Camera Models
If you are interested, this book has it all:

A. Zisserman and R. Hartley

Multiview Geometry

Cambridge University Press, 2003
Camera

- Camera is structurally similar to the eye

[Source: L.W. Kheng]
Camera

- Camera is structurally similar to the eye

Source: L.W. Kheng
Camera

- Remember the pinhole camera from Lecture 2?

[Source: A. Torralba]
Remember the pinhole camera from Lecture 2?

[Source: A. Torralba]
Camera

- Remember the pinhole camera from Lecture 2?
- Size of the pinhole is called **aperture**

[Source: A. Torralba]
Pinhole Camera

(A) Source

Pinhole

Image

(B) Source

Reduced pinhole

Image

[Source: A. Torralba]
Shrinking the Aperture

Why not make the aperture as small as possible?

- Less light gets through
- Diffraction effects...

[Source: N. Snavely]
Shrinking the Aperture

[Source: N. Snavely]
Adding a Lens

- A lens focuses light onto the film
- There is a specific distance at which objects are in focus
Adding a Lens

- A lens focuses light onto the film
- There is a specific distance at which objects are **in focus**
- Changing the shape of the lens changes this distance

[Source: N. Snavely]
A lens focuses light onto the film

There is a specific distance at which objects are in focus

Changing the shape of the lens changes this distance

[Source: N. Snavely]
Some “Cameras” Have Bigger Lenses than Others

http://www.use.com/images/s_2/thick_glasses_13b6941623c255ff400a_1.jpg?
Images are 2D projections of real world scene

Images capture two kinds of information:

- **Geometric**: positions, points, lines, curves, etc.
- **Photometric**: intensity, color

Complex 3D-2D relationships

Camera models approximate these relationships

[Source: L.W. Kheng]
Projection

[Source: N. Snavely]
Projection

[Source: N. Snavely]
3D to 2D Projection

- How are 3D primitives projected onto the image plane?
- We can do this using a linear 3D to 2D projection matrix
How are 3D primitives projected onto the image plane?

We can do this using a linear 3D to 2D projection matrix

Different types:
- Perspective projection
- Orthographic projection
- Scaled orthographic projection
- Paraperspective projection

[source: R. Urtasun]
How are 3D primitives projected onto the image plane?

We can do this using a linear 3D to 2D projection matrix

Different types:

- Perspective projection
- Orthographic projection
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[source: R. Urtasun]
3D to 2D Projection

- How are 3D primitives projected onto the image plane?
- We can do this using a linear 3D to 2D projection matrix
- Different types, most common:
  - Perspective projection
  - Orthographic projection
  - Scaled orthographic projection
  - Paraperspective projection

[source: R. Urtasun]
We will use the pinhole model as an approximation.
Modeling Projection

- We will use the pinhole model as an approximation
We will use the pinhole model as an approximation.
We will use the pinhole model as an approximation.
Focal Length

- Can be thought of as **zoom**
- Related to the **field of view**

![Image showing different focal lengths (24mm, 50mm, 200mm, 800mm)]

**Figure**: Image from N. Snavely

[Source: N. Snavely, slide credit: R. Urtasun]
We will use the pinhole model as an approximation. Since it’s easier to think in a non-upsidedown world, we will work with the virtual image plane, and just call it the image plane.

How do points in 3D project to image plane? If I know a point in 3D, can I compute to which pixel it projects?
Modeling Projection

- First some notation which will help us derive the math
- To start with, we need a coordinate system
We place a coordinate system relative to camera: optical center or camera center $C$ is thus at origin $(0, 0, 0)$.\end{itemize}
The $Z$ axis is called the **optical** or **principal axis**. It is orthogonal to the image plane. Axes $X$ and $Y$ are parallel to the image axes.
We will use a right handed coordinate system
The optical axis intersects the image plane in a point, \( p \). We call this point a **principal point**. It’s worth to remember the principal point since it will appear again later in the math.
The distance from the camera center to the principal point is called **focal length**, we will denote it with $f$. It’s worth to remember the focal length since it will appear again later in the math.
We’ll denote the image axes with $x$ and $y$. An image we see is of course represented with these axes. We’ll call this an **image coordinate system**.

