The number $e$ is transcendental$^1$.

Proof. In the proof we shall use the standard notation $f^{(i)}(x)$ to denote the $i$th derivative of $f(x)$ with respect to $x$.

Suppose that $f(x)$ is a polynomial of degree $r$ with real coefficients. Let $F(x) = f(x) + f^{(1)}(x) + f^{(2)}(x) + \cdots + f^{(r)}(x)$. We compute $(d/dx)(e^{-x}F(x))$; using the fact that $f^{(r+1)}(x) = 0$ (since $f(x)$ is of degree $r$) and the basic property of $e$, namely that $(d/dx)e^x = e^x$, we obtain $(d/dx)(e^{-x}F(x)) = -e^{-x}f(x)$.

The mean value theorem asserts that if $g(x)$ is a continuously differentiable, single-valued function on the closed interval $[x_1, x_2]$ then

\[
\frac{g(x_1) - g(x_2)}{x_1 - x_2} = g^{(1)}(x_1 + \theta(x_2 - x_1)), \quad \text{where } 0 < \theta < 1.
\]

We apply this to our function $e^{-x}F(x)$ which certainly satisfies all the required conditions for the mean value theorem on the closed interval $[x_1, x_2]$ where $x_1 = 0$ and $x_2 = k$, where $k$ is any positive integer. We then obtain that $e^{-k}F(k) - F(0) = -e^{-\theta_k}f(\theta_k k)k$, where $\theta_k$ depends on $k$ and is some real number between 0 and 1. Multiplying this relation through by $e^k$ yields $F(k) - F(0)e^k = -e^{(1-\theta_k)k}f(\theta_k k)$. We write this out explicitly:

\[
\begin{align*}
F(1) - eF(0) & = -e^{(1-\theta_1)}f(\theta_1) = \epsilon_1 \\
F(2) - e^2F(0) & = -2e^{2(1-\theta_2)}f(2\theta_2) = \epsilon_2 \\
& \vdots \\
F(n) - e^nF(0) & = -ne^{n(1-\theta_n)}f(n\theta_n) = \epsilon_n.
\end{align*}
\]

Suppose now that $e$ is an algebraic number; then it satisfies some relation of the form

\[
c_0 e^n + c_{n-1} e^{n-1} + \cdots + c_1 e + c_0 = 0,
\]

where $c_0, c_1, \ldots, c_n$ are integers and where $c_0 > 0$.

In the relations (1) let us multiply the first equation by $c_1$, the second by $c_2$, and so on; adding these up we get $c_1 F(1) + c_2 F(2) + \cdots + c_n F(n) - F(0)(c_1 e + c_2 e^2 + \cdots + c_n e^n) = c_1 \epsilon_1 + c_2 \epsilon_2 + \cdots + c_n \epsilon_n$.

In view of relation (2), $c_1 e + c_2 e^2 + \cdots + c_n e^n = -c_0$, whence the above equation simplifies to

\[
c_0 F(0) + c_1 F(1) + \cdots + c_n F(n) = c_1 \epsilon_1 + \cdots + c_n \epsilon_n.
\]

All this discussion has held for the $F(x)$ constructed from an arbitrary polynomial $f(x)$. We now see what all this implies for a very specific polynomial, one first used by Hermite, namely,

\[
f(x) = \frac{1}{(p-1)!}x^{p-1}(1-x)^p(2-x)^p \cdots (n-x)^p.
\]

Here $p$ can be any prime number chosen so that $p > n$ and $p > c_0$. For this polynomial we shall take a very close look at $F(0)$, $F(1)$, $\ldots$, $F(n)$ and we shall carry out an estimate on the size of $\epsilon_1$, $\epsilon_2$, $\ldots$, $\epsilon_n$.

When expanded, $f(x)$ is a polynomial of the form

\[
\frac{(n!)^p}{(p-1)!}x^{p-1} + \frac{a_0 x^p}{(p-1)!} + \frac{a_1 x^{p+1}}{(p-1)!} + \cdots,
\]

where $a_0, a_1, \ldots$, are integers.

---

$^1$Herstein, Topics in Algebra, p. 176–178.
When $i \geq p$ we claim that $f^{(i)}(x)$ is a polynomial, with coefficients which are integers all of which are multiples of $p$. (Prove!) Thus for any integer $j$, $f^{(i)}(j)$, for $i \geq p$ is an integer and is a multiple of $p$.

Now, from its very definition, $f(x)$ has a root of multiplicity $p$ at $x = 1, 2, \ldots, n$. Thus for $j = 1, 2, \ldots, n$, $f(j) = 0$, $f^{(1)}(j) = 0$, $f^{(2)}(j) = 0$. However, $F(j) = f(j) + f^{(1)}(j) + \cdots + f^{(p-1)}(j)$; by the discussion above, for $j = 1, 2, \ldots, n$, $F(j)$ is an integer and is a multiple of $p$.

What about $F(0)$? Since $f(x)$ has a root of multiplicity $p - 1$ at $x = 0$, $f(0) = f^{(1)}(0) = \cdots = f^{(p-2)}(0) = 0$. For $i \geq p$, $f^{(i)}(0)$ is an integer which is a multiple of $p$. But $f^{(p-1)}(0) = (n!)^p$ and since $p > n$ and is a prime number, $p \nmid (n!)^p$ so that $f^{(p-1)}(0)$ is an integer not divisible by $p$. Since $F(0) = f(0) + f^{(1)}(0) + \cdots + f^{(p-2)}(0) + f^{(p-1)}(0) + f^{(p)}(0) + \cdots + f^{(r)}(0)$, we conclude that $F(0)$ is an integer not divisible by $p$. Because $c_0 > 0$ and $p > c_0$ and because $p \nmid F(0)$ whereas $p \mid F(1), p \mid F(2), \ldots p \mid F(n)$, we can assert that $c_0 F(0) + c_1 F(1) + \cdots + c_n F(n)$ is an integer and is not divisible by $p$.

However, by (3), $c_0 F(0) + c_1 F(1) + \cdots + c_n F(n) = c_1 \epsilon_1 + \cdots + c_n \epsilon_n$. What can we say about $\epsilon_i$? Let us recall that

$$\epsilon_i = \frac{-e^{i(1-\theta_i)}(1 - i\theta_i)^p \cdots (n - i\theta_i)^p (i\theta_i)^{p-1}i}{(p-1)!},$$

where $0 < \theta_i < 1$. Thus

$$|\epsilon_i| \leq e^p \frac{n^p (n!)^p}{(p-1)!}.$$ 

As $p \to \infty$,

$$\frac{e^p n^p (n!)^p}{(p-1)!} \to 0,$$

(Prove!) whence we can find a prime number larger than both $c_0$ and $n$ and large enough to force $|c_1 \epsilon_1 + \cdots + c_n \epsilon_n| < 1$. But $c_1 \epsilon_1 + \cdots + c_n \epsilon_n = c_0 F(0) + \cdots + c_n F(n)$, so must be an integer; since it is smaller than 1 in size our only possible conclusion is that $c_1 \epsilon_1 + \cdots + c_n \epsilon_n = 0$. Consequently, $c_0 F(0) + \cdots + c_n F(n) = 0$; this however is sheer nonsense, since we know that $p \nmid (c_0 F(0) + \cdots + c_n F(n))$, whereas $p \mid 0$. This contradiction, stemming from the assumption that $e$ is algebraic, proves that $e$ must be transcendental.