Support Vector Machines

CSC 411 Tutorial

November 10, 2015

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Many thanks to Jake Snell, Yujia Li and Kevin Swersky for much of the following material.

Brief Review of SVMS

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Geometric Intuition

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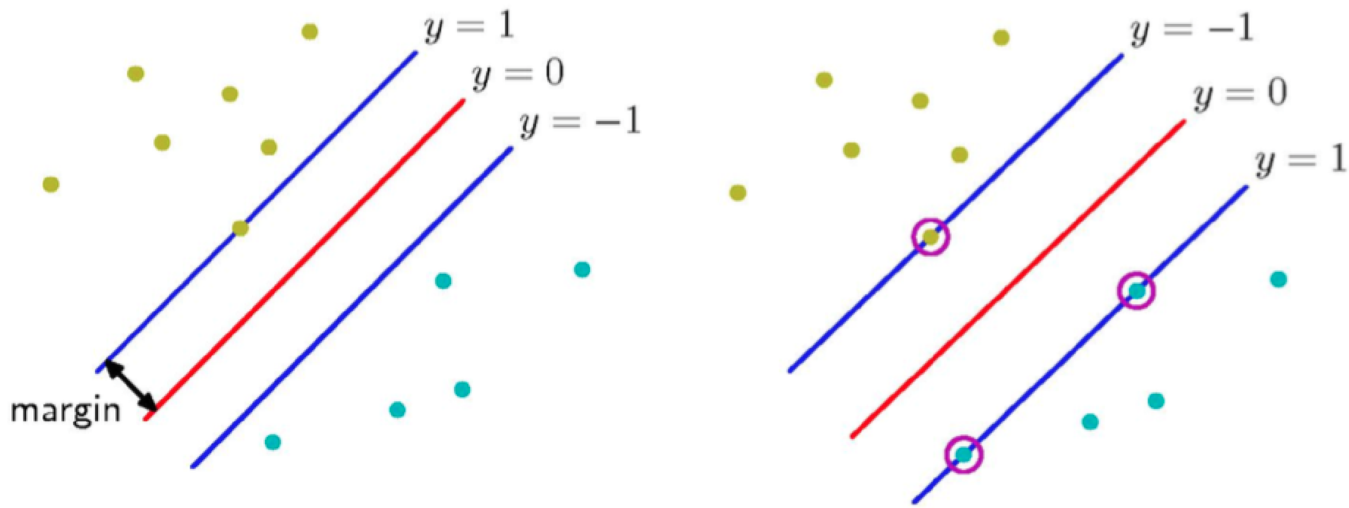

