1. Let $\text{ONE-THIRD-SUM} = \{b_1, \ldots, b_n | \exists S \subseteq \{1, \ldots, n\} : \sum_{i \in S} b_i = (\sum_{1 \leq j \leq n} b_j)/3\}$. Show that $\text{SUBSET-SUM} \leq_p \text{ONE-THIRD-SUM}$ by considering the transformation $f$ defined by $(a_1, \ldots, a_m; t) \mapsto (a_1, \ldots, a_m, A, A + 3t)$ where $A = \sum_{1 \leq i \leq m} a_i$. We assume that all input parameters are positive integers. You must carefully show that $x \in \text{SUBSET-SUM}$ iff $f(x) \in \text{ONE-THIRD-SUM}$. 
2. Let \( \text{FINITE} = \{< M > | \mathcal{L}(M) \text{ is a finite set}\} \). Show that neither \( \text{FINITE} \) nor its complement \( \overline{\text{FINITE}} \) is semi-decidable.

SOLUTION: This is very similar to the proof that \( \text{TOTAL} \) and its complement are not semi-decidable.

We first show \( \text{HB} \leq \text{FINITE} \). To simplify matters assume \( x = < M > \) for some \( M \). We show how to define \( M' \) such that \( M \) halts on blank tape if \( < M' > \in \text{FINITE} \). Then the transformation is the mapping \( f(< M >) = < M' > \). For any input \( W \), \( M' \) simulates \( M \) on blank tape. If \( M \) ever halts then \( M' \) accepts \( w \) and hence \( M' \) will accept infinitely many strings and hence \( < M' > \in \text{FINITE} \). If \( M \) does not halt on blank then \( M' \) accepts no strings and the empty set is certainly finite.

Now we show \( \text{HB} \leq \text{FINITE} \). Again, we simplify matters assume \( x = < M > \) for some \( M \). We now show how to define \( M'' \) such that \( M \) halts on blank tape iff \( < M'' > \in \text{FINITE} \). \( M'' \) on input \( w \) will simulate \( M \) on blank tape for \( |w| \) steps. If \( M \) on blank tape does not halt within \( |w| \) steps then \( M'' \) accepts \( w \) and otherwise it does not accept \( w \). Then it is easy to see that \( M'' \) will accept finitely many strings iff and \( M \) halts on blank tape since once \( w \) is long enough (longer than the number of steps needed for \( M \) to halt on blank tape), \( M'' \) will reject \( w \).