \Pr[h(y_2) = \alpha_2] = \left(\frac{1}{2^m}\right)$$

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- The power of 2-universality
  - Z be the number of solutions in a randomly chosen cell
  - $\mathsf{E}[Z] = \frac{|\mathsf{Sol}(F)|}{2^m}$
  - $\sigma^2[Z] \leq \mathsf{E}[Z]$

2-Universal Hash Functions

- Variables: $X_1, X_2, \cdots X_n$
- To construct $h : \{0,1\}^n \rightarrow \{0,1\}^m$, choose m random XORs
- Pick every X_i with prob. $\frac{1}{2}$ and XOR them
  - $X_1 \oplus X_3 \oplus X_6 \cdots \oplus X_{n-2}$
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- Solutions in a cell: $F \wedge Q_1 \cdots \wedge Q_m$
- Performance of state of the art SAT solvers degrade with increase in the size of XORs (SAT Solvers != SAT oracles)

Improved Universal Hash Functions

- Not all variables are required to specify solution space of F
  - $F := X_3 \iff (X_1 \vee X_2)$
  - X_1 and X_2 uniquely determines rest of the variables (i.e., X_3)
- Formally: if I is independent support, then $\forall \sigma_1, \sigma_2 \in \mathsf{Sol}(F)$, if σ_1 and σ_2 agree on I then $\sigma_1 = \sigma_2$
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Algorithmic procedure to determine I?

- FP^{NP} procedure via reduction to Minimal Unsatisfiable Subset
- Two orders of magnitude runtime improvement

(IMMV; CP15, Constraints16)

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- Independent Support-based 2-Universal Hash Functions

Challenge 2 How many cells?

Question 2: How many cells?

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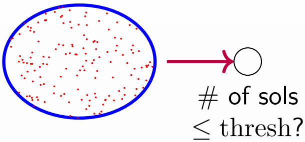
  - Check for every $m = 0, 1, \cdots n$ if the number of solutions $\leq \mathrm{thresh}$

ApproxMC(F, ε, δ)



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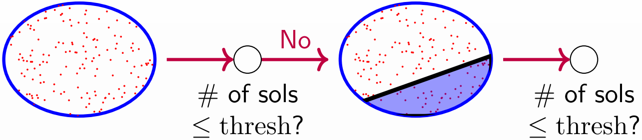

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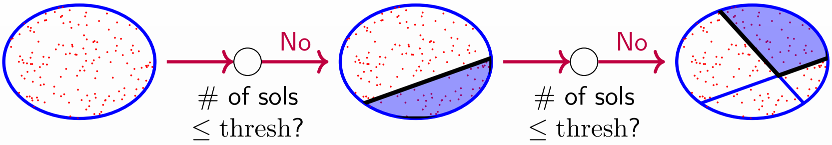

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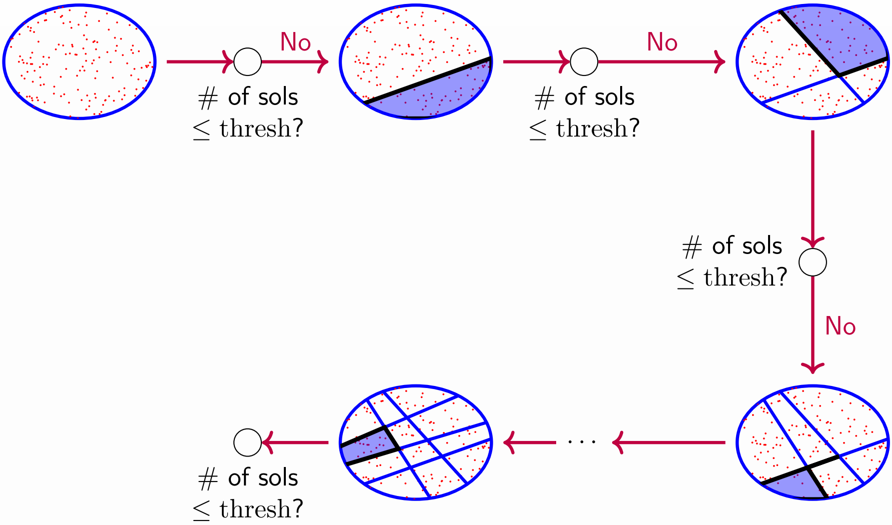

