Scale Mixture of Gaussians and the Statistics of Natural Images

by M.Wainwright, E.Simoncelli
(NIPS ’99)

Darius Braziunas

CSC 2541, Spring 2005

February 4, 2005

Statistics of natural images

- Statistics of coefficients of wavelet bases are non-Gaussian
  - Marginal densities have heavy tails
  - Joint densities have variance dependencies

- Images contain
  - smooth regions

    small filter responses => peak at zero
  - localized features (lines, edges, corners)

    Large amplitude response => heavy tails

Main ideas

- Neighborhoods of wavelet coefficients (adjacent positions, scales, orientations) are modeled as a product of Gaussian vector and scalar multiplier.
- Multiplier modulates local variance of coefficients within neighborhood
- Such models are variance-adaptive (e.g., ARCH)
- Building a Markov tree with hidden multiplier nodes can explain global image statistics

Gaussian scale mixtures (GSMs)

- Random vector **x** is a GSM iff it can be expressed as a product of normal vector **u** (with 0 mean) and an independent positive scalar random variable $\sqrt{z}$

$$\mathbf{x} = \sqrt{z}\,\mathbf{u}$$

- z is the multiplier
- **x** is an infinite mixture of Gaussian vectors

GSMs (cont.)

- GSM density is determined by
  - covariance matrix $\mathbf{C_u}$
  - mixing density $p_z(z)$

$$
\begin{aligned}
p_{\mathbf{x}}(\mathbf{x}) &= \int dz\, p(\mathbf{x}|z)\, p_z(z) \\
&= \int dz\, \frac{\exp\left(-\mathbf{x}^T (z\mathbf{C_u})^{-1}\mathbf{x}/2\right)}{(2\pi)^{N/2}|z\mathbf{C_u}|^{1/2}}\, p_z(z),
\end{aligned}
$$

GSMs (cont.)

- GSMs include
  - α-stale family (e.g., Cauchy distribution)
  - Generalized Gaussian (or Laplacian) family
  - Symmetric Gamma family
- GSM properties:
  - Symmetric
  - Zero-mean
  - Leptokurtotic marginal densities (heavy tails)
  - x is Gaussian when conditioned on z
  - x/sqrt(z) is Gaussian

