Today's motivating example:

Determine explicitly all the numbers x such that \( x + \sqrt{x} = 2 \).

The word "numbers" means "real numbers" from now on (versus "natural numbers", "integers", "rational numbers", etc. We'll discuss those later).

Properly understanding and communicating 'solving' by 'manipulating equations' requires and illuminates the concept of Universally Quantified Implications.

What you might be used to from pre-UofT:

\[
\sqrt{x} = 2 - x \\
x = (2 - x)^2 \\
x = 4 - 4x + x^2 \\
0 = 4 - 5x + x^2 \\
0 = (x - 1)(x - 4) \\
x = 1 \text{ or } x = 4
\]

I read those out in lecture as complete boolean English sentences.

Exercise 2.0: write them out unabbreviated, or at least read them out loud in full English again.

But what was wrong with that collection of sentences?

To answer that questions, let's trace the proof. Note that solving an equation is a proof: but of what kind of claim? That too will emerge from carefully analyzing the meaning of steps we took.
The formal use of "or" is "inclusive". It means at least one of the claims it connects is true. This is how "or" behaves in most programming languages. In English the inclusive-or is sometimes made unambiguous by using "and/or".

More unsettling for some people is when not all the parts of an "or" claim can "actually happen", e.g. "1 = 1 or 1 = 4" is true, but that doesn't claim that 1 = 4 is 'possible', i.e., that there are situations where 1 = 4.

Exercise 2.1: trace the proof with another number.

Some common reactions to the "x = 4":

Algebraic error or typo.

No, but tracing proofs helps catch those (the TAs will do it a lot to help understand your proof and help find the source of errors).

\[ \sqrt{4} = \pm 2 \ ? \]

That is not the convention (in math, you calculator, Python, etc), since then the square root function wouldn't be a function.

Regardless, we would still have to explain this silly but 'within (bad) high school rules proof' of the 'identity' -1 = 1:

\[ -1 = 1 \]

\[ (-1)^2 = 1^2 \]

\[ 1 = 1, \text{ which is true.} \]
Is squaring both sides of an equality 'wrong'? Can't we 'do' the same 'thing' to both sides of an equation?

What does "equation" mean precisely? E.g., this is false:

For all numbers $x$: if $x < 2$ then $x^2 < 4$.

Exercise 2.2: prove that is false, in the way we disprove Universally Quantified Implications.

But we 'squared an equality', not an inequality.

Every result of a 'solution' must be checked, not just to catch typos and algebraic errors.

Is that expected in all your courses?
And how could we ever manage that in situations that produce an infinite number of solutions to check (e.g. for "$x + \sqrt{x} > 2$")?

If 'solving' an equation can produce extra results, what reason is there to believe that it can't miss some results?

The main source of trouble is that the 'solution' was not a paragraph: it was just an unconnected sequence of claims. Let's fix that:

Let $x$ be a number. [programming analogy: parameter to the proof, along with type declaration]
Assume $x + \sqrt{x} = 2$. [programming analogy: a pre-condition to the proof]
Then $\sqrt{x} = 2 - x$.
Then $x = (2 - x)^2$.
Then $x = 4 - 4x + x^2$.
Then $0 = 4 - 5x + x^2$.
Then $0 = (x - 1)(x - 4)$.
Then $x = 1$ or $x = 4$.
Therefore, for any number $x$: if $x + \sqrt{x} = 2$ then $[x = 1$ or $x = 4]$.
 [programming analogy: the post-condition of the proof]

That conclusion is a Universally Quantified Implication.
Take the course's standard approach to exploring it:

....

So is the conclusion true?

Can it miss any solutions? Can it generate extra solutions?
In this case the "Then"s create Universally Quantified Implications by joining two equation claims. E.g. the first one is that:

For all numbers \( x \): if \( x + \sqrt{x} = 2 \) then \( \sqrt{x} = 2 - x \).

That is a special case of this known true Universally Quantified Implication:

For all numbers \( a \) and \( b \) and \( c \): if \( a = b \) then \( a - c = b - c \).

The second "Then" is claiming a special case of:

For all numbers \( a \) and \( b \): if \( a = b \) then \( a^2 = b^2 \).

Exercise 2.3: Use our techniques on the converse of that claim:

For all numbers \( a \) and \( b \): if \( a^2 = b^2 \) then \( a = b \).

Exercise 2.4: Write out the Universally Quantified Implications that the rest of the "Then"s mean.

Each (true) Universally Quantified Implication preserves true claims, so (assuming you did Exercise 2.1) you've now seen the three possibilities, based on the number traced:

I. Starts true and continues to be true.
II. Starts false and continues to be false.
III. Starts false and becomes true.

Exercise 2.5: What should not happen?

In terms of the sets of numbers for which the equation claims are true: the set can only get bigger from claim to claim. Using true Universally Quantified Implications can't make a claim 'less true', but can make it 'more true': enlarge the circle in the Venn Diagram, but not shrink it.

Exercise 2.6: What does this explain about the 'proof' above that \(-1 = 1\)? What does this mean for ALL 'proofs' in the form of a claim that is assumed and then true Universally Quantified Implications are used to prove a known true claim?

So now we know that at there are at MOST two solutions to the original equation, namely: \( x = 1 \) or \( x = 4 \).

Exercise 2.7: Take the course's standard approach to explore the converse of the conclusion:

For all numbers \( x \): if \( [x = 1 \text{ or } x = 4] \) then \( x + \sqrt{x} = 2 \).
But we checked, and found that $x = 4$ is NOT a solution, so the result of the proof can be strengthened:

For all numbers $x$: if $x + \sqrt{x} = 2$ then $x = 1$.

This still does not claim that $x = 1$ IS a solution.

We checked and found that $x = 1$ IS a solution, which can also be summarized as a "converse" Universally Quantified Implication.

For all numbers $x$: if $x = 1$ then $x + \sqrt{x} = 2$.

Together, those two Universally Quantified Implications say that the equation has exactly one solution: when $x = 1$.

Solving an equation means proving a particular Universally Quantified Implication, and its converse.

Note that, while you might have had pre-UofT courses that presented math as 'recipes' and 'what you are allowed to do' [and maybe some UofT courses are or seem that way], math is actually about making claims and determining which are true and which are false, based on knowing that certain other claims are true, and certain other claims are false.

A proof is a convincing argument for a certain audience. First for yourself, often for others. Just as in commenting your code, the goal is to help someone understand: often your future self, which you might have already experienced if you've programmed and read old programs you wrote. Practically, you make sure that you have saved the reader effort. Assume the reader does not know whether the particular claims are true versus false, and don't make the reader have to work it out themselves to check. In particular, imagine a fellow CSC165 student who does not already understand the problem: the TAs will pretend they are not past the level of a good CSC165 student, and pretend that they have not already done the reasoning you are presenting.