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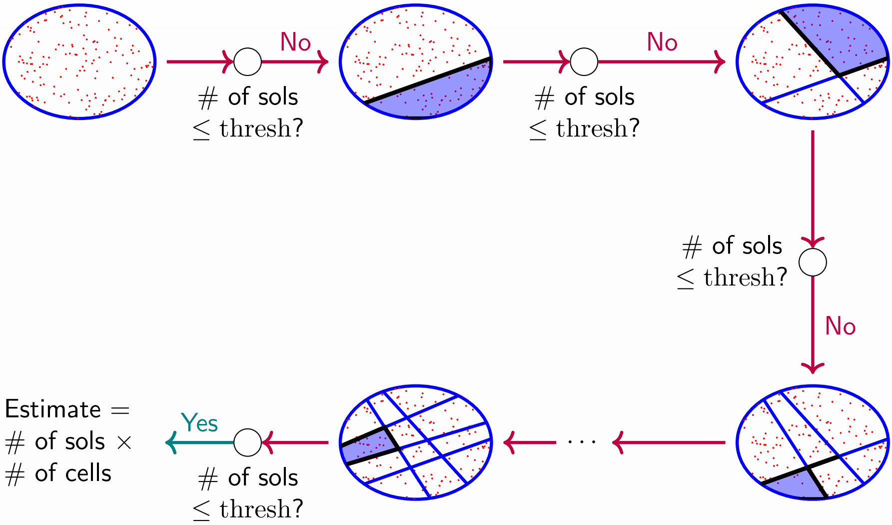

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  - Challenge Query i and Query j are not independent
  - Independence crucial to analysis (Stockmeyer 1983, $\cdots$)
  - Key Insight: The probability of making a bad choice of Q_i is very small for $i \ll m^*$

(CMV, IJCAI16)

Taming the Curse of Dependence

Let $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\text{thresh}}$ $(m^* = \log(\frac{|\mathsf{Sol}(F)|}{\text{thresh}}))$

Lemma (1)

ApproxMC (F, ε, δ) *terminates with* $m \in \{m^* - 1, m^*\}$ *with probability* $\geq$ *0.8*

Lemma (2)

For $m \in \{m^* - 1, m^*\}$, *estimate obtained from a randomly picked cell lies within a tolerance of* ε *of* $|\mathsf{Sol}(F)|$ *with probability* $\geq$ *0.8*

ApproxMC(F, ε, δ)

Theorem (Correctness)

$$\Pr\left[\frac{|\mathrm{Sol}(F)|}{1+\varepsilon} \leq ApproxMC(F, \varepsilon, \delta) \leq |\mathrm{Sol}(F)|(1+\varepsilon)\right] \geq 1 - \delta$$

Theorem (Complexity)

ApproxMC(F, ε, δ) makes $\mathcal{O}(\frac{\log n \log(\frac{1}{\delta})}{\varepsilon^2})$ calls to SAT oracle.

- *Prior work required $\mathcal{O}(\frac{\boldsymbol{n} \log \boldsymbol{n} \log(\frac{1}{\delta})}{\varepsilon})$ calls to SAT oracle (Stockmeyer 1983)*

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Theorem (FPRAS for DNF; (MSV, FSTTCS 17; CP 18, IJCAI-19))

If F is a DNF formula, then ApproxMC is FPRAS – fundamentally different from the only other known FPRAS for DNF (Karp, Luby 1983)

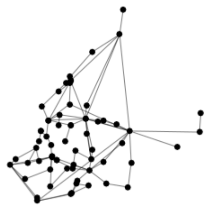
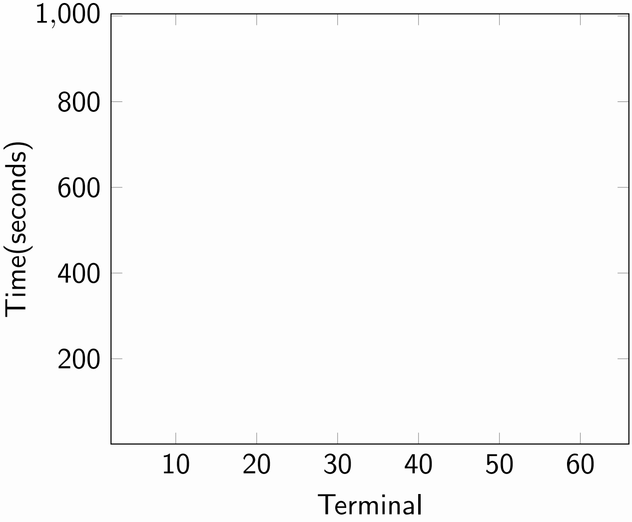


