An Approach to Higher-Order Set Theory

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Foreword

The present paper is a revised and extended version of a manuscript (see also a short note [4]) written by the older author ten years ago when he was trying to make a notion of foundations of set theory suitable for a working mathematician. Currently, this is just a goal of the younger author and, thus, the paper presents a working mathematician view on the set theoretical foundations.
Essentially, its only distinction from the conventional standard is a very strong (in fact, higher-order) replacement axiom formulated *semantically* in contrast to its usual *syntactical* formulations where 'any function' is understood as 'any formula presenting a function'. Of course, to meet formal requirements adopted for working with set theory foundations, all the consideration is done within the framework of some sufficiently respectable set theory like Z or an elementary fragment of type theory *etc* so that "semantic" treatment of notion of 'any function' above turns out to be a syntactical one as required.

1. Motivation and outline of results.

An ordinary mathematician, normally more or less inclined to platonism, usually works not in the framework of model theory (syntax, semantics, interpretation) but rather in the purely semantic terms of Bourbaki mathematical structures considered in some intuitive set theory and intuitive logic. By formalizing the latter in a suitable metatheory, one can put the consideration into a syntactical framework (and thus meet requirements of the Hilbert formalism). In this paper a similar procedure is carried out for some mathematical structure intended to be "the set universe". However, we propose a way of formalizing metatheory where notions of formula, interpretation, satisfaction are left to be intuitive while set-theoretical metanotions are formalized (in contrast to, for example, formalizing metatheory in [1], where models of ZF, ZF itself and the truth predicate are considered the objects of formal Morse theory of classes, M).

In precise terms, in a standard first-order language $\Sigma_{\in}$ with a unique nonlogical binary predicate symbol $\in$ we introduce a certain formula $SU(V,\varepsilon)$ (with two free variables V and ε) which should be substantively understood as an $\in$-proposition: *"The set V with binary relation $\varepsilon \subset V{\times}V$ is a universe of sets, i.e. a model of some set theory with respect to symbol ε"*. Further, we will consider those consequences of the formula which can be obtained if some $\in$-theory of sets, $ST_{\in}$, is held in the language $\Sigma_{\in}$. $ST_{\in}$ is to be understood as the external metatheory of sets. The results of such considerations have the following shape:

$$ST_{\in} \vdash SU(V,\varepsilon) \rightarrow \psi(V,\varepsilon)\,,$$

where $\psi(V,\varepsilon)$ is to be understood as an $\in$-proposition about universe (V,ε). Actually, $SU(V,\varepsilon)$ is a conjunction of *ZF* axioms written down for the symbol ε as the membership predicate symbol and relativized by V. The replacement axiom scheme is replaced by a unique higher-order replacement axiom $\alpha(V,\varepsilon)$, expressible owing to $ST_{\in}$. (But externally $\alpha(V,\varepsilon)$ is a first order formula!) To define the constructions to be involved into $SU(V,\varepsilon)$ it is sufficient to take for $ST_{\in}$ Zermelo theory $Z_{\in}$ in the $\Sigma_{\in}$-language (or even a type theory fragment up to the fourth level inclusively).

In such a way of regarding things the set-theoretical concepts used in the metatheory are fixed explicitely by fixing $ST_{\in}$, and formula $SU(V,\varepsilon)$ may be called *the syntactical model of higher-order set theory.*

The following results will be proved.

If $Z_{\in}$ (or the above mentioned fragment of type theory) is taken for the external set theory $ST_{\in}$, then:

(1) The Morse class theory can be interpreted in $SU(V,\varepsilon)$.

(2) Any two universes (i.e. the models of $ZF^h_{\in}$ with the replacement axiom of higher order) are either ε-isomorphic or one of them can be embedded into another ε-isomorphically. (The exact formulations are presented in sections 3.2, 4.2 below.)