Margin Derivation

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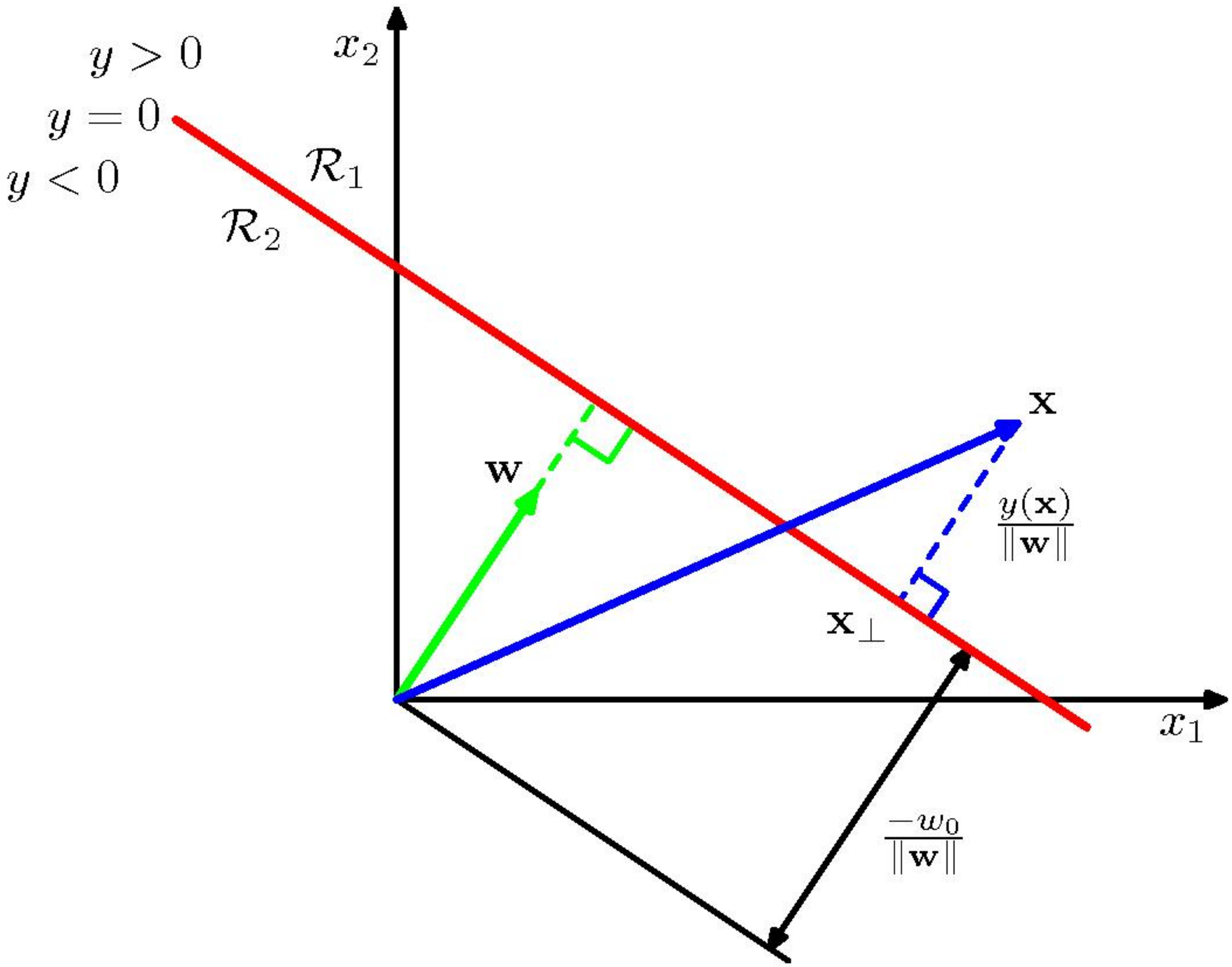

Margin Derivation

Compute the distance d_n of an arbitrary point x_n in the (+) class to the separating hyperplane.

$$w^T \left(x_n - d_n \frac{w}{||w||} \right) + b = 0$$

$$w^T x_n - d_n \frac{w^T w}{||w||} + b = 0$$

$$w^T x_n + b = d_n ||w||$$

$$d_n = \frac{w^T x_n + b}{||w||}$$

If we let $t_n \in \{1, -1\}$ denote the class of x_n, then the distance becomes

$$d_n = \frac{t_n(w^T x_n + b)}{||w||}$$

We can set $d_n = \dfrac{1}{||w||}$ for the point x_n closest to the decision boundary, leading to the problem:

$$\max \frac{1}{||w||}$$

$$\text{s.t. } t_n(w^T x_n + b) \geq 1, \ \text{ for } n = 1 \ldots N$$

SVM Problem

But scaling $w \to \kappa w$ and $b \to \kappa b$ doesn't change $d_n = \dfrac{t_n(w^T x_n + b)}{||w||}$.

or equivalently:

$$\min \frac{1}{2}||w||^2$$
$$\text{s.t. } t_n(w^T x_n + b) \geq 1, \text{ for } n = 1 \ldots N$$

Non-linear SVMs

For a linear SVM, $y(x) = w^T x + b$.

We can just as well work in an alternate feature space: $\tilde{y}(x) = w^T \phi(x) + b$.

