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A note on element-wise matrix sparsification via a matrix-valued Bernstein inequality

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ABSTRACT

Given a matrix $A \in \mathbb{R}^{n\times n}$, we present a simple, element-wise sparsification algorithm that zeroes out all sufficiently small elements of A and then retains some of the remaining elements with probabilities proportional to the square of their magnitudes. We analyze the approximation accuracy of the proposed algorithm using a recent, elegant non-commutative Bernstein inequality, and compare our bounds with all existing (to the best of our knowledge) element-wise matrix sparsification algorithms.

1. Introduction

Element-wise matrix sparsification was pioneered by Achlioptas and McSherry [1,2], who described sampling-based algorithms to select a small number of elements from an input matrix $A \in \mathbb{R}^{n\times n}$ in order to construct a sparse sketch $\widetilde{A} \in \mathbb{R}^{n\times n}$, which is close to A in the operator norm. Such sketches were used in approximate eigenvector computations [1,4,2], semi-definite programming solvers [3,7], and matrix completion problems [5,6]. Motivated by their work, we present a simple matrix sparsification algorithm that achieves the best known upper bounds for element-wise matrix sparsification.

Our main algorithm (Algorithm 1) zeroes out "small" elements of A and randomly samples the remaining elements of A with respect to a probability distribution that favors "larger" entries.

In Algorithm 1, we let $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ denote the standard basis vectors for $\mathbb{R}^n$ (see Section 3.1 for more notation). Our sampling procedure selects s entries from A (note that $\widehat{A}$ from the description of Algorithm 1 is simply A, but with elements less than or equal to $\epsilon/(2n)$ zeroed out) in s independent, identically distributed (i.i.d.) trials with replacement. In each trial, elements of A are retained with probability proportional to their squared magnitude. Note that the same element of A could be selected multiple times and that $\widetilde{A}$ contains at most s non-zero entries. Theorem 1 is our main quality-of-approximation result for Algorithm 1 and achieves sparsity bounds proportional to $\|A\|_F^2$.

Algorithm 1 Matrix sparsification algorithm.

1: **Input:** $A \in \mathbb{R}^{n\times n}$, accuracy parameter $\epsilon > 0$.
2: **Let** $\widehat{A} = A$ and **zero-out** all entries of $\widehat{A}$ that are smaller (in absolute value) than $\epsilon/2n$.
3: **Set** s as in Eq. (1).
4: **For** $t = 1 \ldots s$ (i.i.d. trials with replacement) **randomly sample** indices (i_t, j_t) (entries of $\widehat{A}$), with

$$\mathbb{P}\big((i_t, j_t) = (i, j)\big) = p_{ij},$$

where $p_{ij} := \widehat{A}_{ij}^2 / \|\widehat{A}\|_F^2$ for all $(i, j) \in [n] \times [n]$.

5: **Output:**

$$\widetilde{A} = \frac{1}{s} \sum_{t=1}^{s} \frac{\widehat{A}_{i_t j_t}}{p_{i_t j_t}} e_{i_t} e_{j_t}^T \in \mathbb{R}^{n\times n}.$$

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Table 1
Summary of prior work in matrix sparsification results. Given a matrix $A \in \mathbb{R}^{n\times n}$ and an accuracy parameter $\epsilon > 0$, we seek a sparse $\widetilde{A} \in \mathbb{R}^{n\times n}$ such that $\|A - \widetilde{A}\|_2 \leqslant \epsilon$. The first column indicates the number of non-zero entries in $\widetilde{A}$, whereas the second column indicates whether this number is exact or simply holds in expectation. In terms of notation, we let b denote the $\max_{i,j}|A_{ij}|$ and R denote the $\max_{ij}|A_{ij}|/\min_{A_{ij}\neq 0}|A_{ij}|$. Finally, c_1, c_2, c_3, c_4 denote unspecified constants.