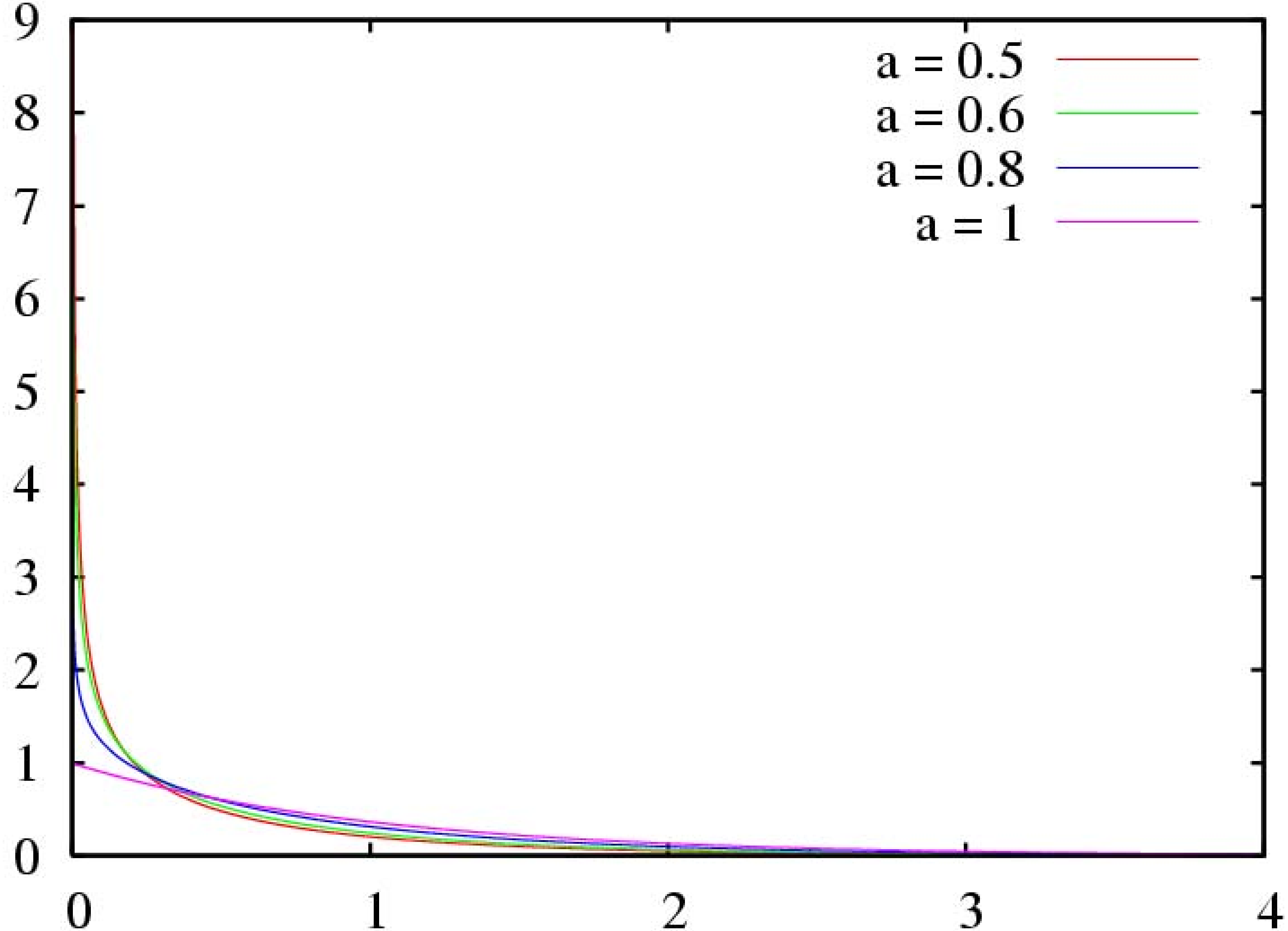
Gamma distribution

z is a gamma variable:

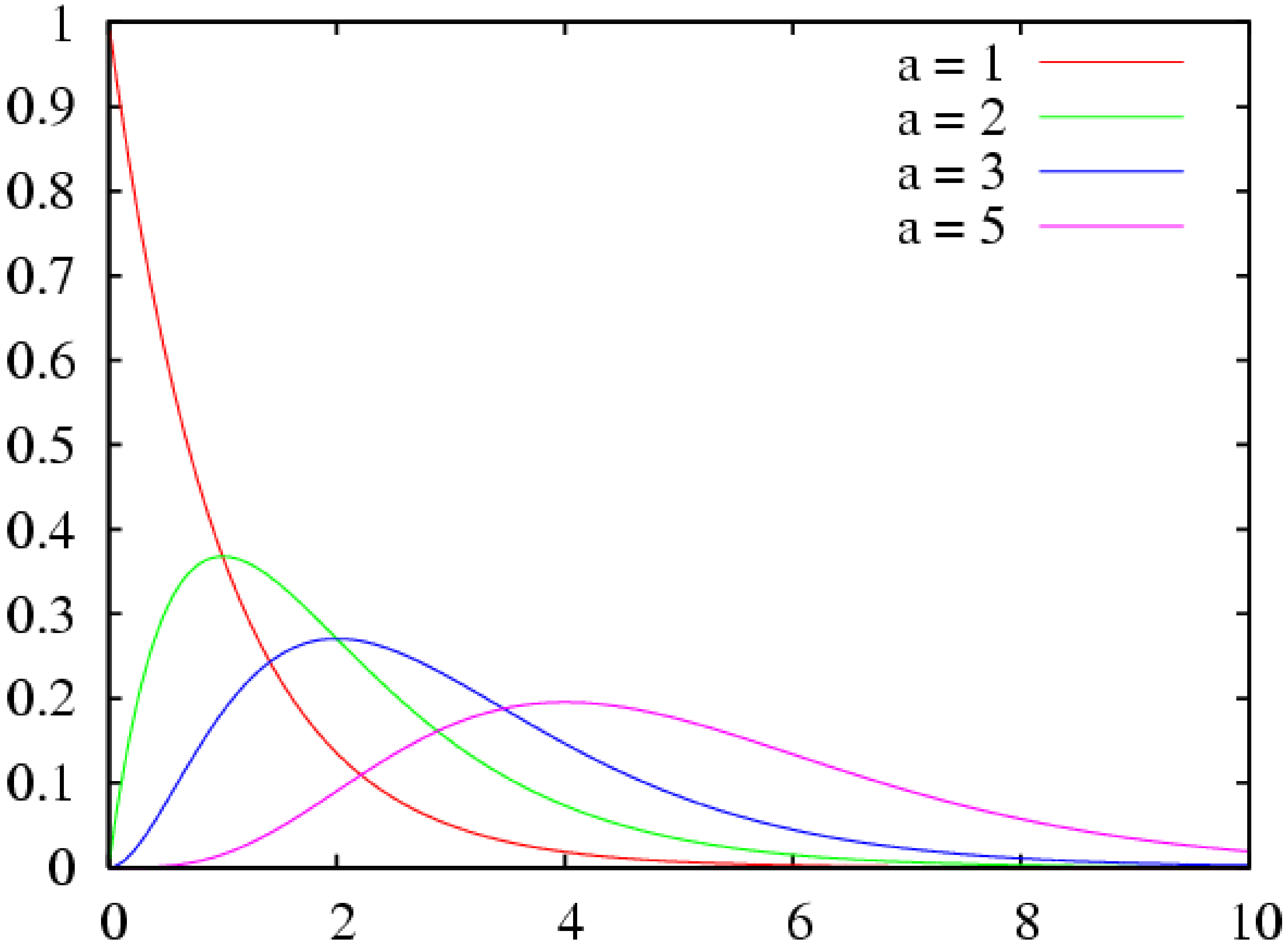
$p(z) = (1/\Gamma(a))\, z^{a-1} \exp(-z)$

