Duration: **50 minutes**
Aids Allowed: **one single-sided handwritten 8.5” x 11” aid sheet**

**Student Number:**

**Last (Family) Name(s):**

**First (Given) Name(s):**

**Do not turn this page until you have received the signal to start.**
In the meantime, please read the instructions below carefully.

This term test consists of 5 questions on 8 pages (including this one), printed on both sides of the paper. **When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, and write your name on the back of the last page.**

Answer each question directly on the test paper, in the space provided, and use one of the “blank” pages for rough work. If you need more space for one of your solutions, use a “blank” page and indicate clearly the part of your work that should be marked.

In your answers, you may use without proof any theorem or result covered in lectures, tutorials, homework, quizzes, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks will be given for showing that you know the general structure of an answer, even if your solution is incomplete.

If you are unable to answer a question (or part of a question), remember that you will get 10% of the marks for any solution that you leave *entirely blank* (or where you cross off everything you wrote to make it clear that it should not be marked).

**Marking Guide**

# 1: ____/10
# 2: ____/ 8
# 3: ____/ 7
# 4: ____/ 7
# 5: ____/ 8

**TOTAL:** ____/40

**Good Luck!**
Question 1. [10 marks]
Let $D = \{\text{all tests and all students}\}$, $S(x)$ mean: “$x$ is a student,” $T(x)$ mean: “$x$ is a test,” and $W(x,y)$ mean: “$x$ wrote $y.$”

For each English statement below, write a standard symbolic representation of the statement, and for each symbolic statement, write a natural English expression of the statement.

Part (a) [2 marks]
$\forall x \in D, S(x) \Rightarrow W(x,y)$

Part (b) [2 marks]
“Some student wrote no test.”

Part (c) [2 marks]
$\forall x \in D, T(x) \Rightarrow \exists y \in D, S(y) \land W(y,x)$

Part (d) [2 marks]
“No test was written by every student.”

Part (e) [2 marks]
$(\exists x \in D, S(x) \land \forall y \in D, T(y) \Rightarrow W(x,y)) \Rightarrow (\forall y \in D, T(y) \Rightarrow \exists x \in D, S(x) \land W(x,y))$
Question 2. [8 marks]
For each equivalence, either give a derivation to show that the equivalence holds (and justify each step of your derivation with the name of a basic equivalence from the course notes), or give an interpretation to show that the equivalence does not hold (and explain why your interpretation makes the equivalence false).

Part (a) [4 marks]
$((\exists x \in D, P(x)) \Rightarrow Q) \iff (\forall x \in D, P(x) \Rightarrow Q)$
(where variable $x$ does not appear in $Q$)

Part (b) [4 marks]
$(Q \Rightarrow (\exists x \in D, P(x))) \iff (\forall x \in D, Q \Rightarrow P(x))$
(where variable $x$ does not appear in $Q$)
Question 3. [7 marks]

Part (a) [4 marks]
Write a detailed proof structure for the following statement about infinite sequences of natural numbers $a_0, a_1, a_2, \ldots$. Note: We are looking for just the proof structure—there is no content to fill in.

$$\forall i \in \mathbb{N}, i \geq 3 \Rightarrow \exists j \in \mathbb{N}, (j < i \land a_j = a_i) \lor (j > i \land a_j < a_i)$$

Part (b) [3 marks]
Give a specific sequence $a_0, a_1, a_2, \ldots$ that makes the statement above True, and give a very brief high-level justification that this is the case.
Question 4. [7 marks]
Part (a) [2 marks]
Explain the error(s) in the following “proof.”

Claim: If \( r^2 < r^4 \), then \( r \geq -1 \).

“Proof”: Suppose \(-(r \geq -1)\), i.e., \( r < -1 \). Then, squaring both sides gives \( r^2 > 1 \) (the inequality is reversed because both sides are negative). Multiplying both sides by \( r^2 \) gives \( r^4 > r^2 \).

Part (b) [5 marks]
Disprove the following statement. Write a detailed, structured proof.

\[ \forall x \in \mathbb{R}, x^3 + x^4 \text{ is odd} \iff x \text{ is odd} \]
Question 5. [8 marks]

Recall that for integers $x, y, z$, the notation “$x \equiv y \mod z$” means “$x - y$ is a multiple of $z$.” Use this definition to prove the following statement. Write a detailed, structured proof.

$$\forall n \in \mathbb{N}, (n^3 - n) \equiv 0 \mod 6$$
Use the space on this “blank” page for scratch work, or for any solution that did not fit elsewhere. Clearly label each such solution with the appropriate question and part number.
On this page, please write nothing except your name.

Last (Family) Name(s): ________________________________

First (Given) Name(s): ________________________________

Total Marks = 40