The tricky part is how to get from the camera’s coordinate system (3D) to the image coordinate system (2D).
Let’s take some point \( Q \) in 3D. \( Q \) “lives” relative to the camera; its coordinates are assumed to be in camera’s coordinate system.
We call the line from \( Q \) to camera center a **projection line**.
The projection line intersects the image plane in a point $q$. This is the point we see in our image.
First thing to notice is that all points from \( Q \)'s projection line project to the same point \( q \) in the image!

**Ambiguity**: It’s impossible to know how far a 3D point is from the camera along the projection line by looking only at the image (point \( q \)).
Modeling projection

From the movie Bone Collector

- **Ambiguity**: It’s impossible to know how far a 3D point is from the camera along the projection line by looking only at the image (point $q$).
- It’s impossible to know the real 3D size of objects just from an image
- Why did the detective put a dollar bill next to the footprint?
- How would you compute the shoe’s dimensions?
Modeling projection

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- **Ambiguity**: It’s impossible to know how far a 3D point is from the camera along the projection line by looking only at the image (point $q$).
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Projection Equations

- Using similar triangles:

\[ Q = (X, Y, Z)^T \rightarrow \left( \frac{f \cdot X}{Z}, \frac{f \cdot Y}{Z}, f \right)^T \]
Projection Equations

- Using similar triangles:

\[
Q = (X, Y, Z)^T \rightarrow \left( \frac{f \cdot X}{Z}, \frac{f \cdot Y}{Z}, f \right)^T
\]

- This is relative to principal point \( p \). To move the origin to \((0, 0)\) in image:

\[
q = (X, Y, Z)^T \rightarrow \left( \frac{f \cdot X}{Z} + p_x, \frac{f \cdot Y}{Z} + p_y, f \right)^T
\]

where \( p = (p_x, p_y) \) is the principal point.
Projection: Ready for Math

Projection Equations

- Using similar triangles:
  \[ \mathbf{Q} = (X, Y, Z)^T \rightarrow \left( \frac{f \cdot X}{Z}, \frac{f \cdot Y}{Z}, f \right)^T \]

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where \( p = (p_x, p_y) \) is the principal point

- Get the projection by throwing the last coordinate:

\[ Q = (X, Y, Z)^T \rightarrow q = \left( \frac{f \cdot X}{Z} + p_x, \frac{f \cdot Y}{Z} + p_y \right)^T \]

- This is NOT a linear transformation as a division by \( Z \) is non-linear
We will use homogeneous coordinates, which simply append a 1 to the vector

\[ (x, y) \implies \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \]

homogeneous image coordinates

\[ (x, y, z) \implies \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \]

homogeneous scene coordinates

Converting *from* homogeneous coordinates

\[ \begin{bmatrix} x \\ y \\ w \end{bmatrix} \implies (x/w, y/w) \]

\[ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \implies (x/w, y/w, z/w) \]

[Source: N. Snavely]
Homogeneous Coordinates!

- We will use homogeneous coordinates, which simply append a 1 to the vector.
- In homogeneous coordinates, scaling doesn’t affect anything:

\[
\begin{bmatrix}
  x \\
  y \\
  1
\end{bmatrix}
\sim
\begin{bmatrix}
  w \cdot x \\
  w \cdot y \\
  w
\end{bmatrix}
\]
Homogeneous Coordinates!

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  x \\
  y \\
  w
  \end{bmatrix}
  \sim
  \begin{bmatrix}
  w \cdot x \\
  w \cdot y \\
  w
  \end{bmatrix}
  \]

- In Projective Geometry, all points are equal under scaling.
Useful Trivia about Homogeneous Coordinates

- Homogeneous coordinates are quite useful in general. Let’s see why
- Let’s look at equation of a line in 2D: \( ax + by + c = 0 \)
Useful Trivia about Homogeneous Coordinates

- Homogeneous coordinates are quite useful in general. Let’s see why
- Let’s look at equation of a line in 2D: \( ax + by + c = 0 \)
- I can represent the line with a homogeneous vector \( \mathbf{l} := (a, b, c)^T \) (why homogeneous?) and a homogeneous vector \( \mathbf{x} := (x, y, 1) \). How?
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- Dot product is 0: $l^T \cdot x = 0$!
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- Dot product is 0: \( \mathbf{l}^T \cdot \mathbf{x} = 0! \)
- So if I have a line and someone gives me a point \( (x, y) \), how do I quickly check if the point lies on the line? Homogeneous coordinates, and check if dot product is 0.
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- Ok, what if I give you two points $(x_1, y_1)$ and $(x_2, y_2)$ and I ask you to write an equation for the line between them?
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- Ok, what if I give you two points $(x_1, y_1)$ and $(x_2, y_2)$ and I ask you to write an equation for the line between them?
- You can solve a linear system. But it’s much easier in homogeneous coordinates: Since both points lie on a line, the homogeneous vectors are both “orthogonal” to the line vector $l$ (dot product 0).
Homogeneous coordinates are quite useful in general. Let’s see why.