$z \sim \mathrm{Gamma}(a,1)$

Gamma distribution



Gamma distribution



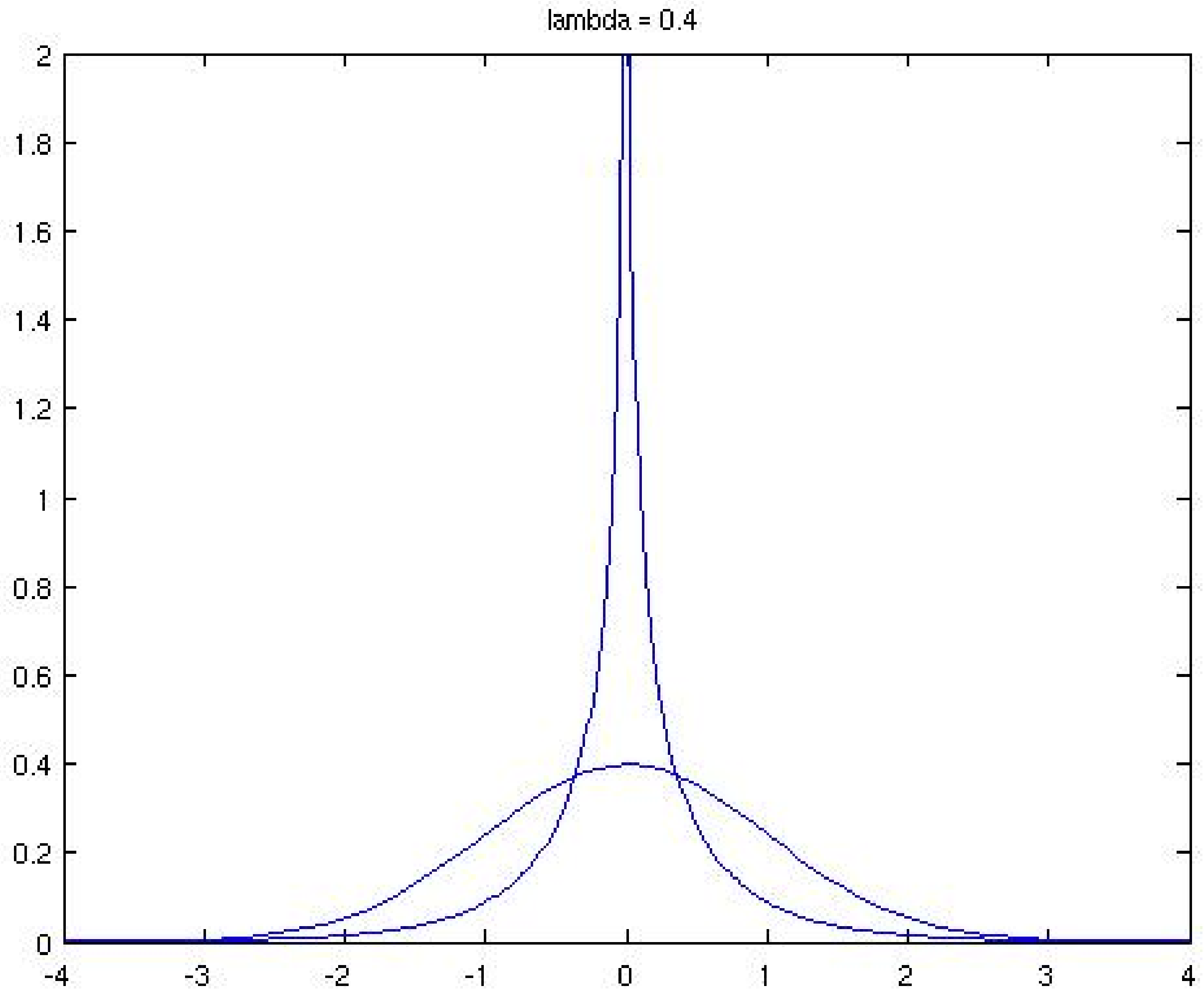
Symmetrized Gamma (log plot)