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http://i.imgur.com/WuxyO.png
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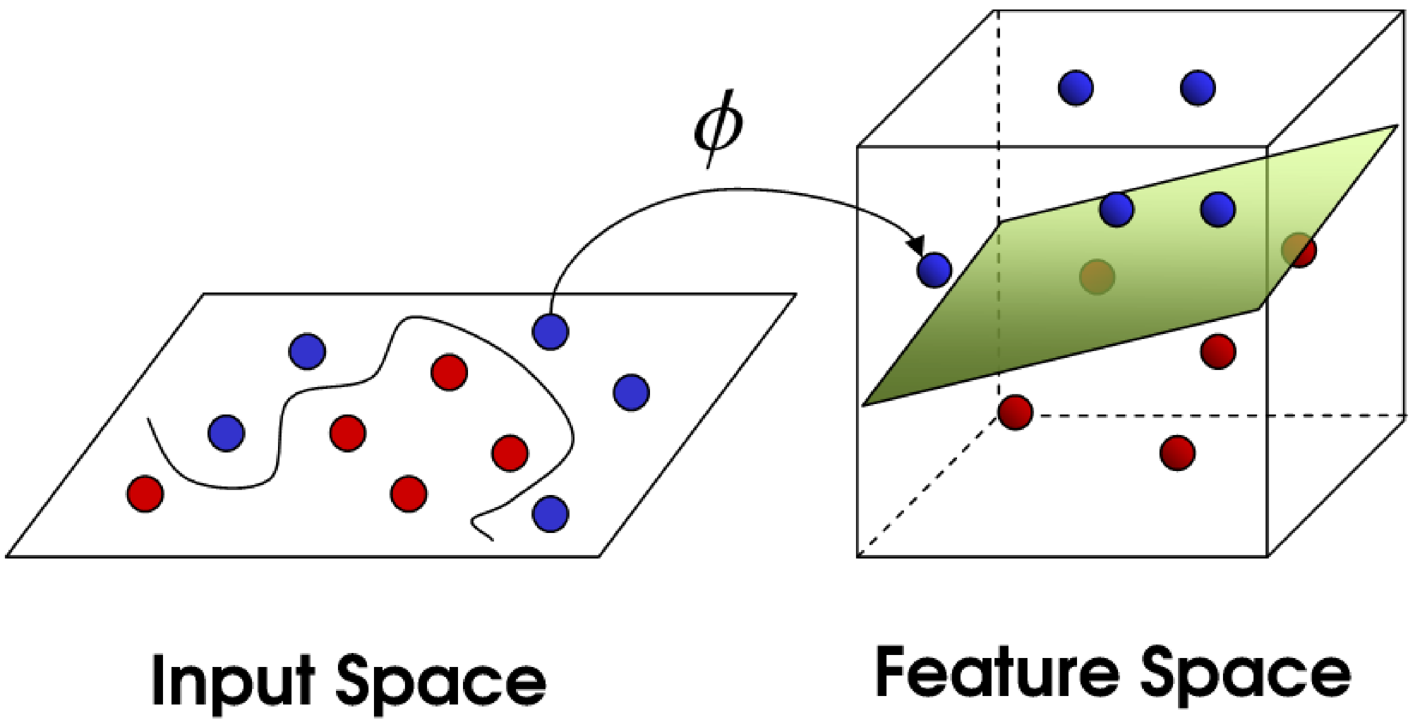
Out[29]:



Non-linear SVMs

http://www.youtube.com/watch?v=3liCbRZPrZA

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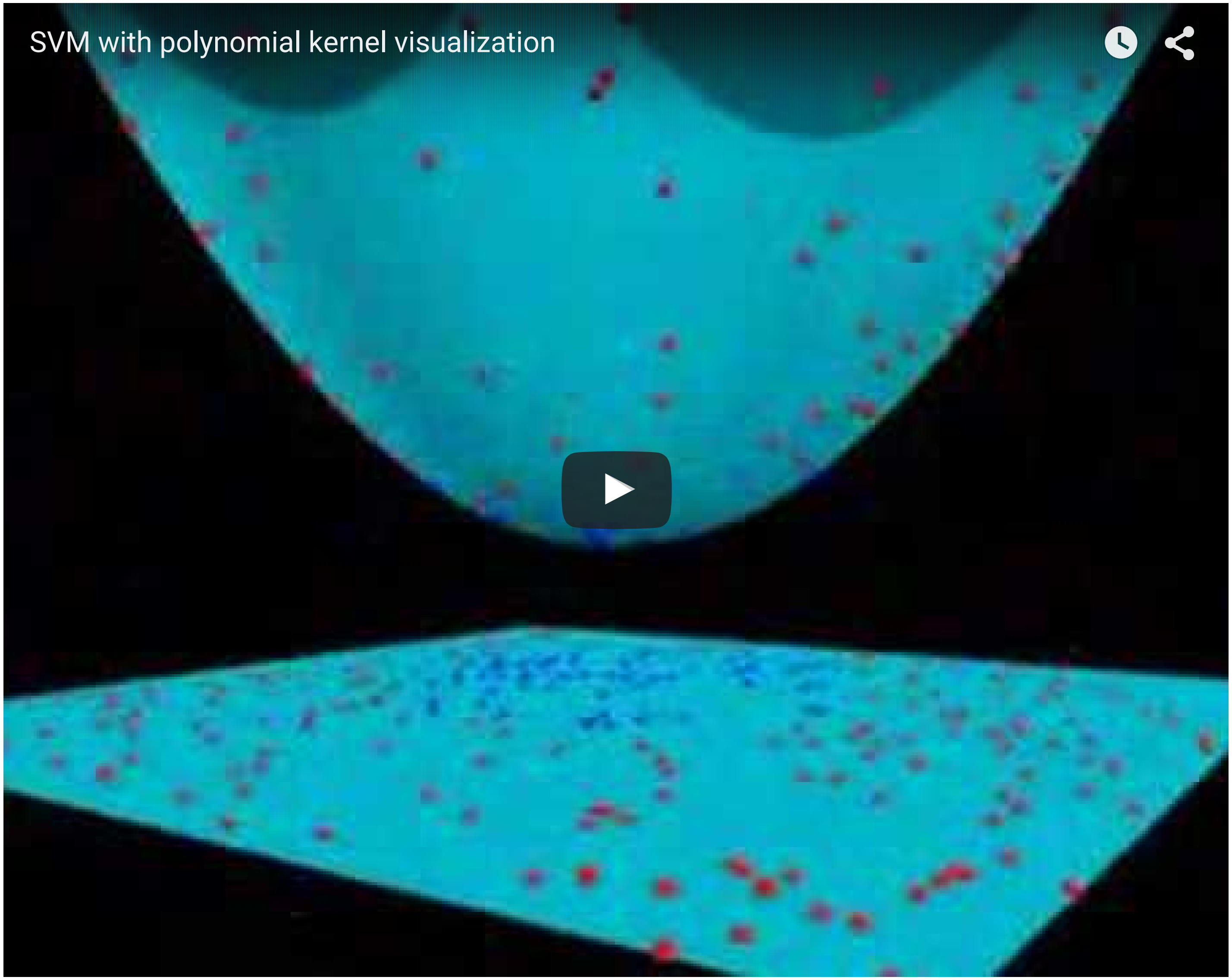

SVMs vs Logistic Regression

Logistic Regression

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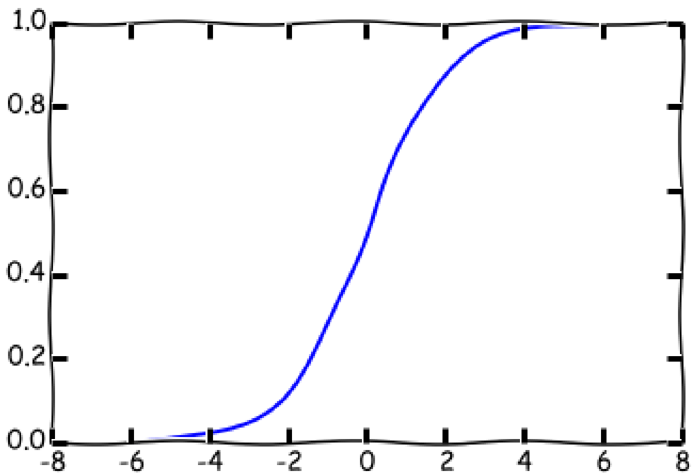


