1(a) For all integers \( k \): if \( k \geq 1 \) then \( \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{(k+1)} < \frac{1}{k^2} \). 

[Just putting in \( k = 0, k = 1, \) and \( k = 2 \) determines everything. 
It breaks easily if \( k = 0 \) is allowed, so use "\( k \geq 1 \)" or "\( k \geq 2 \)."

\( k = 1 \) easily breaks \( \frac{1}{(k+1)^2} \).
\( k = 2 \) easily breaks \( \frac{1}{(k+2)^2} \).
Using "\( k \geq 1 \): 
\( k = 1 \) easily makes each of the following false: 
\( \frac{1}{(k+0)^2} < \frac{1}{k} - \frac{1}{(k+1)} \)
\( \frac{1}{k} - \frac{1}{(k+1)} < \frac{1}{(k+2)^2} \)
\( \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{(k+1)} \)
\( \frac{1}{k} - \frac{1}{(k+1)} < \frac{1}{(k+2)^2} \)

\( k = 1 \) easily shows the first of the following is stronger, as does any other \( k \geq 1 \): 
If \( k \geq 1 \) then \( \frac{1}{(k+1)^2} < \frac{1}{k(k+1)} < \frac{1}{k^2} \).
If \( k \geq 1 \) then \( \frac{1}{(k+2)^2} < \frac{1}{k(k+1)} < \frac{1}{k^2} \).

Using "\( k \geq 2 \): 
The same process applies, since everything false for \( k = 1 \) is still false for \( k = 2 \).
Some things undefined become defined, but numeric conclusions are false or weaker.]

1(b) For all integers \( k \): if \( k \geq 0 \) then \( \frac{1}{(k+2)^2} < \frac{1}{(k+1)} - \frac{1}{(k+2)} < \frac{1}{(k+1)^2} \).

[Done by a similar process.
And writing out the first three relevant instances for 1(a) and for 1(b) produces identical inequalities:
\( \frac{1}{2^2} < \frac{1}{1} - \frac{1}{2} < \frac{1}{1^2} \): From 1(a) for \( k = 1 \), and 1(b) for \( k = 0 \).
\( \frac{1}{3^2} < \frac{1}{2} - \frac{1}{3} < \frac{1}{2^2} \): From 1(a) for \( k = 2 \), and 1(b) for \( k = 1 \).
\( \frac{1}{4^2} < \frac{1}{3} - \frac{1}{4} < \frac{1}{3^2} \): From 1(a) for \( k = 3 \), and 1(b) for \( k = 2 \).

So "calling" 1(a) with "\( k+1 \)" produces 1(b), as we'll see below when proving 1(b) from 1(a).]

1(c) For all integers \( k \): if \( k \geq 1 \) then \( k^2 < k^3 + k < (k + 1)^2 \).

1(d)

Proof of 1(c):

Let \( k \in \mathbb{Z} \).
Assume \( k \geq 1 \).
\( k^2 = k \cdot k \)
\( < k(k+1) \).  # Since \( 0 < 1 \leq k \), and \( k < 1 \).
\( k(k+1) < (k+1)(k+1) \). # Since \( k < k+1 \), and \( 0 < 1 \leq k < k+1 \).
\( = (k+1)^2 \).
\( k(k+1) = k^2 + k \).
\( k^2 + k < k(k+1) = k^3 + k \).
\( k^2 + k = k(k+1) < (k+1)^2 \).

Proof of 1(a):

Let \( k \in \mathbb{Z} \).
Assume \( k \geq 1 \).
\( \frac{1}{k^2} \).  # Since \( 0 < 1 \leq k \) so \( 0 < 1^2 \leq k^2 \).
\( < k^3 + k < (k+1)^3 \). # From the claim for 1(c), since \( k \geq 1 \).
\( 1/k^2 > 1/(k^3+k) > 1/(k+1)^3 \). # From the chain of positive comparisons.
\( 1/k - 1/(k+1) = (k+1-k)/k(k+1) \). # Since \( k \geq 1 \) and \( k+1 > k > 0 \), so \( k \neq 0 \) and \( k+1 \neq 0 \).
\( = 1/(k^3+k) \).
# Now substituting for \( 1/(k^3+k) \) in the previous inequality chain:
\( 1/k^2 > 1/k - 1/(k+1) > 1/(k+1)^3 \).

Disproof of "For all integers \( k \): if \( k \geq 0 \) then \( \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{(k+1)} < \frac{1}{k^2} \)."

The claim is false, by a counter-example.
Let \( k = 0 \).
\( k = 0 \in \mathbb{Z} \).
\( k = 0 \geq 0 \).
\( 1/k \) is undefined.
\( 1/(k+1)^2 \neq 1/k - 1/(k+1) \). # Since \( 1/k \) is undefined.
Disproof of "For all integers $k$: if $k \geq 0$ then $k^2 < k^2 + k < (k + 1)^2$".

The claim is false, by a counter-example.

Let $k = 0$.

$k = 0 \in \mathbb{Z}$.