Comparison with prior results				
Sparsity of $\widetilde{A}$		Failure probability	Citation	Comments
$16n\|A\|_F^2/\epsilon^2 + 8^4 n\log^4 n$	Expected	$e^{-19\log^4 n}$	[2]	$\epsilon > 4\sqrt{n}\cdot b$ $n \geqslant 700\cdot 10^6$
$R\cdot b\cdot n\|A\|_F^2/\epsilon^2$	Expected	$e^{-\Omega(R\cdot n)}$	[9]	$\epsilon > c_1\sqrt{n\cdot R}\cdot b,\ n\geqslant 1$
$c_2 n\log^2(\frac{n}{\log^2 n})\log n\|A\|_F^2/\epsilon^2$	Expected	$1/n$	[12]	$\epsilon > 0,\ n\geqslant 300,$ $c_2 \leqslant 45^2$
$c_3 n\log^3 n\|A\|_F^2/\epsilon^2$	Expected	$1/n$	[13]	$\epsilon > 0,\ n\geqslant 300$ Extends to tensors
$c_4\sqrt{n}\sum_{ij}\|A_{ij}\|/\epsilon$	Exact	$e^{-\Omega(n)}$	[4]	$\epsilon > 0,\ n\geqslant 1$
$28n\ln(\sqrt{2}n)\|A\|_F^2/\epsilon^2$	Exact	$1/n$	Theorem 1	$\epsilon > 0,\ n\geqslant 1$

Theorem 1. *Let $A \in \mathbb{R}^{n\times n}$ be any matrix, let $\epsilon > 0$ be an accuracy parameter, and let $\widetilde{A}$ be the sparse sketch of A constructed via Algorithm* 1. *If*

$$s = \frac{28n\ln(\sqrt{2}n)}{\epsilon^2}\|A\|_F^2, \tag{1}$$

then, with probability at least $1 - n^{-1}$,

$$\|A - \widetilde{A}\|_2 \leqslant \epsilon.$$

$\widetilde{A}$ has at most s non-zero entries and the construction of $\widetilde{A}$ can be implemented in one pass over the input matrix A (see Section 3.2*).*

We conclude this section with Corollary 1, which is a re-statement of Theorem 1 involving the *stable rank* of A, denoted by $\mathbf{sr}(A)$ (recall that the stable rank of any matrix A is defined as the ratio $\mathbf{sr}(A) := \|A\|_F^2/\|A\|_2^2$, which is upper bounded by the rank of A). The corollary guarantees relative error approximations for matrices of – say – constant stable rank, such as the ones that arise in [14,6].

Corollary 1. *Let $A \in \mathbb{R}^{n\times n}$ be any matrix and let $\varepsilon > 0$ be an accuracy parameter. Let $\widetilde{A}$ be the sparse sketch of A constructed via Algorithm* 1 *(with $\epsilon = \varepsilon\|A\|_2$). If $s = 28n\mathbf{sr}(A)\ln(\sqrt{2}n)/\varepsilon^2$, then, with probability at least $1 - n^{-1}$,*

$$\|A - \widetilde{A}\|_2 \leqslant \varepsilon\|A\|_2.$$

It is worth noting that the sampling algorithm implied by Corollary 1 cannot be implemented in one pass, since we would need a priori knowledge of the spectral norm of A in order to implement Step 2 of Algorithm 1.

2. Related work

In this section (as well as in Table 1), we present a head-to-head comparison of our result with all existing (to the best of our knowledge) bounds on matrix sparsification. In [1,2] the authors presented a sampling method that requires in *expectation* $16n\|A\|_F^2/\epsilon^2 + 8^4 n\log^4 n$ non-zero entries in $\widetilde{A}$ in order to achieve an accuracy guarantee ϵ with a failure probability of at most $e^{-19\log^4 n}$. Compared with our result, their bound holds only when $\epsilon > 4\sqrt{n}\cdot\max_{i,j}|A_{ij}|$ and, in this range, our bounds are superior when $\|A\|_F^2/(\max_{i,j}|A_{ij}|)^2 = o(n\log^3 n)$. It is worth mentioning that the constant involved in [1,2] is two orders of magnitude larger than ours and, more importantly, that the results of [1,2] hold only when $n \geqslant 700\cdot 10^6$.