Logistic Regression

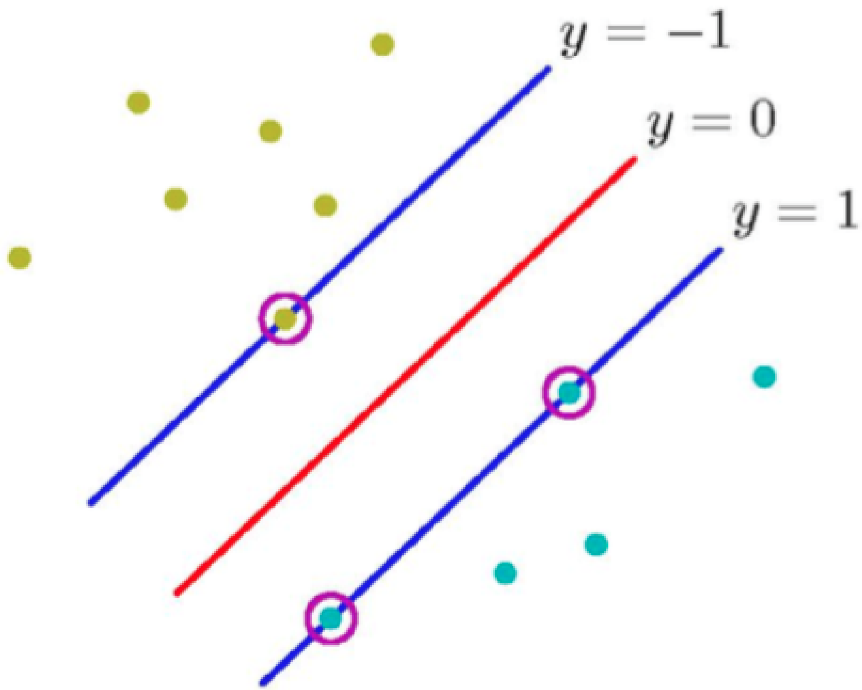
- Train to maximize likelihood
$$\mathcal{L}(w) = \prod_{n=1}^{N} \sigma(w^T x_n + b)^{y_n} (1 - \sigma(w^T x_n + b))^{1-y_n}$$

- Assign probability to each outcome
$$P(y = 1|x) = \sigma(w^T x + b)$$

- Linear decision boundary
$$\hat{y} = I[w^T x + b \geq 0]$$

SVMs

Out[33]:



SVMs

- Enforce a margin of separation
$$y_n(w^T x_n + b) \geq 1,\ \text{for } n = 1 \dots N$$
- Linear decision boundary
$$\hat{y} = I[w^T x + b \geq 0]$$

- Train to find the maximum margin
$$\min \frac{1}{2} ||w||^2$$
$$\text{s.t. } (2y_n - 1)(w^T x_n + b) \geq 1,\ \text{for } n = 1 \dots N$$

Comparison

- **Logistic regression** wants to maximize the probability of the data.
  - The greater the distance from each point to the decision boundary, the better.

- **SVMs** want to maximize the distance from the closest points to the decision boundary.
  - Doesn't care about points that aren't support vectors.

A Different Take

Consider an alternate form of the logistic regression decision function:

$$\hat{y} = \begin{cases} 1 & \text{if } P(y = 1|x) \geq P(y = 0|x) \\ 0 & \text{otherwise} \end{cases}$$

$$P(y = 1|x) \propto \exp(w^T x + b)$$

$$P(y = 0|x) \propto 1$$

A Different Take

Suppose we don't actually care about the probabilities. All we want to do is make the right decision.

We can put a constraint on the likelihood ratio, for some constant $c > 1$:

$$\frac{P(y = 1|x_n)}{P(y = 0|x_n)} \geq c$$

A Different Take

Take the log of both sides:

$$\log P(y = 1|x_n) - \log P(y = 0|x_n) \geq \log c$$

Recalling that $P(y = 1|x_n) \propto \exp(w^T x_n + b)$ and $P(y = 0|x_n) \propto 1$:

$$w^T x_n + b - 0 \geq \log c$$
$$w^T x_n + b \geq \log c$$

But c is arbitrary, so set it s.t. $\log c = 1$:

$$w^T x_n + b \geq 1$$

A Different Take

So now we have $(2y_n - 1)(w^T x_n + b) \geq 1$, for $n = 1 \ldots N$. But this may not have a unique solution, so put a quadratic penalty on the weights to make the solution unique:

$$\min \frac{1}{2} ||w||^2$$
$$\text{s.t. } (2y_n - 1)(w^T x_n + b) \geq 1, \text{ for } n = 1 \ldots N$$

By asking logistic regression to make the right **decisions** instead of maximizing the **probability** of the data, we derived an SVM.