Mixing density	GSM density	GSM char. function
$\sqrt{Z(\gamma)}$	symmetrized Gamma	$(1+\frac{t^2}{2\lambda^2})^{-\gamma}, \quad \gamma > 0$
$1/\sqrt{Z(\beta-\frac{1}{2})}$	Student: $[1/(\lambda^2+y^2)]^{\beta}, \quad \beta > \frac{1}{2}$	No explicit form
Positive, $\sqrt{\frac{\alpha}{2}}$ – stable	α-stable	$\exp(-\lvert\lambda t\rvert^{\alpha}), \quad \alpha \in (0,2]$
No explicit form	generalized Laplacian: $\exp(-\lvert y/\lambda\rvert^{p}), \quad p \in (0,2]$	No explicit form

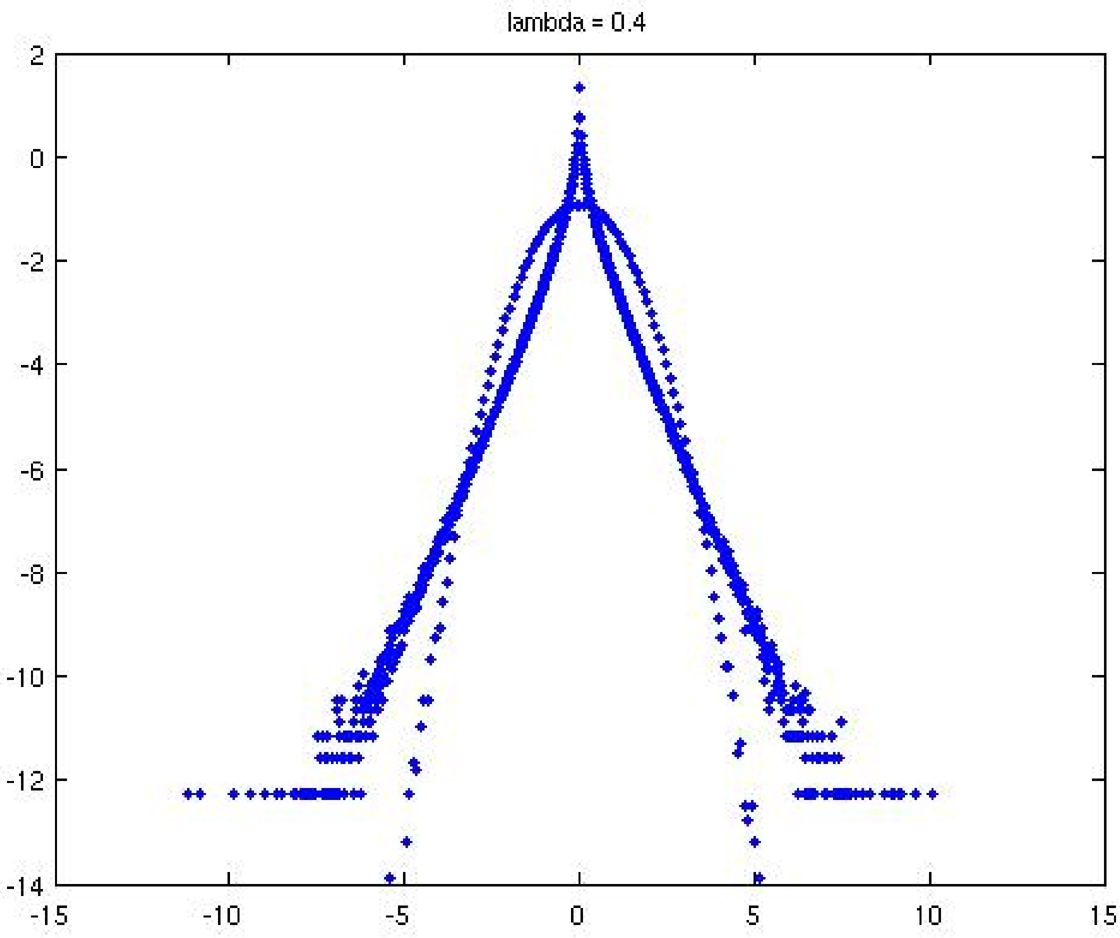
$$f_2(x;p,c) = \frac{1}{Z_2(p,c)} |x|^{p-0.5} K_{(p-0.5)}\left(\sqrt{\frac{2}{c}}|x|\right),$$