$k = 0 \geq 0$.

$k^2 = 0 < 0 = 0^2 + 0 = k^2 + k$.

Proof of "For all integers $k$: if $k \geq 2$ then $k^2 < k^2 + k < (k + 1)^2$" from the original claim.

Let $k \in \mathbb{Z}$.

Assume $k \geq 2$.

$k \geq 2 \geq 1$.

$k^2 < k^2 + k < (k + 1)^2$. # From the original claim, since $k \in \mathbb{Z}$ and $k \geq 1$.

Proof of "For all integers $k$: if $k \geq 2$ then $1/(k+1)^2 < 1/k - 1/(k+1) < 1/k^2$." from the original claim.

Let $k \in \mathbb{Z}$.

Assume $k \geq 2$.

$k \geq 2 \geq 1$.

$1/(k+1)^2 < 1/k - 1/(k+1) < 1/k^2$. # From the original claim, since $k \in \mathbb{Z}$ and $k \geq 1$.

1(e)

Let $k \in \mathbb{Z}$.

Assume $k \geq 0$.

Let $k_0 = k + 1$.

$k_0 = k + 1 \in \mathbb{Z}$. # Since $k \in \mathbb{Z}$.

$k_0 = k + 1 \geq 0 + 1 = 1$. # Since $k \geq 0$.

$1/(k_0+1)^2 < 1/k_0 - 1/(k_0+1) < 1/k_0^2$. # From the original claim, # since $k_0 \in \mathbb{Z}$ and $k_0 \geq 1$.

$1/(k+1)^2 < 1/(k+1) - 1/((k+1)+1) < 1/(k+1)^2$.

$1/(k+2)^2 < 1/(k+1) - 1/(k+2) < 1/(k+1)^2$.

2(a) Proof.

Let $x \in \mathbb{R}$.

Assume $\exists k \in \mathbb{Z} : \exists l \in \mathbb{Z}^* : x = k/l$.

Let $k_0 \in \mathbb{Z}$.

Let $l_0 \in \mathbb{Z}^*$.

Assume $x = k_0/l_0$.

Let $k_i = k_0 + l_0$.

Let $l_i = l_0$.

$k_i = k_0 + l_0 \in \mathbb{Z}$. # Since $k_0 \in \mathbb{Z}$ and $l_0 \in \mathbb{Z}$.

$l_i = l_0 \in \mathbb{Z}^*$.

$x + 1 = (k_0/l_0) + 1 = (k_0 + l_0)/l_0 = k_i/l_i$.

2(b) Prove, by proving the contrapositive: $\forall x \in \mathbb{R} : \neg (Q(x+1) \rightarrow Q(x))$.

Let $x \in \mathbb{R}$.

Assume $\exists k \in \mathbb{Z} : \exists l \in \mathbb{Z}^* : x+1 = k/l$.

Let $k_0 \in \mathbb{Z}$.

Let $l_0 \in \mathbb{Z}^*$.

Assume $x+1 = k_0/l_0$.

Let $k_i = k_0 - l_0$.

Let $l_i = l_0$.

$k_i = k_0 - l_0 \in \mathbb{Z}$. # Since $k_0 \in \mathbb{Z}$ and $l_0 \in \mathbb{Z}$.

$l_i = l_0 \in \mathbb{Z}^*$.

$x = (x+1) - 1 = (k_0/l_0) - 1 = (k_0 - l_0)/l_0 = k_i/l_i$.
2(c)
The claim is false, by the counter-example $\sqrt{2}$ and $\sqrt{2}$.
Let $x = \sqrt{2}$. Let $y = \sqrt{2}$.
$x \in \mathbb{R}$. $y \in \mathbb{R}$.
$\neg Q(x)$. # By given assumption that $\sqrt{2}$ is not rational, i.e. $\neg Q(\sqrt{2})$.
$\neg Q(y)$. # For the same reason.
$xy = \sqrt{2} \cdot \sqrt{2} = 2 = 2/1$ is rational. # Since $2 \in \mathbb{Z}$ and $1 \in \mathbb{Z}^*$.
$Q(xy)$.
$\neg \neg Q(xy)$.

2(d)
[We've allowed a single-sentence format for disproof by counter-example, when the parts of the justification are simple enough to summarize well in one sentence. For the first disproof here, I'll use a somewhat compacted multi-sentence format that works well in such situations.]

The claim is false, by the counter-example $\sqrt{2}$ and 1.
Let $x = \sqrt{2} \in \mathbb{R}$, and let $y = 1 \in \mathbb{R}$.
$x = \sqrt{2}$ is not rational. # Given.
$xy = 1 \cdot \sqrt{2} = \sqrt{2}$ is not rational. # Given.
But it's not true that $x$ is rational and $y$ is irrational.

[For the final one, let's see if it's simple enough to summarize well in a single sentence.]

The converse is false, by counter-example, since $0 = 0/1$ is a rational real number, and $\sqrt{2}$ is an irrational real number (as given), but $0 \cdot \sqrt{2} = 0 = 0/1$ is rational.