If $ZF_{\in}$ is taken for $ST_{\in}$, then for a strongly inaccessible cardinal $\kappa > \omega$ (that is denoted by $InA(\kappa)$) one has:

(3) $ZF_{\in} \vdash InA(\kappa) \rightarrow SU(R_\kappa, \in\restriction R_\kappa)$, where R_κ denotes the κ-level of the von Neumann universe in $ZF_{\in}$; and also

(4) $ZF_{\in} \vdash SU(V,\varepsilon) \rightarrow \exists\kappa\exists h\,(InA(\kappa)\,\&\,$ (h is an isomorphism from (V,ε) to $(R_\kappa, \in\restriction R_\kappa)))$.

These statements imply the equivalence of the axiom asserting existence of a universe (V,ε) (in $ZF_{\in}$) to the axiom asserting existence of a strongly inaccessible cardinal:

(5) $ZF_{\in} \vdash \exists V \exists\varepsilon\, SU(V,\varepsilon) \leftrightarrow \exists\kappa\, InA(\kappa)$.

The categoricity result about ZF^2 (i.e. the Zermelo–Frenkel theory formulated in the second order language) in the sense of theorem 2 is not new (see, for instance, [3]), however, the corresponding proofs are usually developed in the intuitive set theory, and some essential circumstances are lost or implicit. Particularly, the thesis that since models of ZF^2 are isomorphic then the continuum–hypothesis is either true or false, but not independent, protected by Kreisel (the references may be found in [3]), seems to be quite truthful. However, the approach proposed in this paper, when set-theoretical means used in the metatheory are fixed explicitely, demonstrates a scholastic nature of the argumentation like above. Indeed, (4) implies:

$$ZF_{\in} + CH_{\in} \vdash SU(V,\varepsilon) \rightarrow CH(V,\varepsilon),$$

$$ZF_{\in} + \neg CH_{\in} \vdash SU(V,\varepsilon) \rightarrow \neg CH(V,\varepsilon)$$

and, therefore, the question of whether *CH* is true is not solved but rather is transfered into the metatheory (where, certainly, it arises again because of independency of *CH* from $Z_{\in}$).

2. Set universe as a Bourbaki structure.

We assume that there are fixed some intuitive classical logic and some "conventional" intuitive set theory including the basic set-theoretical concepts of empty set, pair, power-set, relation and mapping of sets.

Consider a pair $v = (V, \varepsilon)$, with V a set and $\varepsilon \subset V \times V$ a binary relation on V. Elements of V are called *points* and will be denoted by $x, y, z, \cdots$. We shall call subsets of V *classes* and denote them by capital letters: $A, B, \cdots$. The set of all classes, i.e. subsets of V, will be denoted by $\mathbf{C}$, thus, $\mathbf{C} = 2^V$.

To make formulating axioms for (V, ε) easier we shall introduce the mapping "cap" $\hat{(\cdot)} : V \to \mathbf{C}$ by the rule $\hat{x} = \{y \in V : y \,\varepsilon\, x\}$ for all $x \in V$ (setting this mapping is equivalent to setting ε because $x \,\varepsilon\, y$ iff $x \in \hat{y}$ and we could start from $\hat{(\cdot)}$). The image of V under the mapping "cap" will be denoted by $\mathbf{V}$, $\mathbf{V} = \{\hat{x} : x \in V\}$.

Definition. A pair $v = (V, \varepsilon)$ is said to be *a universe*, if the following axioms (A1)–(A8) hold.

(The logical symbols used in the axiom formulations serve merely as substitutes for the words of the ordinary mathematical language. The following axioms (A1),...,(A6) are nothing but usual axioms of the theory Z in our notation.)

Extensionality (A1). $\forall x \in V \, \forall y \in V \, ((\hat{x} = \hat{y}) \Rightarrow (x = y))$.

Thus, $\hat{(\cdot)}$ is the injection, and the inverse mapping is defined on $\mathbf{V}$, $\check{(\cdot)} : \mathbf{V} \to V$.

