The cut property of Minimum Spanning Trees

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In this handout we use “graph” to mean “undirected, connected graph”, and “tree” to mean “free tree” (i.e., an undirected, connected, acyclic graph).

You already know that a tree with \( n \) nodes has \( n - 1 \) edges. Using this, you can easily prove the following fact: Adding an edge to a tree creates a unique cycle; removing any edge from that cycle yields a tree again. More precisely:

**Fact 1** Let \( T = (V, E) \) be a tree and \( e \) be an edge not in \( E \). Then the graph \( T^+ = (V, E \cup \{e\}) \) has a unique cycle. Furthermore, if \( e' \) is any edge on that cycle, the graph \( T' = (V, (E \cup \{e\}) - e') \) is a tree.

A cut of a graph \( G = (V, E) \) is a partition of the set of nodes \( V \) into two sets of nodes, \( S \subseteq V \) and \( V - S \); we denote this cut as the pair \((S, V - S)\). An edge \( e \) of \( G \) crosses the cut \((S, V - S)\) if one of the endpoints of \( e \) is in \( S \) and the other is in \( V - S \).

**Theorem 2 (Cut property of MSTs)** Let \( G = (V, E) \) be a graph, \((S, V - S)\) be a cut of \( G \), \( F \) be a set of edges of \( G \), and \( e \) be an edge of \( G \) such that

(a) \( F \) is contained in a MST of \( G \),
(b) no edge in \( F \) crosses the cut \((S, V - S)\), and
(c) \( e \) is a minimum weight edge that crosses the cut \((S, V - S)\).

Then \( F \cup \{e\} \) is also contained in a MST of \( G \).

Intuitively, this theorem says that we can extend a partial MST (i.e., a subset of the edges of an MST of a graph) to a better partial MST (i.e., an even bigger subset of the edges of an MST of the graph) as follows: Find any cut that no edge in \( F \) crosses, and pick a minimum weight edge \( e \) that crosses the cut; then \( F \cup \{e\} \) is the expanded partial MST. It is clear how such a property leads to greedy MST algorithms: We start with the trivial partial solution (an empty set of edges), and then we use the cut property to greedily expand this partial solution, until our “partial” solution contains \( n - 1 \) edges — i.e., is a full MST. The correctness of Prim’s and Kruskal’s MST algorithms is based on the cut property. (Seeing how is left as an exercise.) We now prove Theorem 2.

**Proof.** Let \( T \) be a MST that contains the edges in \( F \). (Such a MST exists by hypothesis (a) of the theorem.) If \( T \) contains \( e \), we are done. So, suppose \( T \) does not contain \( e \).

Let \( T^+ \) be the graph that results when we add \( e \) to \( T \). By Fact 1, \( T^+ \) has a unique cycle; let \( u \) and \( v \) be the endpoints of \( e \). By hypothesis (c), \( e \) crosses the cut \((S, V - S)\), so without loss of generality, suppose that \( u \in S \) and \( v \in V - S \). The unique cycle of \( T^+ \) consists of \( e \), and a path of edges in \( T \) that connects \( u \) to \( v \). This path contains an edge, say \( e' \), that has one endpoint in \( S \) and the other in \( V - S \); this is because \( u \in S \) and \( v \in V - S \). So, \( e' \) also crosses the cut \((S, V - S)\).

Since both \( e \) and \( e' \) cross the cut \((S, V - S)\), and \( e \) is a minimum weight edge that crosses this cut (see hypothesis (c)), \( \text{weight}(e) \leq \text{weight}(e') \). By Fact 1, removing \( e' \) from \( T^+ \) results in a spanning tree \( T' \) of \( G \). Recall that \( T' \) is constructed by adding \( e \) to \( T \) and then removing \( e' \) from the resulting graph. So,

\[
\text{weight}(T') = \text{weight}(T) + \text{weight}(e) - \text{weight}(e') \leq \text{weight}(T)
\]
(where the last inequality follows from the fact that $\text{weight}(e) \leq \text{weight}(e')$). Since (i) $T'$ is a spanning tree of $G$, (ii) $T$ is a MST of $G$, and (iii) $\text{weight}(T') \leq \text{weight}(T)$, it follows that

$$T' \text{ is also a MST of } G. \quad (1)$$

Next we argue that $T'$ contains all the edges in $F$. To see this recall that $T$ contains all the edges in $F$, and $e'$, the only edge of $T$ that $T'$ does not contain, is not in $F$. (This is because $e'$ crosses $(S, V - S)$ and no edge in $F$ crosses $(S, V - S)$, by hypothesis (b).) So, the edges of $T$ excluding $e'$ contain the edges in $F$. $T'$ contains all the edges of $T$ except $e'$, and so

$$T' \text{ contains the edges in } F. \quad (2)$$

By its definition,

$$T' \text{ contains the edge } e. \quad (3)$$

By (1), (2), and (3), $T'$ is a MST of $G$ that contains the edges in $F \cup \{e\}$, as wanted. $\square$