K is a Bessel function

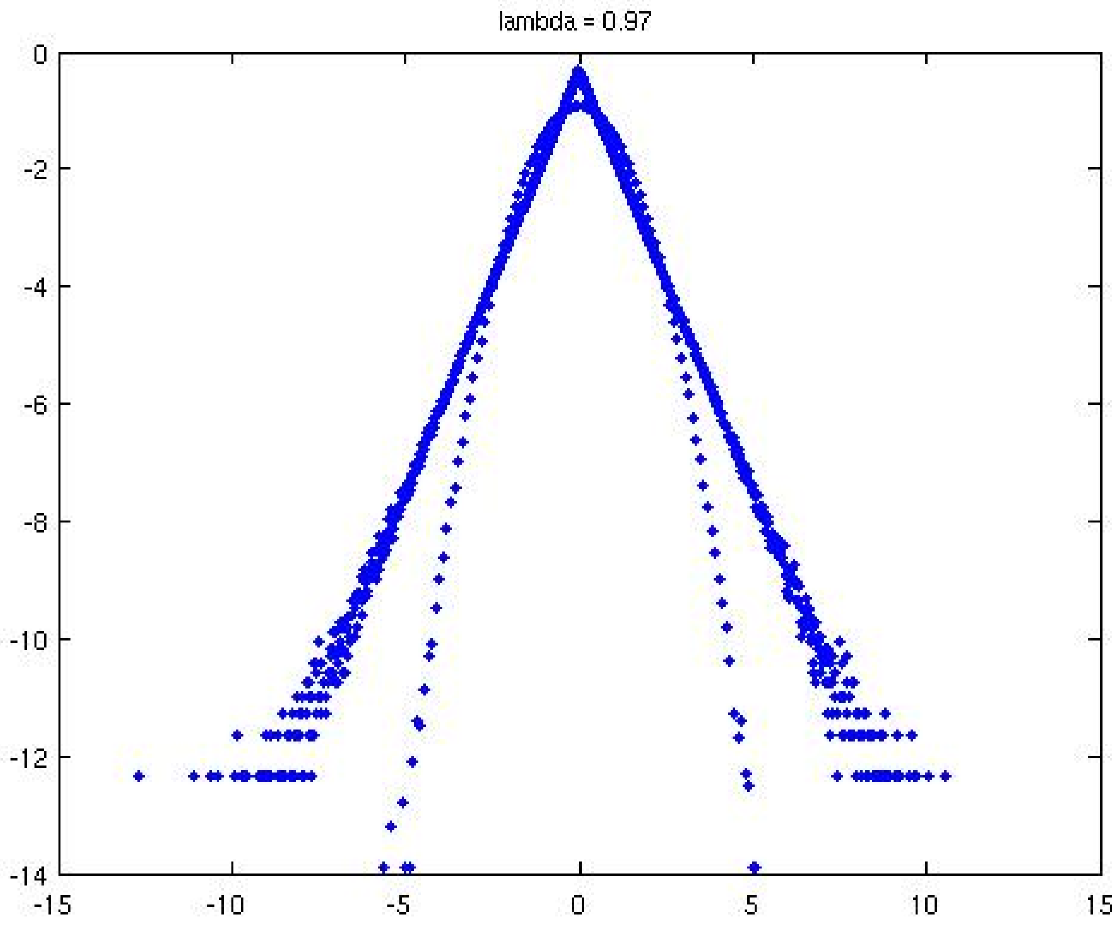
Symmetrized Gamma



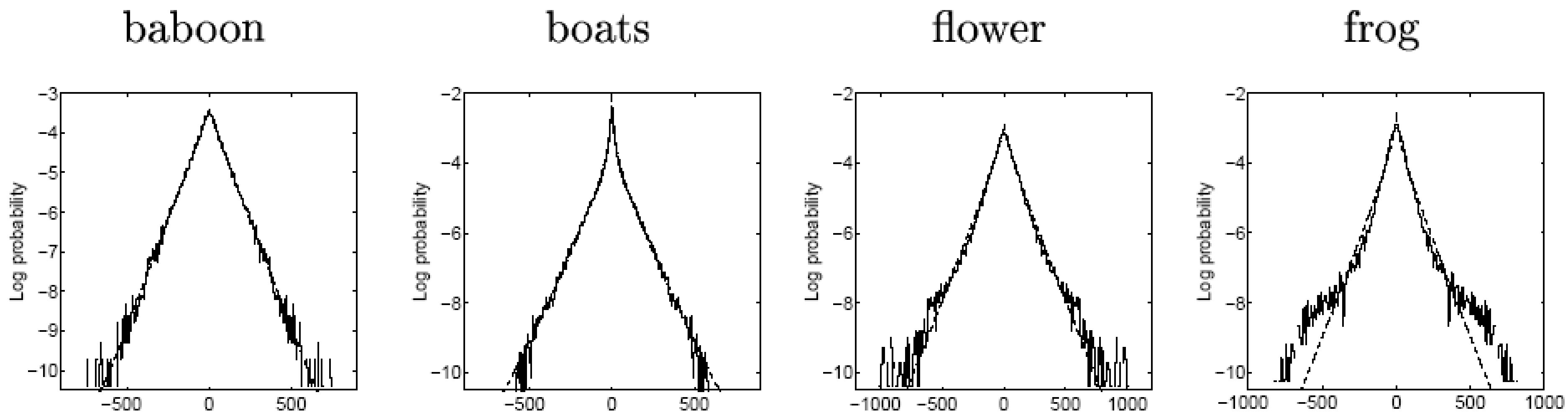
Symmetrized Gamma (log plot)