Let’s look at equation of a line in 2D: \( ax + by + c = 0 \)

I can represent the line with a homogeneous vector \( \mathbf{l} := (a, b, c)^T \) (why homogeneous?) and a homogeneous vector \( \mathbf{x} := (x, y, 1) \). How?

Dot product is 0: \( \mathbf{l}^T \cdot \mathbf{x} = 0! \)

So if I have a line and someone gives me a point \((x, y)\), how do I quickly check if the point lies on the line? Homogeneous coordinates, and check if dot product is 0.

Ok, what if I give you two points \((x_1, y_1)\) and \((x_2, y_2)\) and I ask you to write an equation for the line between them?

You can solve a linear system. But it’s much easier in homogeneous coordinates: Since both points lie on a line, the homogeneous vectors are both “orthogonal” to the line vector \( \mathbf{l} \) (dot product 0).

We know that a vector orthogonal to two vectors is a cross product between them: \( \mathbf{l} = (x_1, y_1, 1)^T \times (x_2, y_2, 1)^T \). And this is easy to compute.
We currently have this (the nasty division by $Z$):

$$Q = (X, Y, Z)^T \rightarrow q = \begin{bmatrix} \frac{f \cdot X}{Z} + p_x \\ \frac{f \cdot Y}{Z} + p_y \end{bmatrix}$$
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Write this with homogeneous coordinates:

$$Q = (X, Y, Z)^T \rightarrow q = \begin{bmatrix} \frac{f \cdot X}{Z} + p_x \\ \frac{f \cdot Y}{Z} + p_y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f \cdot X + Z \cdot p_x \\ f \cdot Y + Z \cdot p_y \\ Z \end{bmatrix}$$
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We can now write this as matrix multiplication:

$$Q = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \begin{bmatrix} f \cdot X + Z \cdot p_x \\ f \cdot Y + Z \cdot p_y \\ Z \end{bmatrix} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$
From previous slide:

\[
Q = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} f \cdot X + Z \cdot p_x \\ f \cdot Y + Z \cdot p_y \\ Z \end{bmatrix} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
\]

Write:

\[
K = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}
\]

This is called a **camera calibration matrix** or **intrinsic parameter matrix**. The parameters in \( K \) are called **internal camera parameters**.
Camera Intrinsics

- From previous slide:

\[
Q = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \begin{bmatrix} f \cdot X + Z \cdot p_x \\ f \cdot Y + Z \cdot p_y \\ Z \end{bmatrix} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
\]

- Write:

\[
K = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}
\]

This is called a **camera calibration matrix** or **intrinsic parameter matrix**. The parameters in \(K\) are called **internal camera parameters**.

- Finally:

\[
\begin{bmatrix} w \cdot x \\ w \cdot y \\ w \end{bmatrix} = K \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow q = \begin{bmatrix} x \\ y \end{bmatrix}
\]

[Source: Zisserman & Hartley]
Camera Intrinsics

- Camera calibration matrix:

\[
K = \begin{bmatrix}
f & 0 & p_x \\
0 & f & p_y \\
0 & 0 & 1
\end{bmatrix}
\]

It can be a little more complicated. Pixels may not be square:

\[
K = \begin{bmatrix}
f_x & 0 & p_x \\
0 & f_y & p_y \\
0 & 0 & 1
\end{bmatrix}
\]

And there might be a skew angle \( \phi \) between \( x \) and \( y \) image axis:

\[
K = \begin{bmatrix}
f_x & f_x \cot \phi & p_x \\
0 & f_y & p_y \\
0 & 0 & 1
\end{bmatrix}
\]

[Source: Zisserman & Hartley]
Camera Intrinsics

- Camera calibration matrix:

\[ K = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \]

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  0 & f_y & p_y \\
  0 & 0 & 1
\end{bmatrix}
\]

- And there might be a skew angle \( \theta \) between \( x \) and \( y \) image axis:

\[
K = \begin{bmatrix}
  f_x & -f_x \cot \theta & p_x \\
  0 & f_y / \sin \theta & p_y \\
  0 & 0 & 1
\end{bmatrix}
\]

[Source: Zisserman & Hartley]
Camera Intrinsics

- Camera calibration matrix:

  \[
  K = \begin{bmatrix}
  f & 0 & p_x \\
  0 & f & p_y \\
  0 & 0 & 1
  \end{bmatrix}
  \]

  We’ll work with this one

- It can be a little more complicated. Pixels may not be square:

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  \]

[Source: Zisserman & Hartley]
Perspective Projection

[Source: N. Snavely]
Dimensionality Reduction Machine (3D to 2D)

3D world

2D image

Point of observation

What have we lost?