In [9], the authors study the $\|\cdot\|_{\infty\to 2}$ and $\|\cdot\|_{\infty\to 1}$ norms in the matrix sparsification context and they also present a sampling scheme analogous to ours. They achieve (in expectation) a sparsity bound of $Rn\|A\|_F^2\max_{i,j}|A_{ij}|/\epsilon^2$ when $\epsilon \geqslant \sqrt{nR}\max_{i,j}|A_{ij}|$; here $R = \max_{ij}|A_{ij}|/\min_{A_{ij}\neq 0}|A_{ij}|$. Thus, our results are superior (in the above range of ϵ) when $R\cdot\max_{i,j}|A_{ij}| = \omega(\log n)$.

It is harder to compare our method to the work of [4], which depends on the $\sum_{i,j=1}^n |A_{ij}|$. The latter quantity is, in general, upper bounded only by $n\|A\|_F$, in which case the sampling complexity of [4] is much worse, namely $O(n^{3/2}\|A\|_F^2/\epsilon)$. Finally, the recent bounds on matrix sparsification via the non-commutative Khintchine's inequality in [12] are inferior compared to ours in terms of sparsity guarantees by at least $O(\ln^2(n/\ln^2 n))$. However, we should mention that the bounds of [12] can be extended to multi-dimensional matrices (tensors), whereas our result does not generalize to this setting; see [13] for details.

3. Background

3.1. Notation

We let $[n]$ denote the set $\{1, 2, \ldots, n\}$. We will use the notation $\mathbb{P}(\cdot)$ to denote the probability of the event in the parentheses and $\mathbb{E}(X)$ to denote the expectation of a random variable X. When X is a matrix, $\mathbb{E}(X)$ denotes the element-wise expectation of each entry of X. For a matrix $X \in \mathbb{R}^{n\times n}$, $X^{(j)}$ will denote the j-th column of X as a column vector and, similarly, $X_{(i)}$ will denote the i-th row of X as a row vector (for any i or j in $[n]$). The Frobenius norm $\|X\|_F$ of the matrix X is defined as $\|X\|_F^2 = \sum_{i,j=1}^n X_{ij}^2$, and the spectral norm $\|X\|_2$ of the matrix X is defined as $\|X\|_2 = \max_{\|y\|_2=1}\|Xy\|_2$. For two symmetric matrices X, Y we say that $Y \succcurlyeq X$ if and only if $Y - X$ is a positive semi-definite matrix. Finally, I_n denotes the identity matrix of size n and $\ln x$ denotes the natural logarithm of x.

Algorithm 2 One-pass SELECT algorithm.

1: **Input:** A_{ij} for all $(i,j) \in [n] \times [n]$, arbitrarily ordered and $\epsilon > 0$.
2: $N = 0$.
3: **For all** $(i,j) \in [n] \times [n]$ **such that** $A_{ij}^2 > \frac{\epsilon^2}{4n^2}$
- $N = N + A_{ij}^2$.
- **Set** $(I,J) = (i,j)$ and $S = A_{ij}$ **with probability** $\frac{A_{ij}^2}{N}$.

4: **Output:** Return (I,J), S and N.

3.2. Implementing the sampling in one pass over the input matrix

We now discuss the implementation of Algorithm 1 in one pass over the input matrix A. Towards that end, we will leverage (a slightly modified version of) Algorithm SELECT (p. 137 of [8]).

We note that Step 3 essentially operates on $\widehat{A}$. Clearly, in a single pass over the data we can run in parallel s copies of the SELECT Algorithm (using a total of $O(s)$ memory) to effectively return s independent samples from $\widehat{A}$. Lemma 1 (p. 136 of [8], note that the sequence of the A_{ij}^2's is all-positive) guarantees that each of the s copies of SELECT returns a sample satisfying:

$$\mathbb{P}\big((i_t,j_t) = (i,j)\big) = \frac{\widehat{A}_{ij}^2}{\sum_{i,j=1}^n \widehat{A}_{ij}^2} = \frac{\widehat{A}_{ij}^2}{\|\widehat{A}\|_F^2}, \quad \text{for all } t = 1,\ldots,s.$$

Finally, in the parlance of Step 5 of Algorithm 1, (i_t,j_t) is set to (I,J) and $p_{i_t j_t}$ is set to S^2/N for all $t \in [s]$.