Symmetrized Gamma (log plot)



Symmetrized Gamma (log plot)



$[\gamma, \lambda^2] = [0.97, 15.04]$ $[0.45, 13.77]$ $[0.78, 26.83]$ $[0.80, 15.39]$

$\Delta H/H = 0.00079$ 0.0030 0.0030 0.0076

Figure 1. GSMs (dashed lines) fitted to empirical histograms (solid lines). Below each plot are the parameter values, and the relative entropy between the histogram (with 256 bins) and the model, as a fraction of the histogram entropy.

Estimating z

- N=11 neighbors (4 adjacent positions, 5 orientations, 2 scales)
- Observe coefficients Y
- Estimate hidden z:

$$
\begin{aligned}
\hat{z} &= \arg\max_z \{ \log p(Y|z) \} \\
&= \arg\min_z \{ N \log(z) + Y^T Q^{-1} Y / 2z^2 \} \\
&= \sqrt{Y^T Q^{-1} Y / N},
\end{aligned}
$$

Normalized coefficient

- **x = √z u**
- x_0 **/** √z is a normalized coeff.

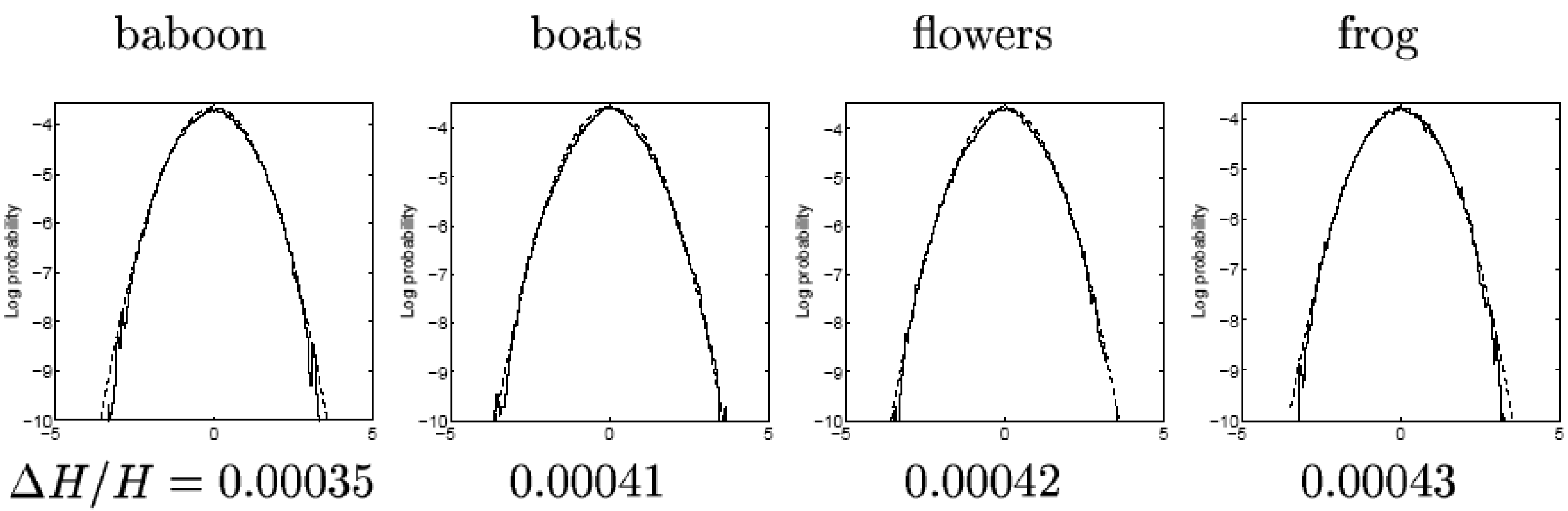


Figure 2. Marginal log histograms (solid lines) of the normalized coefficient ν for a single subband of four natural images. Each shape is close to an inverted parabola, in agreement with Gaussians (dashed lines) of equivalent empirical variance. Below each plot is the relative entropy between the histogram (with 256 bins) and a variance-matched Gaussian, as a fraction of the total histogram entropy.