- Angles
- Distances (lengths)

Slide by A. Efros
Figures © Stephen E. Palmer, 2002
Projection properties

- **Many-to-one**: any points along same ray map to same point in image
Projection properties

- **Many-to-one**: any points along same ray map to same point in image
- Points $\mapsto$ points
Projection properties

- **Many-to-one**: any points along same ray map to same point in image
- Points $\rightarrow$ points
- Lines $\rightarrow$ lines. Why?
Projection properties

Figure: Proof by drawing
Projection properties

Figure: Proof by drawing
Projection properties

- **Many-to-one**: any points along same ray map to same point in image
- Points $\rightarrow$ points
- Lines $\rightarrow$ lines
- But line through principal point projects to a point. Why?

**Figure**: Can you tell where is the principal point?
Projection properties

- **Many-to-one**: any points along same ray map to same point in image
- Points → points
- Lines → lines
- But line through principal point projects to a point. Why?
- Planes → planes
Projection properties

- **Many-to-one**: any points along same ray map to same point in image
  - Points $\rightarrow$ points
  - Lines $\rightarrow$ lines
  - But line through principal point projects to a point. Why?
  - Planes $\rightarrow$ planes
  - But plane through principal point projects to line. Why?
Projection Properties: Cool Facts

Parallel lines converge at a **vanishing point**

- Each different direction in the world has its own **vanishing point**

[Adopted from: N. Snavely, R. Urtasun]
Parallel lines converge at a **vanishing point**

- Each different direction in the world **has its own vanishing point**
- All lines with the same 3D direction intersect at the **same vanishing point**
Projection Properties: Vanishing Point

- All lines with the same 3D direction intersect at the **same vanishing point**. Why?
All lines with the same 3D direction intersect at the same vanishing point. Why?
Projection Properties: Vanishing Point

- All lines with the same 3D direction intersect at the same vanishing point. Why?
Projection Properties: Vanishing Point

- All lines with the same 3D direction intersect at the **same vanishing point**. Why?
- Line that passes through \( \mathbf{V} \) with direction \( \mathbf{D} \): \( \mathbf{X} = \mathbf{V} + t\mathbf{D} \).
Projection Properties: Vanishing Point

- All lines with the same 3D direction intersect at the **same vanishing point**.
- **Why?**

Line that passes through \( \mathbf{V} \) with direction \( \mathbf{D} \): \( \mathbf{X} = \mathbf{V} + t\mathbf{D} \).

- Project it:

\[
\begin{bmatrix}
wx \\
wy \\
wz
\end{bmatrix} = K \begin{bmatrix}
f & 0 & p_x \\
0 & f & p_y \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
V_x + tD_x \\
V_y + tD_y \\
V_z + tD_z
\end{bmatrix} = \begin{bmatrix}
fV_x + ftD_x + p_x V_z + tp_x D_z \\
fV_y + ftD_y + p_y V_z + tp_y D_z \\
V_z + tD_z
\end{bmatrix}
\]
Projection Properties: Vanishing Point

- All lines with the same 3D direction intersect at the **same vanishing point**. *Why?*

- Line that passes through \( \mathbf{V} \) with direction \( \mathbf{D} \): \( \mathbf{X} = \mathbf{V} + t\mathbf{D} \).

