See the Matlab files in a205soln.zip for the solution source code.

1. **Fourier Transform of Comb Functions [10pts]:** Let \( f(n) = \text{Comb}(n; n_s) \). Define \( \omega_k = \frac{2\pi}{N} k \) and \( m = N/n_s \), which is assumed to be an integer. Then, by the definition of the DFT,

\[
\hat{f}(k) \equiv \sum_{n=0}^{N-1} f(n)e^{-i\omega_k n} = \sum_{n=0}^{N/m-1} \text{Comb}(n; n_s)e^{-i\omega_k n} \\
= \sum_{j=0}^{N/m-1} e^{-i\omega_k n_s j} \quad \text{since Comb}(n; n_s) \text{ is 1 only for } n = jn_s \\
= \sum_{j=0}^{m-1} e^{-i\frac{2\pi}{m} n_s j} \\
= \sum_{j=0}^{m-1} e^{-i2\pi m \frac{j}{m}}. \quad (1)
\]

A similar argument to the one given in section 1.3 of the Fourier notes shows that the last sum above is equal to \( m\text{Comb}(k; m) \). To show this, there are two cases to consider, depending on whether \( k/m \) is an integer or not.

First, suppose \( k/m \) is an integer. Then every term in the sum in equation (1) is \( e^{-i2\pi \frac{j}{m}} = 1 \). So the sum is the number of terms, namely \( m = N/n_s \).

Alternatively, suppose \( k/m \) is not an integer. Let \( z \) be the sum in (1). Then,

\[
z e^{-i2\pi \frac{k}{m}} = \sum_{j=0}^{m-1} e^{-i2\pi \frac{1}{m} j} e^{-i2\pi \frac{k}{m}} = \sum_{j=0}^{m-1} e^{-i2\pi \frac{1}{m} (j+1)} = \sum_{j=1}^{m-1} e^{-i2\pi \frac{k}{m} j} \\
= e^{-i2\pi \frac{k}{m}} + \sum_{j=1}^{m-1} e^{-i2\pi \frac{k}{m} j} = e^{-i2\pi k} + \sum_{j=1}^{m-1} e^{-i2\pi \frac{k}{m} j} \\
= 1 + \sum_{j=1}^{m-1} e^{-i2\pi \frac{k}{m} j} = \sum_{j=0}^{m-1} e^{-i2\pi \frac{k}{m} j} = z, \quad \text{by equation 1.}
\]

Therefore we have shown that \( ze^{-i2\pi \frac{k}{m}} = z \). That is, \( z(e^{-i2\pi \frac{k}{m}} - 1) = 0 \). Since we are assuming that \( \frac{k}{m} \) is not an integer, \( e^{-i2\pi \frac{k}{m}} \neq 1 \), and therefore it follows that \( z \) must be zero.

Therefore, we have shown that

\[
\hat{f}(k) = \begin{cases} 
  m & \text{for } k/m \text{ an integer,} \\
  0 & \text{otherwise.} 
\end{cases} \quad (2)
\]

That is, \( \hat{f}(k) = m\text{Comb}(k; m) = \frac{N}{n_s}\text{Comb}(k; N/n_s) \).

2. **Convolutions of Fourier Transforms [10pts]:** Let \( s(n) = f(n)g(n) \). Then

\[
\mathcal{F}(s) = \hat{s}(k) = \sum_{n=0}^{N-1} f(n)g(n)e^{-i\omega_k n}, \quad (3)
\]
for \( \omega_k = \frac{2\pi}{N} k \).

On the other hand, let \( \hat{f}(k) \) and \( \hat{g}(k) \) be the Fourier transforms of \( f(n) \) and \( g(n) \), respectively. By the definition of convolution of \( N \)-periodic functions we have

\[
(\hat{f} \ast \hat{g})(k) = \sum_{j=0}^{N-1} \hat{f}(j)\hat{g}(k-j),
\]

\[
= \sum_{j=0}^{N-1} \sum_{n=0}^{N-1} f(n)e^{-i\frac{2\pi}{N}jn} g(m)e^{-i\frac{2\pi}{N}(k-j)m} \quad \text{by the definition of the DFT,}
\]

\[
= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(n)g(m) e^{-i\frac{2\pi}{N}(jn+(k-j)m)},
\]

\[
= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(n)g(m)e^{-i\frac{2\pi}{N}km} \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j(n-m)}. \quad (4)
\]

Notice that since \( 0 \leq n, m \leq N - 1 \) it must be the case that \((n - m)/N\) is not an integer unless \( n = m \). When \( n = m \) it follows that the last sum in (4) is equal to \( N \) (since each term is simply 1 and there are \( N \) terms). Alternatively, when \( n \neq m \) a similar argument to the one in Question 1 shows that this sum is 0. Therefore, equation (4) simplifies to

\[
(\hat{f} \ast \hat{g})(k) = N \sum_{n=0}^{N-1} f(n)g(n)e^{-i\frac{2\pi}{N}kn} = N\hat{s}(k) \quad \text{from eqn (3),}
\]

which is what we were asked to show.


4. **Perform the Warp by Looping Over Pixels [10pts]**: Aliasing will occur in situations where the output image involves a significant shrinkage of a textured region in one of the original images. In such a case this textured region will be downsampled (but not blurred) in the warped image. An example of such a region is Reagan’s upper lip, which is shrunk significantly for small values of \( s \). [2pts] (Remaining [8pts] for the implementation.)

5. **Perform the Warps Using Interp2 [10pts]**: See morphSoln.zip.