1. Prove or disprove each of the following statements. Write detailed proof structures and justify your work.

We use the rule names provided in square brackets in lecture notes. For example, \([\Rightarrow I]\) stands for implication introduction. \([M]\) is for obvious mathematical results, additional justification where necessary. Of course, the solutions provided are only one possible way of proving things.

(a) \(\forall m \in \mathbb{R}, \forall n \in \mathbb{R}, m = 0 \lor n = 0 \iff mn = 0\)

This is true.

Assume \(m \in \mathbb{R}, n \in \mathbb{R}\).

# To prove an equivalence, we prove the implication in each direction.

Assume \(m = 0 \lor n = 0\)

# Do a sub-proof by cases.
Case 1: Assume \(m = 0\)

Then, \(mn = 0 \times n = 0\) \(\# \([M]\)\)

Case 1: Assume \(n = 0\)

Then, \(mn = m \times 0 = 0\) \(\# \([M]\)\)

In either case, \(mn = 0\) \(\# \text{[Proof by Cases, } m = 0 \lor n = 0\] \)

Then, \(m = 0 \lor n = 0 \Rightarrow mn = 0\) \(\# \Rightarrow I\)

Assume \(mn = 0\)

# Want to prove \(m = 0 \lor n = 0\)

Assume \(m \neq 0\)

Then, \(mn = 0\).

Divide both sides by \(m\) to get \(n = 0\)

Then, \(m \neq 0 \Rightarrow n = 0\). \(\# \Rightarrow I\)

Then, \(m = 0 \lor n = 0 \iff mn = 0\) \(\# \iff I\)

Then, \(\forall m \in \mathbb{R}, \forall n \in \mathbb{R}, m = 0 \lor n = 0 \iff mn = 0\) \(\# \forall I\)

(b) \(4x^2 - 12x + 20 = 0\) has a real solution.

(Note: do not use derivatives, or formulas you might know about the roots of quadratic equations)

First, formalize this:

\(\exists x \in \mathbb{R}, 4x^2 - 12x + 20 = 0\)

This is false.

To disprove this, prove the negation:

\(\neg(\exists x \in \mathbb{R}, 4x^2 - 12x + 20 = 0)\)

We will use proof by contradiction.

Assume \(\exists x \in \mathbb{R}, 4x^2 - 12x + 20 = 0\)

Let \(y \in \mathbb{R}\) be such that \(4y^2 - 12y + 20 = 0\) \(\# \exists E\)

Then, \(4y^2 - 12y + 9 + 11 = 0\)

But, \(4y^2 - 12y + 9 = (2y - 3)^2\)

So, \((2y - 3)^2 + 11 = 0\)

So, \((2y - 3)^2 = -11 < 0\)

But \((2y - 3)^2 > 0\) \(\# \text{Since a square of a real number is positive}\)

This is a contradiction.

Then, \(\neg(\exists x \in \mathbb{R}, 4x^2 - 12x + 20 = 0)\) \(\# \neg I\)
2. Consider the following two functions that are defined for any $x \in \mathbb{R}$:

$$\lfloor x \rfloor = \text{the largest integer less than or equal to } x,$$

$$\lceil x \rceil = \text{the smallest integer greater than or equal to } x.$$

Prove or disprove each of the following statements. Write detailed proof structures and justify your work.

(a) $\forall n \in \mathbb{R}, n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$

This is false, since the right hand side is always an integer.

To disprove it, prove the negation: $\exists n \in \mathbb{R}, n \neq \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$

Let $y = 2.5$

Then $y \in \mathbb{R}$

Then $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = \lfloor \frac{2.5}{2} \rfloor + \lceil \frac{2.5}{2} \rceil = \lfloor 1.25 \rfloor + \lceil 1.25 \rceil = 1 + 2 = 3$

But, $2.5 \neq 3$

Then, $\exists n \in \mathbb{R}, n \neq \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$ # $3 I$

(b) $\forall n \in \mathbb{N}, n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$

This is true.

Assume $n \in \mathbb{N}$

# Do a sub-proof by cases.

Case 1: Assume $n$ is even

Then, $\exists k \in \mathbb{N}, n = 2k$ # Definition of even

Let $q \in \mathbb{N}$ be such that $n = 2q$ # $\exists E$

Then, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = \lfloor \frac{2q}{2} \rfloor + \lceil \frac{2q}{2} \rceil = \lfloor q \rfloor + \lceil q \rceil$

Then, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = 2q = n$ # $q \in \mathbb{N}$

Case 1: Assume $n$ is odd

Then, $\exists k \in \mathbb{N}, n = 2k + 1$ # Definition of odd

Let $q \in \mathbb{N}$ be such that $n = 2q + 1$ # $\exists E$

Then, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = \lfloor \frac{2q+1}{2} \rfloor + \lceil \frac{2q+1}{2} \rceil = \lfloor q + 0.5 \rfloor + \lceil q + 0.5 \rceil$

Then, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = q + (q + 1) = 2q + 1 = n$ # $q \in \mathbb{N}$

In either case, $n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$ # Proof by cases: $n \in \mathbb{N}$, so it is either even or odd

Then, $\forall n \in \mathbb{N}, n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$ # $\forall I$