- Project it:

\[
\begin{bmatrix}
wx \\
w y \\
w
\end{bmatrix} = K \begin{bmatrix}
f & 0 & p_x \\
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V_x + tD_x \\
V_y + tD_y \\
V_z + tD_z
\end{bmatrix} = \begin{bmatrix}
fV_x + ftD_x + p_x V_z + tp_x D_z \\
fV_y + ftD_y + p_y V_z + tp_y D_z \\
V_z + tD_z
\end{bmatrix}
\]

- Send \( t \to \infty \) and compute \( x \) and \( y \):

\[
x = \lim_{t \to \infty} \frac{fV_x + ftD_x + p_x V_z + tp_x D_z}{V_z + tD_z} = \frac{fD_x + p_x D_z}{D_z}
\]

\[
y = \lim_{t \to \infty} \frac{fV_y + ftD_y + p_y V_z + tp_y D_z}{V_z + tD_z} = \frac{fD_y + p_y D_z}{D_z}
\]
All lines with the same 3D direction intersect at the same vanishing point.

Why?

Line that passes through $\mathbf{V}$ with direction $\mathbf{D}$: $\mathbf{X} = \mathbf{V} + t\mathbf{D}$.

Project it:

$$
\begin{bmatrix}
wx \\
wy \\
w
\end{bmatrix} = K \mathbf{X} =
\begin{bmatrix}
f & 0 & p_x \\
0 & f & p_y \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
V_x + tD_x \\
V_y + tD_y \\
V_z + tD_z
\end{bmatrix} =
\begin{bmatrix}
fV_x + ftD_x + p_x V_z + tp_x D_z \\
fV_y + ftD_y + p_y V_z + tp_y D_z \\
V_z + tD_z
\end{bmatrix}
$$

Send $t \to \infty$ and compute $x$ and $y$:

$$
x = \lim_{t \to \infty} \frac{fV_x + ftD_x + p_x V_z + tp_x D_z}{V_z + tD_z} = \frac{fD_x + p_x D_z}{D_z}
$$

$$
y = \lim_{t \to \infty} \frac{fV_y + ftD_y + p_y V_z + tp_y D_z}{V_z + tD_z} = \frac{fD_y + p_y D_z}{D_z}
$$

This doesn’t depend on $\mathbf{V}$! So all lines with direction $\mathbf{D}$ go to this point!
Projection Properties: Vanishing Point

- All lines with the same 3D direction intersect at the same vanishing point.
Projection Properties: Vanishing Point

- All lines with the same 3D direction intersect at the **same vanishing point**.
- The easiest way to find this point: Translate line with direction $\mathbf{D}$ to the camera center. This line intersects the image plane in the vanishing point corresponding to direction $\mathbf{D}$! Why?
Projection Properties: Cool Facts

Parallel lines converge at a **vanishing point**

- Each different direction in the world **has its own vanishing point**
- Lines parallel to image plane are also parallel in the image (we say that they intersect at infinity). Why?
Projection Properties: Cool Facts

- Lines parallel to image plane are also parallel in the image. We say that they intersect at infinity.

\[ \text{doesn't intersect image plane! So no vanishing point!} \]

\[ \text{camera center} \]

\[ \text{parallel lines in the world, parallel to image plane} \]
Projection Properties: Cool Facts

- Lines parallel to image plane are also parallel in the image. We say that they intersect at infinity.
Projection Properties: Cool Tricks

- This picture has been recorded from a car with a camera on top. We know the camera intrinsic matrix $K$.
- Can we tell the incline of the hill we are driving on?
- How?
Can we tell the incline of the hill we are driving on?

Figure: This is the 3D world behind the picture.
Can we tell the incline of the hill we are driving on?

Figure: If we compute the 3D direction of the house’s vertical lines relative to camera, we have the incline! How can we do that?
Projection Properties: Cool Tricks

- Can we tell the incline of the hill we are driving on?

**Figure:** Extract “vertical” lines and compute vanishing point. How can we compute direction in 3D from vanishing point (if we have $K$)?
Can we tell the incline of the hill we are driving on?

**Figure**: This picture should help.
Projection Properties: Cool Tricks

Can we tell the incline of the hill we are driving on?

We have:

\[
\begin{bmatrix}
  w \cdot vp_x \\
  w \cdot vp_y \\
  w
\end{bmatrix} = KD \rightarrow D = wK^{-1} \begin{bmatrix}
  vp_x \\
  vp_y \\
  1
\end{bmatrix} \rightarrow \text{normalize } D \text{ to norm 1}
\]
Vanishing Points Can be Deceiving

- Parallel lines converge at a **vanishing point**.
- But intersecting lines in 2D are not necessary parallel in 3D.