Figure: Plantersville, SC

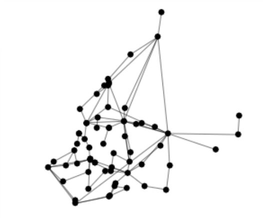


- $G = (V, E)$; source node: s
- Compute Pr[t is disconnected]?

Timeout = 1000 seconds

(DMPV, AAAI17)

Reliability of Critical Infrastructure Networks



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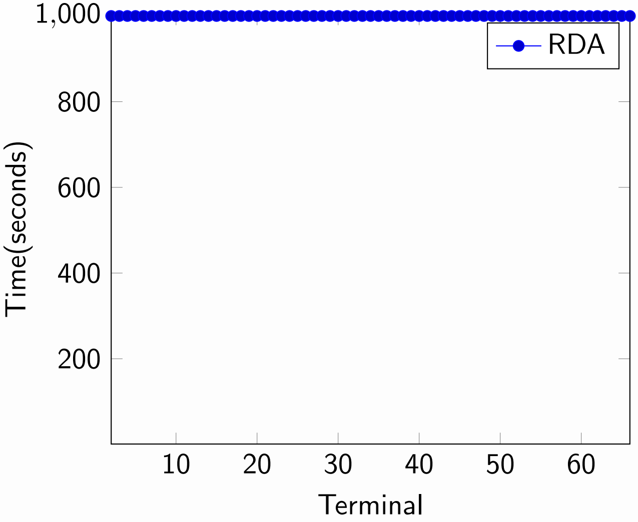

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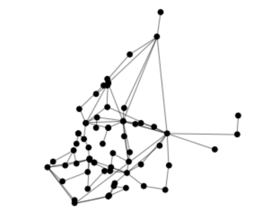

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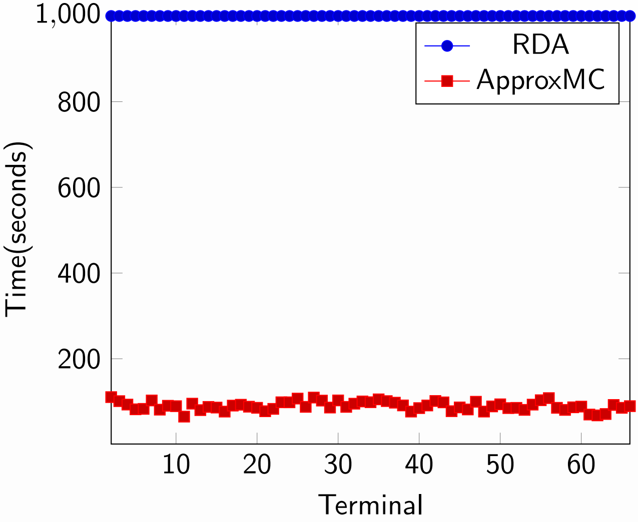

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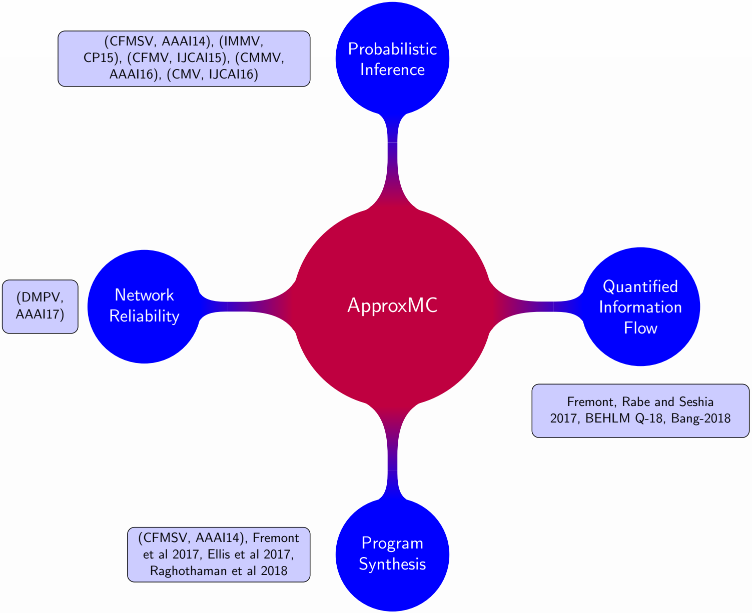
Beyond Network Reliability