Empty set (A2). $\emptyset \in \mathbf{V}$.

$\check{\emptyset}$ will be denoted by 0_v.

Pairing (A3). $\forall x \in V \, \forall y \in V \, (\{x, y\} \in \mathbf{V})$.

Let us denote: $\{x, y\}_v = (\{x, y\})^\vee$, $\{x\}_v = \{x, x\}_v$, $\langle x, y \rangle_v = \{\{x\}_v, \{x, y\}_v\}_v$.

Union (A4). $\forall x \in V \, (U_x = \bigcup \{\hat{y} : y \in \hat{x}\} \in \mathbf{V})$.

We shall denote $\check{U}_x$ by $\bigcup_v(x)$ and, as usually, $x \bigcup_v y = \bigcup_v(\{x, y\}_v)$. It is easy to see that $\widehat{x \bigcup_v y} = \hat{x} \bigcup \hat{y}$.

Power-set (A5). $\forall x \in V \, (P_x = \{y : \hat{y} \subset \hat{x}\} \in \mathbf{V})$.

$\check{P}_x$ will be denoted by $\mathbf{P}_v(x)$.

Foundation (A6). $\forall x \neq 0_v \, \exists y \in \hat{x} \, (\hat{x} \bigcap \hat{y} = \emptyset)$.

Let us define on V the unary operation $Sc_v : x \mapsto x \bigcup_v \{x\}_v$. $Sc_v(x)$ will be denoted also by $x +_v 1$ or x'.

Infinity (A7). $\Omega = \{0_v, \, 0'_v, \, 0''_v, \, \cdots\} \in \mathbf{V}$.

$\check{\Omega}$ will be denoted by ω_v.

Replacement (A8). $\forall F \, (F$ is a mapping $V \to V) \, \forall x \in V \, (F''(\hat{x}) \in \mathbf{V})$,

where $F'' : 2^V \to 2^V$ is the image mapping generated by F.

Axiom (A8) implies immediately the following one.

Comprehension (A8'). $\forall x \in V \, \forall A \subset \hat{x} \, (A \in \mathbf{V})$.

We note that by the definition of $\mathbf{P}_v$ in (A5) and by (A8) for $x \in V$ the sets $\widehat{\mathbf{P}_v(x)}$ and $2^{\hat{x}}$ are isomorphic. Therefore, if we accept the axiom of choice in our intuitive set theory, then the corresponding (V, ε)-statement will be a theorem for (V, ε).

In the usual way (with obvious modifications owing to a specific character of the replacement axiom) one can reproduce the standard set of set-theoretical constructions for the universe (V, ε): classes-relations (and points-relations), ordinals, cardinals, etc. (for example, what Th.J.Jech ([2]) calls classes we treat as subsets of V, and sets are classes belonging to $\mathbf{V}$, i.e., in fact, points from V (see section 4 for more precise elaboration)). We shall only describe here the construction of the von Neumann universe in (V, ε). It is a function $R_v : On_v \to V$, defined on the ordinals from V by recursion, where:

$$On_v = \{\alpha \in V : \hat{\alpha} \textit{ is a transitive set ordered totally by } \varepsilon\},$$

$R_v(0_v) = 0_v$, $R_v(\alpha') = \mathbf{P}_v(R_v(\alpha))$ and if α is a limit ordinal, then:

$$R_v(\alpha) = \bigcup\nolimits_v \{R_v(\beta) : \beta \in \hat{\alpha}\}^\vee = \left(\bigcup \{\widehat{R_v(\beta)} : \beta \in \hat{\alpha}\}\right)^\vee$$

$\widehat{R_v(\alpha)}$ will be denoted by V_α. As it is easy to see, the standard proof of equivalence between the axiom of foundation and the statement: "$V = \bigcup \{V_\alpha : \alpha \in On_v\}$" is valid.

