1. I claim that there is no integer \( n \geq 0 \) such that \( n^2 \) is even and \( n \) is odd. Fill in the outline below to prove I'm right. Notice that my proof structure requires you to find some contradiction — something that has to be true yet has to be false, which is impossible.

Assume that there is an integer \( n \geq 0 \) such that \( n^2 \) is even and \( n \) is odd.

Since \( n \) is odd, it must be true that \( n = 2k + 1 \), for some integer \( k \).

So \( n^2 = (2k + 1)(2k + 1) \)
\[ = 4k^2 + 4k + 1 \]
\[ = 2(2k^2 + 2k) + 1 \]

So \( n^2 \) is one greater than an integer multiple of 2, which means it is odd.

But we \( n^2 \) is even, by our assumption.

We have a contradiction, so our assumption cannot be valid.

So there is no integer \( n \geq 0 \) such that \( n^2 \) is even and \( n \) is odd.

2. Fill in the outline below to prove that \( n^2 + n \) is even, for any integer \( n \).

Let \( n \) by any integer.

\( n \) must be either even or odd.

Case 1: Assume \( n \) is even.

Then \( n = 2k \) for some integer \( k \).

So \( n^2 + n = 4k^2 + 2k \)
\[ = 2(2k^2 + k) \]

So \( n^2 + n \) is an integer multiple of 2, which means it is even.

Case 2: Assume \( n \) is odd.

Then \( n = 2k + 1 \) for some integer \( k \).

So \( n^2 + n = (2k + 1)^2 + (2k + 1) \)
\[ = 4k^2 + 4k + 1 + 2k + 1 \]
\[ = 4k^2 + 6k + 2 \]
\[ = 2(2k^2 + 3k + 1) \]

So \( n^2 + n \) is an integer multiple of 2, which means it is even.

So \( n^2 + n \) is even (regardless of whether \( n \) is even or odd).

So for any integer \( n \), \( n^2 + n \) is even.
3. Fill in the outline below to prove that for any integer \( n \), if \( n \) is not a multiple of 3, \( n^2 \) must be one greater than a multiple of 3. Note that inside the assumption, you may use any valid reasoning strategy that leads to the desired conclusion: \( n^2 = 3p + 1 \), for some integer \( p \).

Let \( n \) be any integer.

Assume that \( n \) is not a multiple of 3.

So either \( n = 3k + 1 \) or \( n = 3k + 2 \), for some integer \( k \).

Case 1: \( n = 3k + 1 \)

So \( n^2 = (3k + 1)(3k + 1) \)
\[ = 9k^2 + 6k + 1 \]
\[ = 3(3k^2 + 2k) + 1 \]

So \( n^2 = 3p + 1 \), for some integer \( p \).

Case 2: \( n = 3k + 2 \)

So \( n^2 = (3k + 2)(3k + 2) \)
\[ = 9k^2 + 12k + 4 \]
\[ = 3(3k^2 + 4k + 1) + 1 \]

So \( n^2 = 3p + 1 \), for some integer \( p \).

Therefore \( n^2 = 3p + 1 \), for some integer \( p \) (regardless of whether \( n = 3k + 1 \) or \( n = 3k + 2 \)).

Therefore \( n^2 \) is one greater than a multiple of 3.

So if \( n \) is not a multiple of 3, \( n^2 \) is one greater than a multiple of 3.

So for any integer \( n \), if \( n \) is not a multiple of 3, \( n^2 \) is one greater than a multiple of 3.