Joint statistics

- Coefficients are nearly decorrelated, (to second-order) but not independent
- Dependency of coefficients across scales, positions, orientations
- GSMs can model such random fields with spatially fluctuating variance
- Local variance is governed by a continuous multiplier variable
- Peaks and cusps can be explained by presence of “objects” in images (sharp discontinuities)

Joint statistics

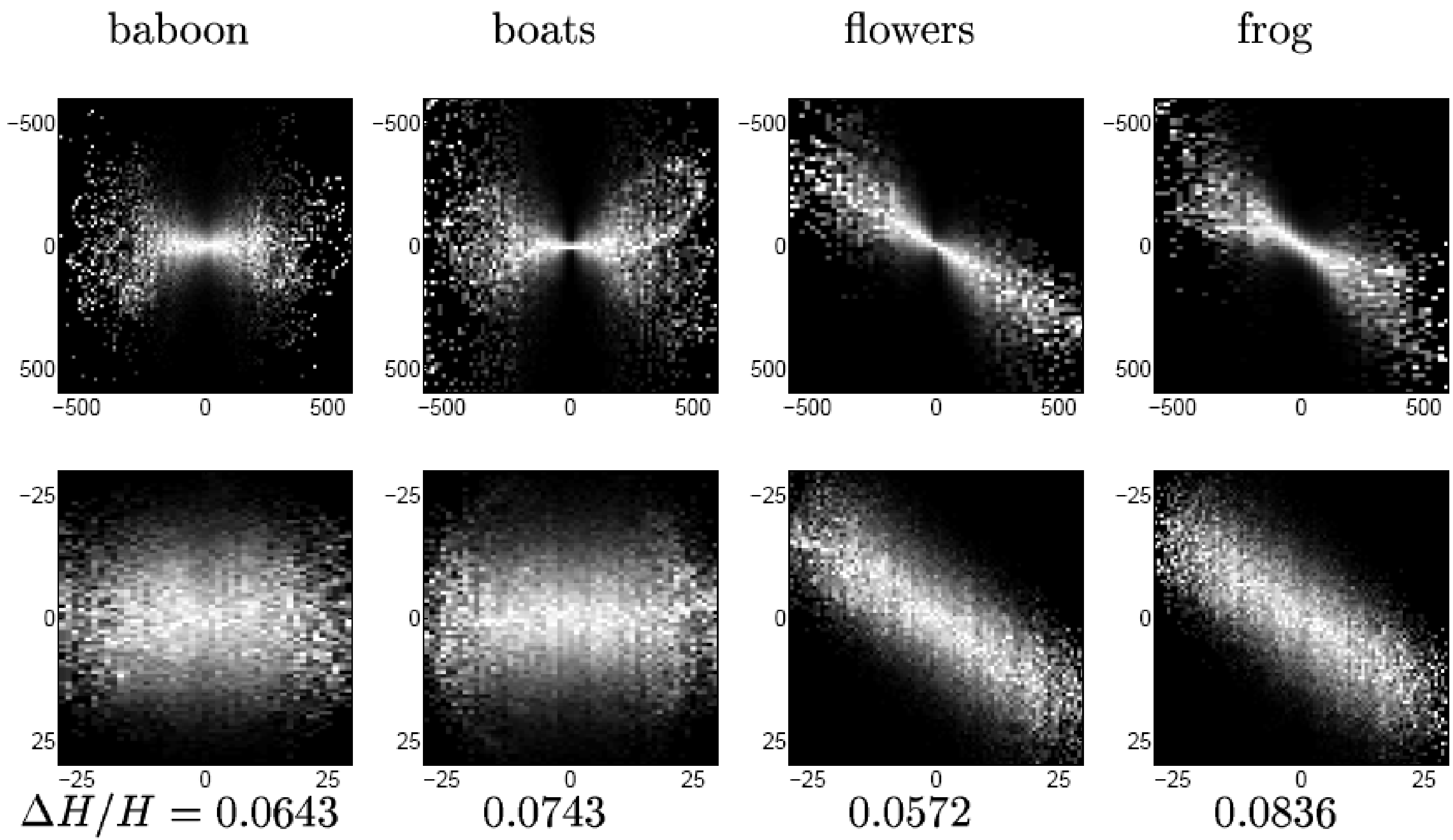
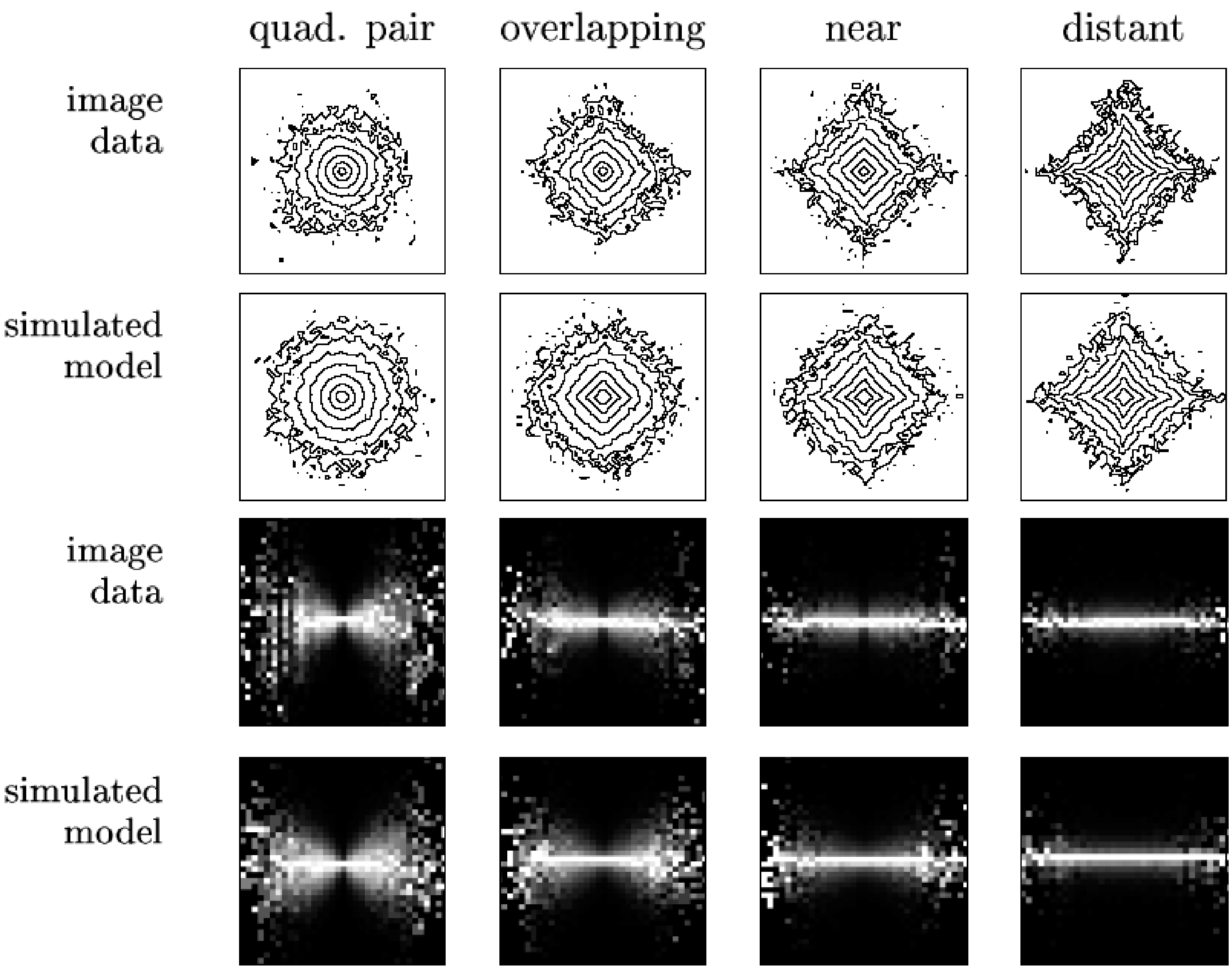


Figure 3. Top row: joint conditional histograms of raw wavelet coefficients for four natural images. Bottom row: joint conditional histograms of normalized pairs of coefficients. Below each plot is the relative entropy between the joint histogram (with 256×256 bins) and a covariance-matched Gaussian, as a fraction of the total histogram entropy.

Joint statistics



Markov structure

- Dependency between coefficients decreases as their spatial separation increases; therefore GSM is not enough
- Need graphical model to specify relations between multipliers
- Coefficients are linked by hidden scaling variables which govern local image structure
- Random cascades on a multiresolution tree

Markov structure

- At node s: $y(s) \stackrel{d}{=} \|x(s)\|\, u(s)$

$$y(s) = \left\| \mu^{d(s,s\wedge t)}\, x(s\wedge t) + v_1(s) \right\| \, u(s)$$

$$y(t) = \left\| \mu^{d(t,s\wedge t)}\, x(s\wedge t) + v_2(t) \right\| \, u(t)$$

Issues, questions

- Applications: denoising, compression
- Can GSM fully model image statistics?
- GSM is still state-of-the-art
- Joint coefficient distribution shapes not well explained
- How would you learn a tree of hidden scaling variables?