Network Reliability

Probabilistic Inference

Constrained Counting

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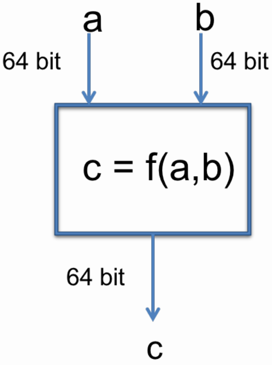
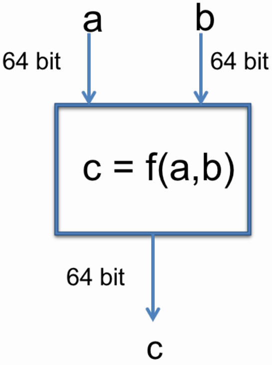


- Design is simulated with test vectors (values of *a* and *b*)
- Results from simulation compared to intended results

Hardware Validation



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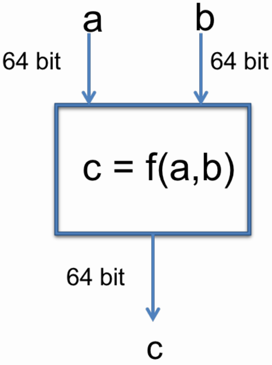

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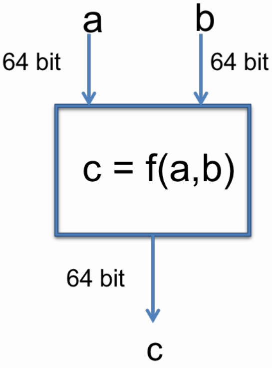
- Use constraints to represent *interesting* verification scenarios



Constraints

- Designers:
  - $a +_{64} 11 * 32b = 12$
  - $a <_{64} (b >> 4)$
- Past Experience:
  - $40 <_{64} 34 + a <_{64} 5050$
  - $120 <_{64} b <_{64} 230$
- Users:
  - $232 * 32a +_{64} b! = 1100$
  - $1020 <_{64} (b/_{64}2) +_{64} a <_{64} 2200$

Test vectors: random solutions of constraints

Constrained Sampling

- Given:
  - Set of Constraints F over variables $X_1, X_2, \cdots X_n$
- Uniform Sampler

$$\forall y \in \mathsf{Sol}(F), \mathsf{Pr}[\text{y is output}] = \frac{1}{|\mathsf{Sol}(F)|}$$

- Almost-Uniform Sampler

$$\forall y \in \mathsf{Sol}(F), \frac{1}{(1+\varepsilon)|\mathsf{Sol}(F)|} \leq \mathsf{Pr}[\text{y is output}] \leq \frac{(1+\varepsilon)}{|\mathsf{Sol}(F)|}$$

Prior Work

Strong guarantees but poor scalability

- Polynomial calls to NP oracle (Bellare, Goldreich and Petrank,2000)
- BDD-based techniques (Yuan et al 1999, Yuan et al 2004, Kukula and Shiple 2000)
- Reduction to approximate counting (Jerrum, Valiant and Vazirani 1986)

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- Randomization in SAT solvers (Moskewicz 2001, Nadel 2011, Dutra Bachrach and Sen 2018)
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How to bridge this gap between theory and practice?