Likelihood Ratio

The **likelihood ratio** drives this derivation:

$$r = \frac{P(y = 1|x)}{P(y = 0|x)} = \frac{\exp(w^T x + b)}{1} = \exp(w^T x + b)$$

Different classifiers assign **different costs** to r.

LR Cost

Choose $\text{cost}(r) = \log\left(1 + \frac{1}{r}\right)$

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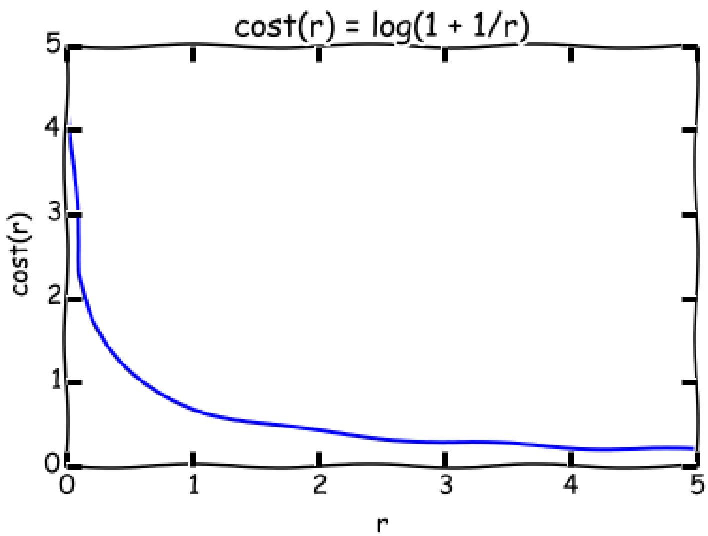


LR Cost

$$\begin{aligned}
\log\left(1+\frac{1}{r}\right) &= \log\left(1+\exp(-(w^T x+b))\right) \\
&= -\log\frac{1}{1+\exp(-(w^T x+b))} \\
&= -\log\sigma(w^T x+b)
\end{aligned}$$

Minimizing $\mathrm{cost}(r)$ is the same as minimizing the negative log-likelihood objective for logistic regression!

