CSC 263 Homework 2

Due June 22, 2004

1. (20 points) Consider a binary tree $T$. Let $|T|$ be the number of nodes in $T$. Let $x$ be a node in $T$, let $L_x$ be the left subtree of $x$ and let $R_x$ be the right subtree of $x$. We say that $x$ has the “approximately balanced property,” $ABP(x)$, if

\[ |R_x| \leq 2|L_x| \quad \text{and} \quad |L_x| \leq 2|R_x|. \]

(a) What is the maximum height of a binary tree $T$ on $n$ nodes where $ABP(root)$ holds?

(b) We call $T$ an ABP-tree if $ABP(x)$ holds for every node $x$ in $T$. Prove that if $T$ is an ABP-tree, then the height of $T$ is $O(\log n)$. More precisely, show that

\[ \text{height}(T) \leq \log_2 n / \log_2 \frac{3}{2}. \]

2. (15 points) In class, we analyzed the worst-case running times for the dictionary operations on BST trees. Now we will consider one possible notion of average case. Assume we have an empty tree $T$ and we do the following operations

\[ \text{INSERT}(T, 1), \text{INSERT}(T, 2), \ldots, \text{INSERT}(T, n) \]

in some order. Assume all orders are equally likely. Since all operations on BST trees depend on the height of the tree, we are interested in $H_{\text{avg}}(n)$—the expected height of $T$ after the $n$ inserts.

(a) Define the probability space that we are dealing with.

(b) Let $A_i$ be the event that we insert $i$ first. Write an expression for $t(A_i)$, the expected height of the final tree given that we insert $i$ first, in terms of $H_{\text{avg}}$.

(c) Write a recurrence relation for $H_{\text{avg}}(n)$.

(d) (Extra Credit) Solve the recurrence relation from part (c). Hint: it should be $\Theta(\log n)$.

3. (15 points) We want to augment Red-Black Trees to support the following query, $\text{Average}(x)$, which returns the average key-value in the subtree rooted at node $x$ (including $x$ itself). The query should work in worst-case time $\Theta(1)$.

(a) What extra information needs to be stored at each node?

(b) Describe how to modify $\text{INSERT}$ to maintain this information. What is its worst-case running time now?
(c) Describe how to modify \texttt{DELETE} to maintain this information. What is its worst-case running time now?

4. (25 points) \textbf{Programming Question}: Coming soon!