4. Proof of Theorem 1

The proof of Theorem 1 will combine Lemmas 1 and 4 in order to bound $\|A - \widetilde{A}\|_2$ as follows:

$$\|A - \widetilde{A}\|_2 = \|A - \widehat{A} + \widehat{A} - \widetilde{A}\|_2 \leqslant \|A - \widehat{A}\|_2 + \|\widehat{A} - \widetilde{A}\|_2 \leqslant \epsilon/2 + \epsilon/2 = \epsilon.$$

The failure probability of Theorem 1 emerges from Lemma 4, which fails with probability at most n^{-1} for the choice of s in Eq. (1). The proof of Lemma 4 will involve an elegant matrix-valued Bernstein bound proven in [14]. See also [10] or [15, Theorem 2.10] for similar bounds.

4.1. Bounding $\|A - \widehat{A}\|_2$

Lemma 1. *Using the notation of Algorithm* 1, $\|A - \widehat{A}\|_2 \leqslant \epsilon/2$.

Proof. Recall that the entries of $\widehat{A}$ are either equal to the corresponding entries of A or they are set to zero if the corresponding entry of A is (in absolute value) smaller than $\epsilon/(2n)$. Thus,

$$\|A - \widehat{A}\|_2^2 \leqslant \|A - \widehat{A}\|_F^2 = \sum_{i,j=1}^n (A - \widehat{A})_{ij}^2 \leqslant \sum_{i,j=1}^n \frac{\epsilon^2}{4n^2} \leqslant \frac{\epsilon^2}{4}. \qquad \square$$

4.2. Bounding $\|\widehat{A} - \widetilde{A}\|_2$

In order to prove our main result in this section (Lemma 4) we will leverage a powerful matrix-valued Bernstein bound originally proven in [14] (Theorem 3.2). We restate this theorem, slightly rephrased to better suit our notation.

Theorem 2. *(See Theorem* 3.2 *of* [14].*) Let* $M_1, M_2, \ldots, M_s$ *be independent, zero-mean random matrices in* $\mathbb{R}^{n\times n}$. *Suppose* $\max_{t\in[s]}\{\|\mathbb{E}(M_t M_t^T)\|_2, \|\mathbb{E}(M_t^T M_t)\|_2\} \leqslant \rho^2$ *and* $\|M_t\|_2 \leqslant \gamma$ *for all* $t \in [s]$. *Then, for any* $\tau > 0$,

$$\left\| \frac{1}{s}\sum_{t=1}^s M_t \right\|_2 \leqslant \tau$$

holds, subject to a failure probability of at most

$$2n \exp\left(-\frac{s\tau^2/2}{\rho^2 + \gamma\tau/3} \right).$$

In order to apply the above theorem, using the notation of Algorithm 1, we set $M_t = \frac{\widehat{A}_{i_t j_t}}{p_{i_t j_t}} e_{i_t} e_{j_t}^T - \widehat{A}$ for all $t \in [s]$ to obtain

$$\frac{1}{s}\sum_{t=1}^s M_t = \frac{1}{s}\sum_{t=1}^s \left[\frac{\widehat{A}_{i_t j_t}}{p_{i_t j_t}} e_{i_t} e_{j_t}^T - \widehat{A} \right] = \widetilde{A} - \widehat{A}. \tag{2}$$

Let $0_{n\times n}$ denote the $n \times n$ matrix of all-zeros. It is easy to argue that $\mathbb{E}(M_t) = 0_{n\times n}$ for all $t \in [s]$. Indeed, if we consider that $\sum_{i,j=1}^n p_{ij} = 1$ and $\widehat{A} = \sum_{i,j=1}^n \widehat{A}_{ij} e_i e_j^T$ we obtain

$$\mathbb{E}(M_t) = \sum_{i,j=1}^n p_{ij} \left(\frac{\widehat{A}_{ij}}{p_{ij}} e_i e_j^T - \widehat{A} \right) = \sum_{i,j=1}^n \widehat{A}_{ij} e_i e_j^T - \sum_{i,j=1}^n p_{ij} \widehat{A} = 0_{n\times n}.$$

Our next lemma bounds $\|M_t\|_2$ for all $t \in [s]$.