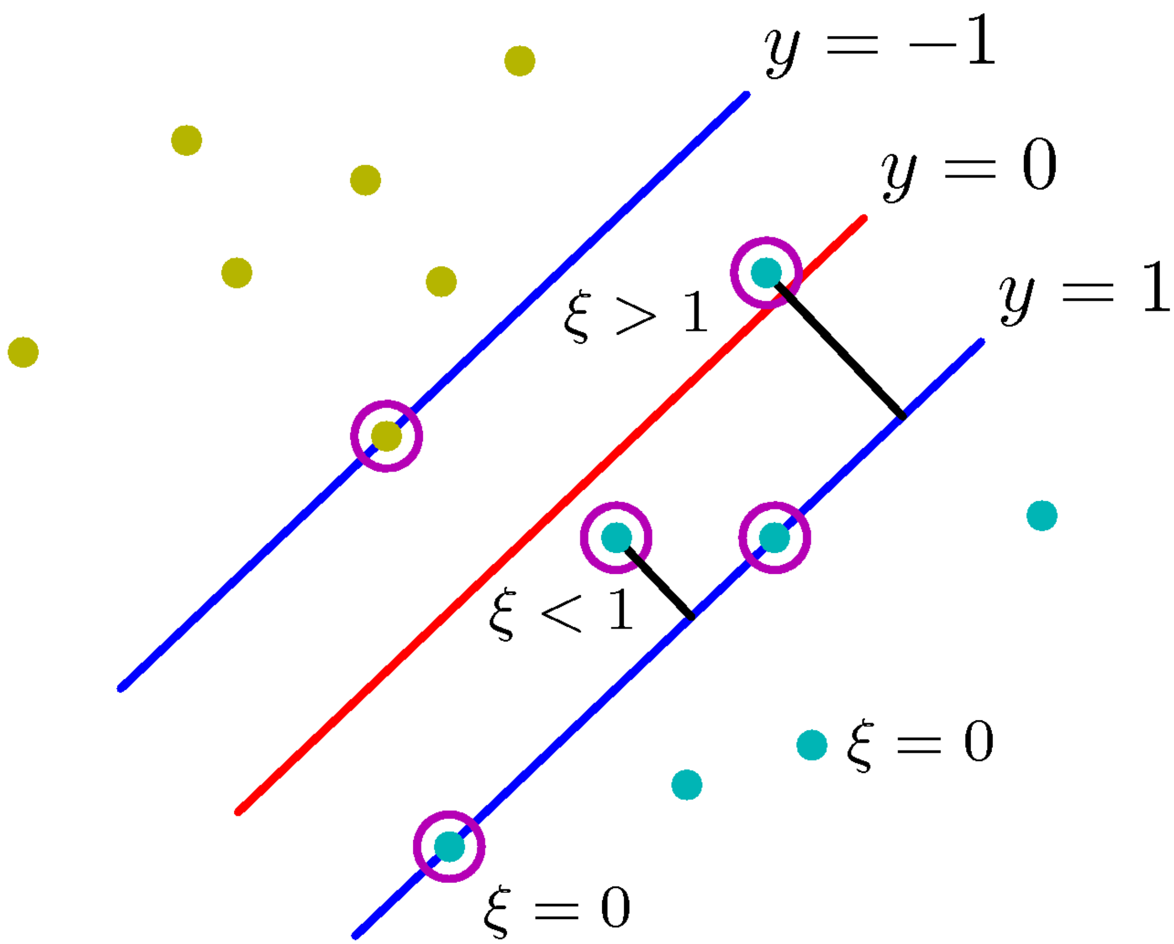
SVM with Slack Variables

If the data is not linearly separable, we can introduce slack variables.

$$\min \frac{1}{2}||w||^2 + C\sum_{n=1}^{N}\xi_n$$
$$\text{s.t. } (2y_n - 1)(w^T x_n + b) \geq 1 - \xi_n, \text{ for } n = 1 \ldots N$$
$$\text{and } \xi_n \geq 0, \text{ for } n = 1 \ldots N$$

SVM with Slack Variables

Out[58]:



SVM Cost

Choose $\mathrm{cost}(r) = \max(0, 1 - \log(r)) = \max(0, 1 - (w^T x + b))$

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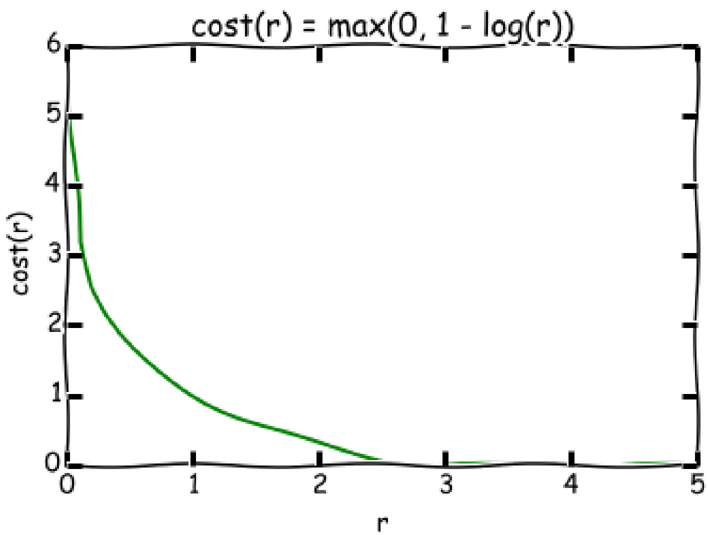