[Source: A. Jepson]
Parallel lines converge at a **vanishing point**

- Each different direction in the world **has its own vanishing point**
- For lines on the same 3D plane, the vanishing points lie on a **line**. We call it a **vanishing line**. Vanishing line for the ground plane is a **horizon line**.
Parallel lines converge at a **vanishing point**

- For lines on the same 3D plane, the vanishing points lie on a **line**. We call it a **vanishing line**. Vanishing line for the ground plane is a **horizon line**.
- Some horizon lines are nicer than others ;)

![Image of a beach scene with palm trees and a horizon line indicated]
Projection Properties: Cool Facts

Parallel lines converge at a **vanishing point**

- For lines on the same 3D plane, the vanishing points lie on a **line**. We call it a **vanishing line** or a **horizon line**.

- Parallel planes in 3D have the **same horizon line** in the image.

![Image of parallel lines in a tunnel](http://upload.wikimedia.org/wikipedia/commons/d/d8/Frankfurt_Airport_tunnel.JPG)
Can I tell how much above ground this picture was taken?
Projection Properties: Cool Facts

- Can I tell how much above ground this picture was taken?
Projection Properties: Cool Facts

- Same distance as where the horizon intersects a building
Projection Properties: Cool Facts

- Same distance as where the horizon intersects a building: 50 floors up
Projection Properties: Cool Facts

- This is only true when the camera (image plane) is orthogonal to the ground plane. And the ground plane is flat.
- A very nice explanation of this phenomena can be find by Derek Hoiem here: https://courses.engr.illinois.edu/cs543/sp2011/materials/3dscene_book_svg.pdf
Orthographic Projection

Requires no division and simply drops the $Z$ coordinate.

Orthographic projection:

\[
\begin{align*}
Q &= 2 \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix} X Y Z \\
1 &= 3 \begin{bmatrix} 7 \\ 7 \\ 5 \end{bmatrix} !
\end{align*}
\]

Special case of perspective projection where the distance from the camera center to the image plane is infinity.

[Source: R. Urtasun]
Orthographic Projection

Requires no division and simply drops the Z coordinate.
Orthographic Projection

- Requires no division and simply drops the $Z$ coordinate.
- Orthographic projection:

\[
Q = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}
\]
Orthographic Projection

- Requires no division and simply drops the Z coordinate.
- Orthographic projection:

\[
Q = \begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix} \rightarrow \begin{bmatrix}
X \\
Y \\
1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix}
\]

- Special case of perspective projection where the distance from the camera center to the image plane is infinity
Orthographic Projection

[Source: N. Snavely]
Orthographic Projection

- For perspective projection lines parallel in 3D are **not** parallel in the image.
- For orthographic projection lines parallel in 3D are parallel in the image.

[Source: A. Torralba]
We are not yet done with projection. To fully specify projection, we need to:

- Describe its **internal parameters** (we know this, this is our $K$)
Camera Parameters

We are not yet done with projection. To fully specify projection, we need to:

- Describe its **internal parameters** (we know this, this is our $K$)
- Describe its **pose in the world**. Two important coordinate systems:
  - World coordinate system
  - Camera coordinate system
Camera Parameters

- Why two coordinate systems?

**Figure:** Imagine this is your room.
Camera Parameters

- Why two coordinate systems?

**Figure:** When you were furnishing you measured everything in detail.
Camera Parameters

- Why two coordinate systems?

Figure: Thus you know all coordinates relative to a special point (origin) and coordinate system in the room. This is your room’s (world) coordinate system.
Camera Parameters

- Why two coordinate systems?

**Figure:** Now you take a picture and you wonder how points project to camera. In order to project, you need all points in camera’s coordinate system.
Camera Parameters

- Why two coordinate systems?

**Figure:** For e.g. self-driving cars, 3D points are typically measured with Velodyne.
Camera Parameters

- Why two coordinate systems?

Figure: We want to be able to project the 3D points in Velodyne’s coordinate system onto an image captured by a camera.
Camera Parameters

- Why two coordinate systems?

