Determine whether each of the following statements is true. Write a detailed structured proof to prove or disprove the statement.

Note: “g is O(f)” means the same as “g ∈ O(f)”.

1. $7n^9 - 6n^2 + 20n$ is $O(8n^{10} - 5n + 20)$.

**Proof outline:** By definition of “O”, we have to show

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 7n^9 - 6n^2 + 20n \leq c(8n^{10} - 5n + 20).$$

This can be done using the following proof structure.

Let $c' = \ldots$ Then $c' \in \mathbb{R}^+$.
Let $B' = \ldots$ Then $B' \in \mathbb{N}$.

Assume $n \in \mathbb{N}$ and $n \geq B'$.

... show that $7n^9 - 6n^2 + 20n \leq c'(8n^{10} - 5n + 20)$.
Then $\forall n \in \mathbb{N}, n \geq B' \Rightarrow 7n^9 - 6n^2 + 20n \leq c'(8n^{10} - 5n + 20)$.
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 7n^9 - 6n^2 + 20n \leq c(8n^{10} - 5n + 20)$.

**Scratch work:** Working “forward” from the left-hand side, we get:

$$7n^9 - 6n^2 + 20n \leq 7n^9 + 20n \leq 7n^9 + 20n^9 \quad \text{if } n \geq 1$$

$$\leq 27n^9 \leq n^{10} \quad \text{if } n \geq 27$$

(Note that there are other inequalities we could have used, e.g., $27n^9 \leq 27n^{10}$ for all $n \geq 1$.)

Working “backward” from the right-hand side, we get:

$$8n^{10} - 5n + 20 \geq 8n^{10} - 5n \geq 8n^{10} - 5n^{10} \quad \text{because } -n \geq -n^{10}$$

$$\geq 3n^{10} \geq n^{10}$$

Since both chains of inequalities connect, we are done: we can pick $B = 27$ (because we require $n \geq 27$ in our first chain) and $c = 1$.

**Proof:** (This is the actual final “solution”. We skip the formal introduction of $B'$ and $c'$ and instead, simply use their values directly. This style of proof is fine, and it is a little less verbose than using “$B''$” and “$c''$” throughout the argument.)

Assume $n \in \mathbb{N}$ and $n \geq 27$.

Then, $7n^9 - 6n^2 + 20n$

$$\leq 7n^9 + 20n \leq 7n^9 + 20n^9 \quad \text{if } n \geq 1$$

$$\leq 27n^9$$

$$\leq n^{10} \quad \text{if } n \geq 27$$

$$\leq 3n^{10} \leq 8n^{10} - 5n^{10} \quad \text{because } -n \geq -n^{10}$$

$$\leq 8n^{10} - 5n + 20 \leq 8n^{10} - 5n + 20$$

Then $\forall n \in \mathbb{N}, n \geq 27 \Rightarrow 7n^9 - 6n^2 + 20n \leq 8n^{10} - 5n + 20$

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 7n^9 - 6n^2 + 20n \leq c(8n^{10} - 5n + 20)$.

Then $7n^9 - 6n^2 + 20n \in O(8n^{10} - 5n + 20)$, by definition.

# (pick $B = 27$ and $c = 1$)
2. \((6n^2 - 7)^2\) is \(\mathcal{O}(n^3 + n - 3)\).

**Proof outline:** The statement is false.

From the negated definition of \(\mathcal{O}\), we have to prove

\[
\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land (6n^2 - 7)^2 > c(n^3 + n - 3)
\]

This can be done with the following proof structure:

Assume \(c \in \mathbb{R}^+\) and \(B \in \mathbb{N} \).

Let \(n_0 = \ldots \) # an expression containing \(B\) and \(c\)

...show that \(n_0 \in \mathbb{N} \).

...show that \(n_0 \geq B \).

...show that \((6n_0^2 - 7)^2 > c(n_0^3 + n_0 - 3)\).

Then, \(\exists n \in \mathbb{N}, n \geq B \land (6n^2 - 7)^2 > c(n^3 + n - 3)\).

Then, \(\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land (6n^2 - 7)^2 > c(n^3 + n - 3)\).

**Scratch work:** The property of \(n_0\) that will be most difficult to show is \((6n_0^2 - 7)^2 > c(n_0^3 + n_0 - 3)\), so we focus on it first. It is tempting to try to solve for \(n_0\) — and if the expression were simpler, this would yield an appropriate value. But it will be complicated in this case, and it is not necessary. Remember that, intuitively, we are simply trying to prove that \((6n^2 - 7)^2\) is larger than \(n^3 + n - 3\) by more than a constant factor.

Working “forward” from the left-hand side, we get:

\[
(6n^2 - 7)^2 = 36n^4 - 84n^2 + 49 > 36n^4 - 84n^2
\]

Note: we can’t simply use \(84n^2 \leq 84n^4\), because in that case we will overshoot (the resulting value will become negative). So, let’s use a bound which makes the result reasonable. For example, \(n^2 \geq 16\) if \(n \geq 4\). In that case,

\[
84n^2 = 84n^2 \frac{n^2}{n^2} \leq 84n^2 \frac{n^2}{16} = 5.25n^4 \leq 6n^4
\]

So,

\[
36n^4 - 84n^2 \geq 36n^4 - 6n^4
\]

if \(n \geq 4\)

\[
= 30n^4
\]

Working “backward” from the right-hand side, we get:

\[
n^3 + n - 3 < n^3 + n \leq 2n^3
\]

if \(n \geq 1\)

Now, we want \(30n^4 > c(2n^3)\), i.e., \(15n > c\). This will be true as long as \(n > c/15\). In fact, we can even say simply \(n > c\). Since \(c \in \mathbb{R}^+\), to ensure \(n \in \mathbb{N}\), we can pick any value \(n \geq \lfloor c \rfloor + 4\). This guarantees \(n \geq 4\), which is needed for the inequalities above to hold. Finally, we also need \(n \geq B\), which can be achieved simply by picking \(n = B + \lfloor c \rfloor + 4\).
Proof:

Assume \( c \in \mathbb{R}^+ \) and \( B \in \mathbb{N} \).