Plotted in terms of $\mathbf{r}$

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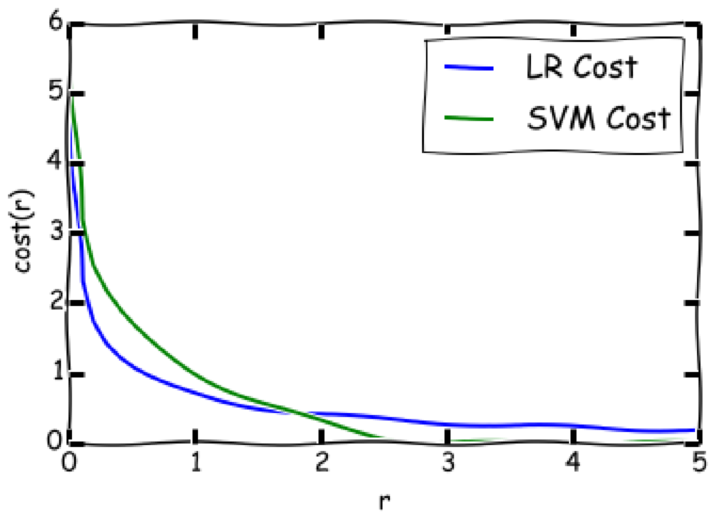

Plotted in terms of $\mathbf{w^T x + b}$

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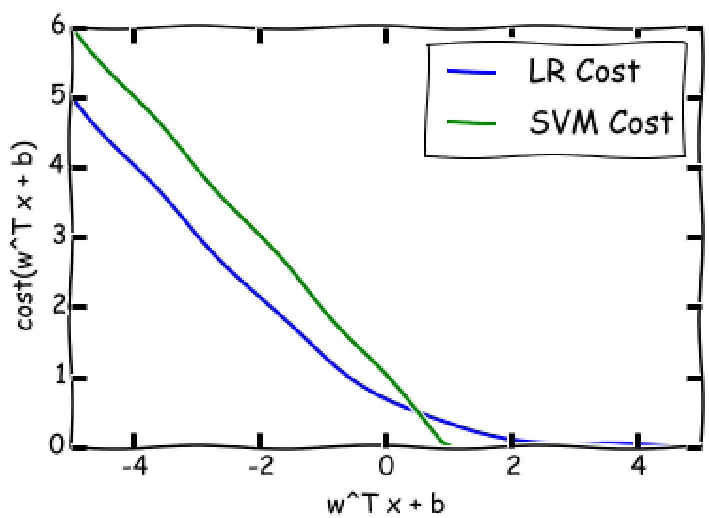

Exploiting the Connection between LR and SVMs

Kernel Trick for LR

In the dual form, the SVM decision boundary is

$$y(x) = w^T \phi(x) + b = \sum_{n=1}^{N} \alpha_n t_n K(x, x_n) + b = 0$$

We could plug this into the LR cost:

$$\log\left(1 + \exp\left(-\sum_{n=1}^{N} \alpha_n t_n K(x, x_n) - b\right)\right)$$

Multi-class SVMS

Recall multi-class logistic regression

$$P(y = i|x) = \frac{\exp(w_i^T x + b_i)}{\sum_k \exp(w_k^T x + b_k)}$$

Multi-class SVMS

Suppose that we just want the decision rule to satisfy

$$\frac{P(y = i|x)}{P(y = k|x)} \geq c, \text{ for } k \neq i$$

Taking logs as before,

$$(w_i^T x + b_i) - (w_k^T x + b_k) \geq 1, \text{ for } k \neq i$$

Multi-class SVMS

Now we have the quadratic program for multi-class SVMs.

$$\min \frac{1}{2}||w||^2$$
$$\text{s.t. } (w_{y_n}^T x_n + b_{y_n}) - (w_k^T x_n + b_k) \geq 1, \text{ for } n = 1 \ldots N, k \neq y_n$$

LR and SVMs are closely linked

- Both can be viewed as taking a probabilistic model and miminizing some cost associated with the likelihood ratio.
- This allows use to extend both models in principled ways.

Which to Use?

Logistic regression

- Gives calibrated probabilities that can be interpreted as confidence in a decision.
- Unconstrained, smooth objective.
- Can be used within Bayesian models.

SVMs

- No penalty for examples where the correct decision is made with sufficient confidence, which can lead to good generalization.
- Dual form gives sparse solutions when using the kernel trick, leading to better scalability.