Question 1. [8 marks]
Suppose we have the linear programming problem (LP): max $c^T x$ subject to $Ax \leq b$ and $x \geq 0$. Here $A$ is a $m \times n$ matrix, $b$ is an $m$-vector, and $x$, $c$ are both $n$-vectors. We wish to make some changes to the constraints on $x$. For each separate cases below, indicate how $A$, $b$, $c$, and/or $x$ should be modified in order to reformulate the LP to include the desired changes. In each case, your **new linear program must be written in standard form**. If this is not possible for any of these constraints, clearly explain why. (You do not need to combine the constraints below with each other)

**Part (a) [2 marks]**
Add the constraint $u^T x = 1$ to the original LP, where $u$ is some given (constant) $n$-vector, and write the new LP (if possible) in standard form. Explain your solution.

**Part (b) [3 marks]**
Add the constraint $|w^T x - 1| \leq 2$ to the original LP, where $w$ is some given (constant) $n$-vector, and write the new LP (if possible) in standard form. Explain your solution.
Question 1.  (CONTINUED)

Part (c)  [3 MARKS]

Remove the constraint $x_1 \geq 0$. That is, we now wish to maximize $c^T x$ with respect to $x^T = (x_1, x_2, \ldots, x_n)$, where $x$ is subject to $Ax \leq b$, $x_k \geq 0$ for $k = 2, 3, \ldots n$, and with $x_1$ taking on any real value. Remember your new LP must be written in standard form. Explain your solution.
Question 2. [12 marks]

Given a standard max s-t flow problem \( G = (V, E, c) \) (see example below), with positive integer-valued capacities, \( c(e) \), we are interested enhancing the Ford-Fulkerson algorithm in order to find some alternative integer-valued maximal flows, if they exist.

In the example above, the edge capacities \( c(e) \) are denoted by the numbers on the edges, and the flow values \( f(e) \) are denoted by the numbers near the capacities (i.e., the value of the flow shown above is 1).

Part (a) [2 marks]
Sketch the residual graph \( G_f \) for the flow \( f(e) \) above. Is the given flow maximal? Briefly explain why your answer is true.

Part (b) [2 marks]
Give the set \( A \) for an s-t min cut \( (A, B) \) which would be computed given the algorithm described in class using the above flow \( f(e) \). Briefly explain.
Question 2. (CONTINUED)
Part (c) [8 MARKS]

Consider the general network flow problem \((V, E, c)\) with source and sink vertices \(s, t \in V\) (i.e., \(s\) has no incoming-edges and \(t\) no outgoing edges), and with positive integer-valued capacities \(c(e)\). Suppose we compute a max flow \(f(e)\) using Ford-Fulkerson with flow value \(v(f) > 0\). Now consider any particular saturated edge \(e_0 \in E\). Precisely describe an \(O(n+m)\) algorithm which returns true or false depending on whether an alternative integer-valued flow \(g(e)\) exists for which \(g(e_0) < f(e_0)\) and yet the value of the flow is still maximal (i.e., \(v(g) = v(f)\)). (Hint: Such a \(g(e)\) exists for the example flow given above for \(e_0 = (a, c)\) but not for \(e_0 = (z, t)\).)

In this algorithm description you can assume the reader knows about the residual graph, breadth/depth first search, a module for finding an augmenting path and its bottleneck flow. Do not describe these items in detail. Only explain how you might use them to solve this problem. **The description of your algorithm does not need to be in pseudo-code.** Briefly explain why your algorithm is \(O(n + m)\) and correct. Continue writing on the back of this page if necessary.