Lemma 2. *Using our notation,* $\|M_t\|_2 \leqslant 4n\epsilon^{-1}\|\widehat{A}\|_F^2$ *for all* $t \in [s]$.

Proof. First, using the definition of M_t and the fact that $p_{i_t j_t} = \widehat{A}_{i_t j_t}^2 / \|\widehat{A}\|_F^2$,

$$\|M_t\|_2 = \left\| \frac{\widehat{A}_{i_t j_t}}{p_{i_t j_t}} e_{i_t} e_{j_t}^T - \widehat{A} \right\|_2 \leqslant \frac{\|\widehat{A}\|_F^2}{|\widehat{A}_{i_t j_t}|} + \|\widehat{A}\|_2 \leqslant \frac{2n\|\widehat{A}\|_F^2}{\epsilon} + \|\widehat{A}\|_F.$$

The last inequality follows since all entries of $\widehat{A}$ are at least $\epsilon/(2n)$ and the fact that $\|\widehat{A}\|_2 \leqslant \|\widehat{A}\|_F$. We can now assume that

$$\|\widehat{A}\|_F \leqslant \frac{2n\|\widehat{A}\|_F^2}{\epsilon} \tag{3}$$

to conclude the proof of the lemma. To justify our assumption in Eq. (3), we note that if it is violated, then it must

be the case that $\|\widehat{A}\|_F < \epsilon/(2n)$. If that were true, then all entries of $\widehat{A}$ would be equal to zero. (Recall that all entries of $\widehat{A}$ are either zero or, in absolute value, larger than $\epsilon/(2n)$.) Also, if $\widehat{A}$ were identically zero, then (i) $\widetilde{A}$ would also be identically zero and, (ii) all entries of A would be at most $\epsilon/(2n)$. Thus,

$$\|A - \widetilde{A}\|_2 = \|A\|_2 \leqslant \|A\|_F \leqslant \sqrt{n^2 \frac{\epsilon^2}{4n^2}} = \frac{\epsilon}{2}.$$

Thus, if the assumption of Eq. (3) is not satisfied, the resulting all-zeros $\widetilde{A}$ still satisfies Theorem 1. □

Our next step towards applying Theorem 2 involves bounding the spectral norm of the expectation of $M_t M_t^T$. The spectral norm of the expectation of $M_t^T M_t$ admits a similar analysis and the same bound and is omitted.

Lemma 3. *Using our notation,* $\|\mathbb{E}(M_t M_t^T)\|_2 \leqslant n\|\widehat{A}\|_F^2$ *for any* $t \in [s]$.

Proof. We start by evaluating $\mathbb{E}(M_t M_t^T)$; recall that $p_{ij} = \widehat{A}_{ij}^2/\|\widehat{A}\|_F^2$:

$$\begin{aligned}
&\mathbb{E}\big(M_t M_t^T\big) \\
&= \mathbb{E}\left(\left(\frac{\widehat{A}_{i_t j_t}}{p_{i_t j_t}} e_{i_t} e_{j_t}^T - \widehat{A}\right)\left(\frac{\widehat{A}_{i_t j_t}}{p_{i_t j_t}} e_{j_t} e_{i_t}^T - \widehat{A}^T\right)\right) \\
&= \sum_{i,j=1}^n p_{ij}\left(\frac{\widehat{A}_{ij}}{p_{ij}} e_i e_j^T - \widehat{A}\right)\left(\frac{\widehat{A}_{ij}}{p_{ij}} e_j e_i^T - \widehat{A}^T\right) \\
&= \sum_{i,j=1}^n \left(\frac{\widehat{A}_{ij}^2}{p_{ij}} e_i e_i^T - \widehat{A}_{ij}\widehat{A} e_j e_i^T - \widehat{A}_{ij} e_i e_j^T \widehat{A}^T + p_{ij}\widehat{A}\widehat{A}^T\right) \\
&= \|\widehat{A}\|_F^2 \sum_{i=1}^n m_i \cdot e_i e_i^T - \sum_{j=1}^n \widehat{A} e_j \sum_{i=1}^n \widehat{A}_{ij} e_i^T \\
&\quad - \sum_{j=1}^n \left(\sum_{i=1}^n \widehat{A}_{ij} e_i\right)(\widehat{A} e_j)^T + \sum_{i,j=1}^n p_{ij}\widehat{A}\widehat{A}^T,
\end{aligned}$$

where m_i is the number of non-zeroes of the i-th row of $\widehat{A}$. We now simplify the above result using a few simple observations: $\sum_{i,j=1}^n p_{ij} = 1$, $\widehat{A} e_j = \widehat{A}^{(j)}$, $\sum_{i=1}^n \widehat{A}_{ij} e_i = \widehat{A}^{(j)}$, and $\sum_{j=1}^n \widehat{A}^{(j)}(\widehat{A}^{(j)})^T = \widehat{A}\widehat{A}^T$. Thus, we get

$$\begin{aligned}
\mathbb{E}\big(M_t M_t^T\big) &= \|\widehat{A}\|_F^2 \sum_{i=1}^n m_i \cdot e_i e_i^T - \sum_{j=1}^n \widehat{A}^{(j)}\big(\widehat{A}^{(j)}\big)^T \\
&\quad - \sum_{j=1}^n \widehat{A}^{(j)}\big(\widehat{A}^{(j)}\big)^T + \widehat{A}\widehat{A}^T \\
&= \|\widehat{A}\|_F^2 \sum_{i=1}^n m_i \cdot e_i e_i^T - \widehat{A}\widehat{A}^T.
\end{aligned}$$

Since $0 \leqslant m_i \leqslant n$ and using Weyl's inequality (Theorem 4.3.1 of [11]), which states that by adding a positive semi-definite matrix to a symmetric matrix all its eigenvalues will increase, we get that

$$-\widehat{A}\widehat{A}^T \preccurlyeq \mathbb{E}\big(M_t M_t^T\big) \preccurlyeq n\|\widehat{A}\|_F^2 I_n.$$

Consequently $\|\mathbb{E}(M_t M_t^T)\|_2 = \max\{\|\widehat{A}\|_2^2, n\|\widehat{A}\|_F^2\} = n\|\widehat{A}\|_F^2$. □

We can now apply Theorem 2 on Eq. (2) with $\tau = \epsilon/2$, $\gamma = 4n\epsilon^{-1}\|\widehat{A}\|_F^2$ (Lemma 2), and $\rho^2 = n\|\widehat{A}\|_F^2$ (Lemma 3). Thus, we get that $\|\widehat{A} - \widetilde{A}\|_2 \leqslant \epsilon/2$ holds, subject to a failure probability of at most

$$2n \exp\left(-\frac{\epsilon^2 s/8}{(1 + 4/6)n\|\widehat{A}\|_F^2}\right).$$

Bounding the failure probability by δ and solving for s, we get that

$$s \geqslant \frac{14}{\epsilon^2} n\|\widehat{A}\|_F^2 \ln\left(\frac{2n}{\delta}\right).$$

Using $\|\widehat{A}\|_F \leqslant \|A\|_F$ (by construction) concludes the proof of the following lemma, which is the main result of this section.

Lemma 4. *Using the notation of Algorithm* 1, *if* $s \geqslant 14n\epsilon^{-2} \times \|A\|_F^2 \ln(2n/\delta)$, *then, with probability at least* $1 - \delta$,

$$\|\widehat{A} - \widetilde{A}\|_2 \leqslant \epsilon/2.$$

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