**Figure**: We want to be able to project the 3D points in Velodyne’s coordinate system onto an image captured by a camera.
To project a point \((X, Y, Z)\) in world coordinates on the image plane, we need to:

1. Transform \((X, Y, Z)\) into camera coordinates.
2. Need to know camera intrinsics.
3. Camera position (in world coordinates)
4. Camera orientation (in world coordinates)

These can all be described with matrices!
To project a point \((X, Y, Z)\) in world coordinates on the image plane, we need to:

- Transform \((X, Y, Z)\) into camera coordinates. We thus need:
  
- Camera position (in world coordinates)
  
- Camera orientation (in world coordinates)

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Projection

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[Source: N. Snavely, slide credit: R. Urtasun]
Projection

To project a point \((X, Y, Z)\) in world coordinates on the image plane, we need to:

- Transform \((X, Y, Z)\) into camera coordinates. We thus need:
  - Camera \textit{position} (in world coordinates)
  - Camera \textit{orientation} (in world coordinates)

- To project into the image plane
  - Need to know \textit{camera intrinsics}
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- To project into the image plane
  - Need to know **camera intrinsics**

- These can all be described with matrices!

[Source: N. Snavely, slide credit: R. Urtasun]
Camera Extrinsic

Figure: We first need our camera position and orientation in the room’s world.
Camera Extrinsics

What is Q in camera’s coordinate system??

c ... camera position in room coordinate system

u, v, w ... 3 orthogonal directions of camera in room coordinate system
Camera Extrinsics

What is Q in camera’s coordinate system??

\[ Q - c \quad \text{makes position relative to camera} \]

\[ \begin{align*}
    c & \quad \text{camera position in room coordinate system} \\
    u, v, w & \quad 3 \text{ orthogonal directions of camera in room coordinate system}
\end{align*} \]
Camera Extrinsic

What is \( Q \) in camera’s coordinate system??

\[
Q - c \quad \text{makes position relative to camera}
\]

\[
R \begin{bmatrix} u & v & w \end{bmatrix} = I \quad \text{(looking for rotation to canonical orientation)}
\]

c \quad \text{camera position in room coordinate system}

\( u, v, w \) \quad \text{3 orthogonal directions of camera in room coordinate system}
What is Q in camera’s coordinate system?

\[ Q - c \quad \text{makes} \quad \text{position} \quad \text{relative to camera} \]

\[ R \begin{bmatrix} u & v & w \end{bmatrix} = I \quad \text{(looking for rotation to canonical orientation)} \]

\[ R \cdot R^T = I \quad \text{(since orientation is orthogonal matrix)} \]

c \quad \text{... camera position in room coordinate system}

\( u, v, w \) \quad \text{... 3 orthogonal directions of camera in room coordinate system}
Camera Extrinsic

What is $Q$ in camera’s coordinate system??

- $Q - c ...$ makes position relative to camera
- $R \begin{bmatrix} u & v & w \end{bmatrix} = I$ (looking for rotation to canonical orientation)
- $R \cdot R^T = I$ (since orientation is orthogonal matrix)
- $R = \begin{bmatrix} u^T & v^T & w^T \end{bmatrix}$

$c ...$ camera position in room coordinate system

$u, v, w ...$ 3 orthogonal directions of camera in room coordinate system
Camera Extrinsic

What is Q in camera’s coordinate system??

\[ Q - c \] makes **position** relative to camera

\[
R \begin{bmatrix} u & v & w \end{bmatrix} = I
\]

(looking for rotation to canonical orientation)

\[
R \cdot R^T = I
\]

(since orientation is orthogonal matrix)

\[
R = \begin{bmatrix} u^T & v^T & w^T \end{bmatrix}
\]

\[
\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R \left( \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - c \right) = \begin{bmatrix} R & -Rc \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}
\]

**Figure:** Final Transformation
Projection Equations

- **Projection matrix** $P$ models the cumulative effect of all intrinsic and extrinsic parameters. We use homogeneous coordinates for 2D and 3D:

$$q = \begin{bmatrix} ax \\ ay \\ a \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$
Projection Equations

- **Projection matrix** $P$ models the cumulative effect of all intrinsic and extrinsic parameters. We use homogeneous coordinates for 2D and 3D:

$$q = \begin{bmatrix} ax \\ ay \\ a \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- It can be computed as

$$P = \begin{bmatrix} f & 0 & px \\ 0 & f & py \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} I_{3\times3} & T_{3\times1} \end{bmatrix}$$

- intrinsics $K$  
- projection  
- rotation  
- translation
Projection Equations

- **Projection matrix** $P$ models the cumulative effect of all intrinsic and extrinsic parameters. We use homogeneous coordinates for 2D and 3D:

\[
q = \begin{bmatrix} ax \\ ay \\ a \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}
\]

- It can be computed as

\[
P = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} I_{3\times3} \\ T_{3\times1} \end{bmatrix}
\]