Close Cousins: Counting and Sampling

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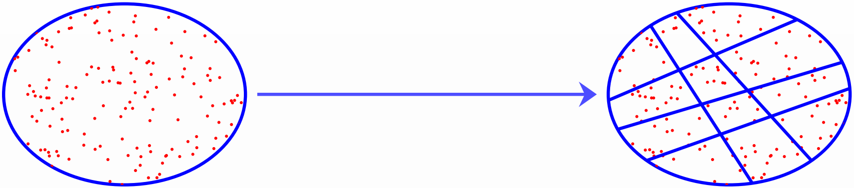
- Approximate counting and almost-uniform sampling are inter-reducible (Jerrum, Valiant and Vazirani, 1986)
- Is the reduction efficient?
  - Almost-uniform sampler (JVV) require linear number of approximate counting calls

Key Ideas



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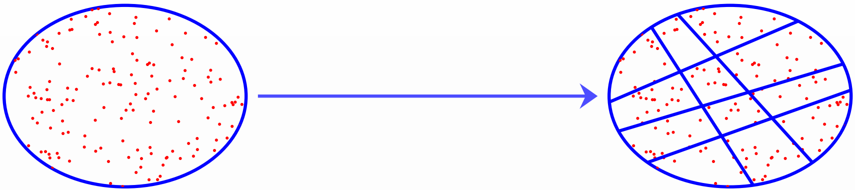

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  - Not just a practical hack required non-trivial proof

(CMV; DAC14),

(CFMSV; AAAI14, TACAS15),

(SGRM; LPAR18,TACAS19)

Theoretical Guarantees

Theorem (Almost-Uniformity)

$$\forall y \in \mathsf{Sol}(F),\ \frac{1}{(1+\varepsilon)|\mathsf{Sol}(F)|} \leq \Pr[y \textit{ is output}] \leq \frac{1+\varepsilon}{|\mathsf{Sol}(F)|}$$

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- *Prior work required* **n** *calls to approximate counter (Jerrum, Valiant and Vazirani, 1986)*

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Desired Uniform Generator	10

Experiments over 200+ benchmarks

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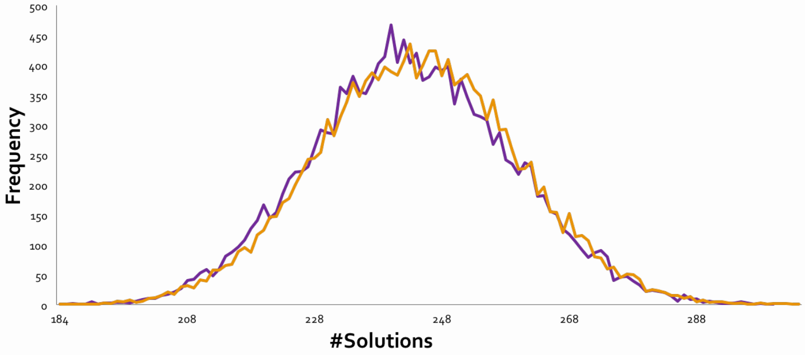
Experiments over 200+ benchmarks

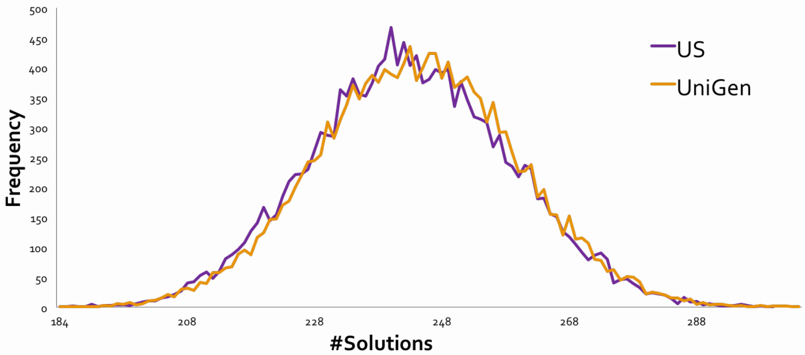
Closer to technical transfer

Quiz Time: Uniformity



- Benchmark: case110.cnf; #var: 287; #clauses: 1263
- Total Runs: 4×10^6; Total Solutions : 16384

