Question 2.  [12 marks]

Given an array of \( n \) integers \([x(1), \ldots, x(n)]\) define an increasing subsequence of \( x \), to be a subsequence \( x(j(1)) < x(j(2)) < \ldots < x(j(p)) \) where the indices \( j(i) \) are also strictly increasing with \( i \), that is, \( 1 \leq j(1) < j(2) < \ldots < j(p) \leq n \). We are interested in a longest increasing subsequence of \( x \), i.e., with the maximum length \( p \). Define \( LLIS(x) \) to be the length of a longest increasing subsequence of \( x \) (there may be more than one increasing subsequence with this length).

For example, given the array \( x = [22, 5, 8, -3, 10, 1] \) the longest increasing subsequence is \([5, 8, 10]\) which has length 3. Therefore, for this example, \( LLIS(x) = 3 \).

In addition, define the array \([q(1), \ldots, q(n)]\) to have elements \( q(k) \) which equal the length of the longest increasing subsequence of \( x \) ending with the value \( x(k) \). In particular, such a subsequence only uses values from the first \( k \) elements of \( x \), namely \([x(1), \ldots, x(k)]\) and must end with \( x(k) \) itself.

For the example array \( x \) above, \([q(1), \ldots, q(6)] = [1, 1, 2, 1, 3, 2] \).

Part (a)  [5 marks]

Give an equation which expresses \( q(k) \) for \( 1 < k \leq n \) in terms of the values of \( q(j) \) for various \( j < k \) (and possibly other simple expressions involving elements of the array \( x \)). Explain.

Solution:
\[
q(k) = 1 + \max \{0 \cup \{q(j) \mid 1 \leq j < k \text{ and } x(j) < x(k)\}\}
\]

Explanation: Two cases for the longest increasing subsequence ending at \( x(k) \):
Case 1: The subsequence is length 1, consisting of \( x(k) \) alone. This case is dealt with by including the term \( \{0\} \) in the max above (the second set could be empty).
Case 2: The longest subsequence ending at \( x(k) \) has length larger than 1.

Let the second last item in the subsequence be \( x(j) \) for some \( j < k \).

For the subsequence to be increasing, we require that \( x(j) < x(k) \).

Since we are looking for the longest subsequence, we must use the longest subsequence ending at \( x(j) \), which has length \( q(j) \).

With \( x(k) \) appended, the length of the resulting subsequence is \( 1 + q(j) \).

Therefore \( q(k) \geq 1 + q(j) \) for each such carefully chosen \( j \) (if any).

Finally, since we are looking for the longest increasing subsequence we want to maximize over all possible choices from these two cases.

Part (b)  [2 marks]

Express \( LLIS(x) \) in terms of the array \( q \). Explain.

Solution:
\[
LLIS(x) = \max\{q(k) \mid 1 \leq k \leq n\}.
\]

Explanation: The longest increasing subsequence has to end at some \( x(k) \), in which case it has length \( q(k) \). So we should choose the max \( q(k) \).
**Question 2. (continued)**

**Part (c) [5 marks]**

Given the arrays $x$ and $q$ defined above, provide pseudo-code for an algorithm which extracts a longest increasing subsequence. (Precisely worded English sentences describing simple steps are allowed in your pseudo-code.) In the previous notation, the algorithm should return $[x(j(1)), x(j(2)), \ldots, x(j(p))]$ for the maximum $p$. The $j(k)$’s themselves do not need to be returned, just the corresponding $x$ values (in order). Briefly explain why your algorithm is correct.

**Solution:**

```plaintext
[s] = LIS(x, q, n)

Find an index $k$ such that $q(k)$ is the maximum of $[q(1), \ldots, q(n)]$.

$s = []$  // $s$ is initially an empty list.

while $q(k) > 1$

    $s = [x(k) \ s]$  // Prepend $s$ with $x(k)$.

    if $q(k) == 1$
        return $s$

    $k = j$

end

Explanation:

As in part (b) above, the longest increasing subsequence ends at $x(k)$, for a $k$ where $q(k)$ is maximal. For $q(k) > 1$, the second-last item $x(j)$ must have $q(j) = q(k) - 1$, since the subsequence ending at $x(j)$ is just one shorter.

This second-last item $x(j)$ must also satisfy $x(j) < x(k)$ since the subsequence is increasing.

The algorithm finds exactly such a $j$ and iterates.

Each iteration decreases $q(k)$ by one, until the $q(k) = 1$.

For $q(k) = 1$ we know the first element of the sub-sequence is $x(k)$.