- $P$ is a $3 \times 4$ matrix. This gives me a $3 \times 1$ vector. Now I divide all coordinates with the third coordinate (making the third coordinate equal to 1), and then drop the last coordinate. As simple as that.
The Projection Matrix

- The projection matrix is defined as

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
R_{3 \times 3} & 0_{3 \times 1} \\
0_{1 \times 3} & 1
\end{bmatrix}
\begin{bmatrix}
I_{3 \times 3} & T_{3 \times 3} \\
0_{1 \times 3} & 1
\end{bmatrix}
\]

- More compactly

\[
P = K \begin{bmatrix} R & t \end{bmatrix}
\]

- Sometimes you will see notation:

\[
P = K \begin{bmatrix} R | t \end{bmatrix}
\]

It’s the same thing.
The Projection Matrix

- The projection matrix is defined as

\[
P = K \cdot R_t
\]

- More compactly

\[
P = K [R \mid t]
\]

- Sometimes you will see notation:

\[
P = K [R \mid t]
\]

It's the same thing.

- This might look complicated. Truth is, in most cases you don’t have P at all, so you can’t really compute any projections. When you have a calibrated camera, then someone typically gives you P. And then projection is easy.
A Short Note on Camera Calibration

The general procedure:

- Place a 3D pattern (for which you know all distances) in front of camera.
- Take a picture. Detect corners in image and find correspondences with the points in the pattern.
- Go to the internet and check out the math that tells you how to compute $K$ from these 2D-3D correspondences. ;) We won’t cover in class.

[Pic from: R. Duraiswami]
Let’s say you have an image but you don’t know \textbf{anything} about the camera (for example, image downloaded from the web).

For images where you see lines corresponding to 3 orthogonal directions, like cubes or rooms, you can compute the camera matrix $K$ as well as $R$ and $t$!

How to do this is explained in the Zisserman & Hartley book.
As a consequence, for scenes with lots of lines (e.g. man-made scenes) one can reconstruct the scene in 3D from a single image!
As a consequence, for scenes with lots of lines (e.g. man-made scenes) one can reconstruct the scene in 3D from a **single image**!
Projection Properties: Cool Facts

- As a consequence, for scenes with lots of lines (e.g. man-made scenes) one can reconstruct the scene in 3D from a **single image**!
- For those interested, check out the math here:

A. Criminisi, I. Reid, and A. Zisserman

*Single View Metrology*


Inserting Objects

K. Karsch, V. Hedau, D. Forsyth, D. Hoiem, Rendering synthetic objects into legacy photographs, SIGGRAPH’11

link to video   link to paper/code
From a longer **video** in which the sun travels across the sky you can compute the camera intrinsic matrix, as well as extrinsic, i.e., the GPS location where you are! Well, up to a 100km accuracy...
From a longer **video** in which the sun travels across the sky you can compute the camera intrinsic matrix, as well as extrinsic, i.e., the GPS location where you are! Well, up to a 100km accuracy...

Is this useful? Maybe, to catch terrorists that record their videos outside.
Exercise (Not Very Easy, But Fun)

- We want to render (project) a 3D CAD model of a car to this image in a realistic way
- How?
Exercise

- First get a CAD model. There are tones of them, e.g. 3D Warehouse (free)
Exercise

- We downloaded this model. Now what?

**Figure:** A CAD model is a collection of 3D vertices and faces that connect the vertices. Each face represents a small triangle. It typically has color.
Exercise

- Our image was collected with a car on the road:
  - A camera was on top of the car, approximately 1.7m above ground
  - Image plane is orthogonal to the ground
  - We have the internal parameters of the camera, $K$. 

Exercise

- Our image was collected with a car on the road:
  - A camera was on top of the car, approximately 1.7m above ground
  - Image plane is orthogonal to the ground
  - We have the internal parameters of the camera, $K$.

- With a little bit of math, we can compute the ground plane in 3D, relative to camera. With a bit more math we can compute which point on the ground plane projects to an image point $(x, y)$.

**How?**

- We can now “place” our CAD model to this point (compute $R$ and $t$)
Exercise

- Our image was collected with a car on the road:
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**How?**

- We can now “place” our CAD model to this point (compute $R$ and $t$)

- Rendering:
Exercise

- That’s it. Make a video for more fun

(click on image to play video)
Exercise

- That’s it. Make a video for more fun

(click on image to play video)
Exercise

- That’s it. Make a video for more fun

(click on image to play video)
A Little More on Camera Models