Statistically Indistinguishable



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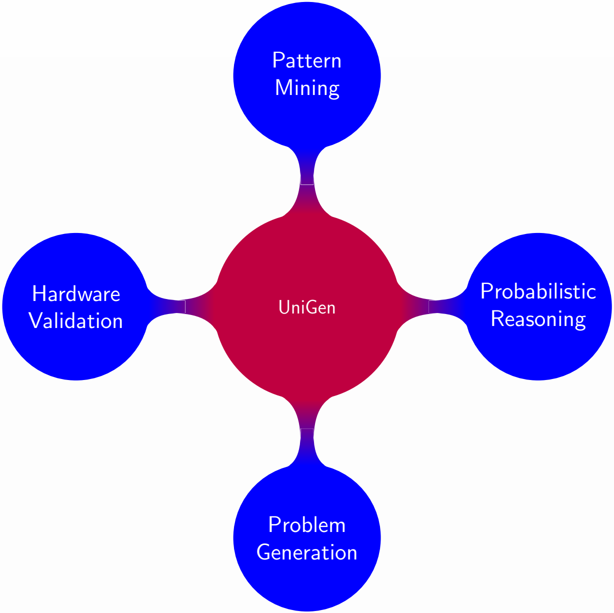
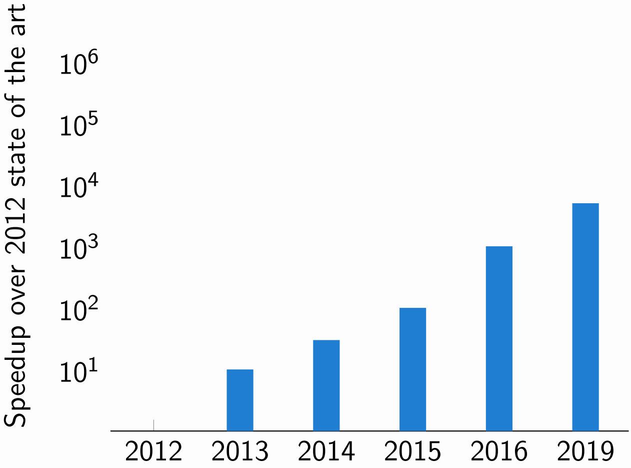
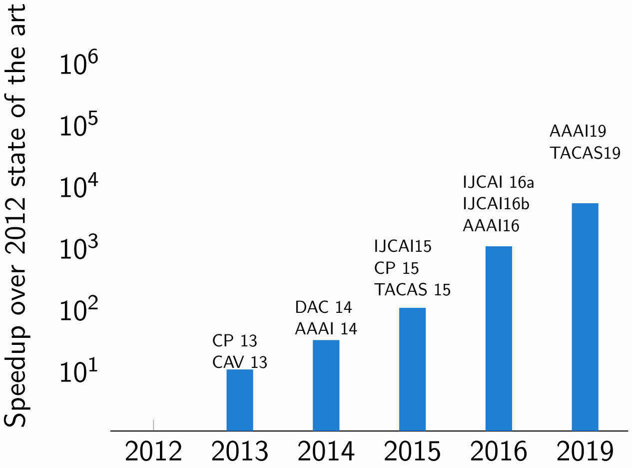
Usages of Open Source Tool: UniGen




Speedup over 2012 state of the art
10^6
10^5
10^4
10^3
10^2
10^1
2012
2013
2014
2015
2016
2019

Mission 2025: Constrained Counting and Sampling Revolution



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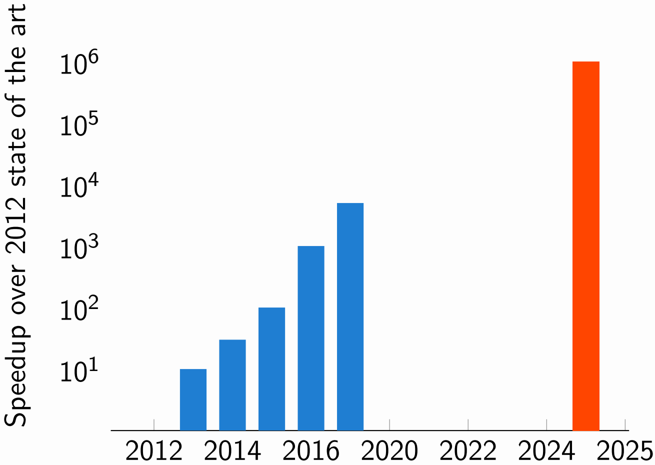

Requires combinations of ideas from theory, statistics and systems

Mission 2025: Constrained Counting and Sampling Revolution

Challenge Problems

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Civil Engineering Reliability for Los Angeles Transmission Grid

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- Handling weighted distributions: Connections to theory of integration
- Verification of sampling and counting (CM, AAAI19)
- Designing hardware accelerators – similar to advances in deep learning
- Understanding and applying sampling and counting to real world use-cases

We can only see a short distance ahead but we can see plenty there that needs to be done (Turing, 1950)