Let \( n_0 = B + \lceil c \rceil + 4 \).

Then \( n_0 \in \mathbb{N} \) because \( B \in \mathbb{N} \) and \( \lceil c \rceil \in \mathbb{N} \) for all \( c \in \mathbb{R}^+ \).

Then \( n_0 \geq B \) (in fact, \( n_0 > B \)).

Then \((6n_0^2 - 7)^2 = 36n_0^4 - 84n_0^2 + 49 \)
\[ > 36n_0^4 - 84n_0^2 \]
\[ \geq 36n_0^4 - 6n_0^4 \quad \text{since } n_0 \geq 4 \]
\[ = 30n_0^4 \]
\[ \geq 30n_0^3c \quad \text{since } n_0 \geq c \]
\[ \geq 2n_0^3c \]
\[ \geq c(n_0^3 + n_0) \quad \text{since } n_0 \geq 1 \]
\[ \geq c(n_0^3 + n_0 - 3) \]

Then, \( \exists n \in \mathbb{N}, n \geq B \wedge (6n^2 - 7)^2 > c(n^3 + n - 3) \).

Then, \( \forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge (6n^2 - 7)^2 > c(n^3 + n - 3) \).

Then \((6n^2 - 7)^2 \notin \mathcal{O}(n^3 + n - 3)\), by definition.
3. \(\sqrt{n^3 + 5n - 9}\) is \(O\left(\frac{n^4 + 5n^2 - 1}{n^2 - 7n + 1}\right)\).

**Proof outline:** The statement is true.

To be able to immediately see that, remember that the lower terms and constants can be ignored. So, \(\sqrt{n^3 + 5n - 9}\) will asymptotically be the same as \(x^{3/2}\), and that \(\frac{n^4 + 5n^2 - 1}{n^2 - 7n + 1}\) will behave as \(\frac{n^4}{n}\), which is \(x^2\), which grows faster than \(x^{3/2}\).

These observations help easily determine whether the statement is true, and get an idea for the proof – but, of course, now we will need to write out the formal proof. From the definition of “\(O\)”, we have to prove

\[\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \sqrt{n^3 + 5n - 9} \leq c\left(\frac{n^4 + 5n^2 - 1}{n^2 - 7n + 1}\right)\]

The proof structure will be similar as before.

**Scratch work:** We start with the left-hand side, and formalize our intuition that the lower terms shouldn’t matter.

\[
\sqrt{n^3 + 5n - 9} \\
\leq \sqrt{n^3 + 5n} \\
\leq \sqrt{n^3 + 5n^3} \\
\leq \sqrt{6n^3} \\
\leq \sqrt{6n^{3/2}}
\]

Working “backward” from the right-hand side, we get:

\[
\frac{n^4 + 5n^2 - 1}{n^2 - 7n + 1} \\
\geq \frac{n^4 - 1}{n^2 - 7n + 1} \\
\geq \frac{n^4 - 1}{n^2 + 1} \\
\geq \frac{(1/2)n^4}{n^2 + 1} \\
\geq \frac{(1/2)n^4}{n^2 + n^2} \\
= \frac{(1/2)n^4}{2n^2} \\
= (1/4)n^2 \\
\geq (1/4)n^{3/2}
\]

Now, we want \(\sqrt{6n^{3/2}} \leq c(1/4)n^{3/2}\). This should work for any \(c \geq 4\sqrt{6}\). Note: \(c \geq 4\sqrt{6}\) is a positive real number, and we can use that value for \(c\). But we don’t have to. For convenience, we can pick, for example, \(c = 12 = 4 \times 3 = 4\sqrt{9} \geq 4\sqrt{6}\). (of course, 100 would work as well!) Finally, we need \(B \geq 2\), since we used that in the second part of the proof.
**Proof:** Assume $n \in \mathbb{N}$ and $n \geq 2$.

Then, $\sqrt{n^3 + 5n - 9}$

\[
\leq \sqrt{n^3 + 5n} \\
\leq \sqrt{n^3 + 5n^3} \quad \text{if } n \geq 1 \\
\leq \sqrt{6n^3} \\
\leq \sqrt{6n^{3/2}} \\
\leq 3 \cdot n^{3/2} \\
\leq 12(1/4n^{3/2}) \\
\leq 12(1/4)n^2 \quad \text{if } n \geq 1 \\
= 12 \frac{(1/2)n^4}{2n^2} \\
\leq 12 \frac{(1/2)n^4}{n^2 + 1} \quad \text{if } n \geq 1 \\
\leq 12 \frac{n^4 - 1}{n^2 + 1} \quad \text{if } n \geq 2 \\
\leq 12 \frac{n^4 - 1}{n^2 - 7n + 1} \\
\leq 12 \frac{n^4 + 5n^2 - 1}{n^2 - 7n + 1}
\]

Then $\forall n \in \mathbb{N}, n \geq 2 \Rightarrow \sqrt{n^3 + 5n - 9} \leq 12 \frac{n^4 + 5n^2 - 1}{n^2 - 7n + 1}$

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \sqrt{n^3 + 5n - 9} \leq c \frac{n^4 + 5n^2 - 1}{n^2 - 7n + 1}$.

# pick $B = 2$ and $c = 12$

Then $\sqrt{n^3 + 5n - 9} \in \mathcal{O}\left(\frac{n^4 + 5n^2 - 1}{n^2 - 7n + 1}\right)$, by definition.