For this assignment you may work in a group of up to two other students in the course.
You must type up your solutions and produce a PDF. Submission details will be posted later on the course web page. If you worked in a group, the your group will submit one copy only.
You have one grace day usable for one of the three assignments. It allows you to submit up to 24 hours after the due time. If you work in a group and use the grace day, then that uses the grace day for everyone in the group.
You may not discuss this assignment with anyone outside your group, except the course instructors, and course TAs, and the CS Help Centre TAs. If you discuss this assignment with a CS Help Centre TA then you must inform them that you are discussing an assignment.
All proofs and disproofs must be in this course’s standard format. Each line (which could span multiple physical lines if it is a chain of equalities and/or inequalities) must be readable as an English sentence. Logical notation may only be used in the approved forms. Each line must be annotated as (exactly one of): the introduction of a variable, an assumption, or (possibly by default) a deduction (which includes stating well-known true claims).
Marks will be taken off for each line that violates the course standards, and the marker will stop reading when that makes the rest of the proof or disproof essentially meaningless or incorrect or the mark has already been reduced to zero, even if later parts of the proof or disproof are “correct”.

1. (a) Fill the blanks in the following claim with numbers \(-2, -1, 0, 1,\) and/or 2. More than one blank can contain the same number. Do not “make” new numbers (e.g. “-21”): the given numbers are not expressions nor digits so don’t “concatenate” them. Make the claim true and as strong as possible.

\[
\text{For all integers } k: \text{ if } k \geq \_ \text{ then } \frac{1}{(k + \_)^2} < \frac{1}{k} - \frac{1}{k + 1} < \frac{1}{(k + \_)^2}.
\]

(b) Do the same as in (a), but for the following claim:

\[
\text{For all integers } k: \text{ if } k \geq \_ \text{ then } \frac{1}{(k + \_)^2} < \frac{1}{k + 1} - \frac{1}{k + 2} < \frac{1}{(k + \_)^2}.
\]

(c) Do the same as in (a), but for the following claim:

\[
\text{For all integers } k: \text{ if } k \geq \_ \text{ then } (k + \_)^2 < k^2 + k < (k + \_)^2.
\]

(d) For each of (a) and (c): prove your claim.

If there is a stronger claim that would have been stronger (only) because of a different choice for the blank in “\(k \geq \_\)”, then disprove the weakest one of those.

If there is a weaker claim that would have been weaker (only) because of a different choice for the blank in “\(k \geq \_\)”, then prove the strongest one of those by using your claim.

(e) For (b): prove your claim by using (not repeating the proof of) your claim from (a).

2. For each real number \(x\) let \(Q(x)\) be defined as: “\(\exists k \in \mathbb{Z} : [\exists l \in \mathbb{Z}^* : x = \frac{k}{l}]\) ”.
In other words, for each real number \(x\) : \(Q(x)\) means “\(x\) is rational”. You may assume that \(\sqrt{2}\) is not rational.

(a) Prove or disprove:

\[
\forall x \in \mathbb{R} : [Q(x) \rightarrow Q(x + 1)].
\]

(b) Prove or disprove:

\[
\forall x \in \mathbb{R} : [[\neg Q(x)] \rightarrow \neg Q(x + 1)].
\]

(c) Prove or disprove:

\[
\forall x, y \in \mathbb{R} : [[\neg Q(x) \land \neg Q(y)] \rightarrow \neg Q(xy)].
\]

(d) Prove or disprove the following claim, and prove or disprove its converse:

\[
\forall x, y \in \mathbb{R} : [[\neg Q(xy)] \rightarrow [Q(x) \land \neg Q(y)]].
\]