It is hoped that a mathematician-platonist will agree to work with such structure (V, ε), if he believes in its existence. An adherent of the Hilbert formalism will probably require to explain what the expressions "the set V", "a subset of V", "a function $F : V \to V$", etc. mean and also what the words "any", "exists", etc. mean. The following section provides answers to these and similar questions.

3. Embedding of the (V, ε)-universe theory into formal set theory.

3.1. According to the point of view described above, the external set theory $ST_\in$ is intended to formalize the intuitive set theory in which the mathematical structures are considered. Therefore, methodologically it is important to make the external formalism as weak as required to make it "intuitively clear"; in any case, it is not desirable to use any transfinite constructions. On the other hand, there is required definability of the set-theoretical concepts used in (A1),...,(A8) in the external formalism.

Generally speaking, the type theory fragment up to the fourth level inclusively is sufficient for this purpose. In this way, the axioms (A1),...,(A8) idicated in section 2 can be written down as $\in$-formulas $\phi_1, \dots, \phi_8$. Then, it is possible to introduce the following $\in$-formula with two free variables V and ε:

$$SU(V, \varepsilon) = (\varepsilon \subset V \times V) \& \phi_1 \& \phi_2 \& \cdots \& \phi_8.$$

Now, the development of the (V, ε)-universe theory (informally understood as proving theorems of the form: "let (V, ε) be the universe, then ψ holds", where ψ is some statement about the universe) takes the form of deductions in $Z_\in$ of the following kind:

$$Z_\in \vdash SU(V, \varepsilon) \to \psi(V, \varepsilon), \qquad (*)$$

where $\psi(V, \varepsilon)$ is the translation of the statement ψ into the $\in$-language.

3.2. What are relations between the formalism $Z_\in + SU(V, \varepsilon)$, i.e. the statements $\psi(V, \varepsilon)$ satisfying (*), and commonly accepted set theories ZF, GB, M?

To write the axioms of these theories, we introduce a standard first order language Σ_e with equality and signature consisting of a unique binary predicate symbol e. This language is said to be *e-language*, its formulas we will call *e-formulas*. The collection of all e-formulas will be denoted by Φ_e. (Certainly, we can take $\Sigma_\in$ for Σ_e.) Now, let two variables be fixed in Σ_e. We define a mapping $\#_{V,\varepsilon} : \Phi_e \to \Phi_\in$, which assigns to any e-formula ϕ its interpretation $\phi^\#$, $\phi \mapsto \phi^\#$, as follows. (Details and numerous "hygienic" reservations are omitted here in a hope that this will not be confusily lead to misunderstanding).
Every atomic e-formula of the form $x\,\varepsilon\,y$ is interpreted by $\in$-formula $\langle x, y\rangle \in \varepsilon$; the equalities are left without changes, i.e. we take $x = y$ for $(x =_e y)^\#$. Then, every compound e-formula is interpreted by "the same" $\in$-formula (with respect to dispositions of connectives and quantifiers), which is additionally relativized with respect to V: all the quantifiers are bounded by the unary predicate $\ldots \in V$ (so that, e.g., $(\exists x\,(\psi))^\#$ is $\exists x \in V\,(\psi^\#)$). A result of the interpretation is said to be *$e^\#$-formula*. The mapping $\#_{V,\varepsilon}$ just defined is such that for any closed e-formula ϕ the corresponding $e^\#$-formula $\phi^\#$ "says the same" but about the universe (V, ε). Any such $\phi^\#$ includes free variables V and ε, and, therefore, it might be written in the form $\phi^\#(V, \varepsilon)$. It is easy to see that for any axiom ϕ of theory $ZF_e \subset \Phi_e$,

$$Z_\in \vdash SU(V,\varepsilon) \to \phi^\#(V,\varepsilon),$$

i.e. $Z_\in + SU(V,\varepsilon)$ is stronger than ZF.

To consider the relationship between $Z_\in + SU(V,\varepsilon)$ and the Morse class theory M, we define the relation $\varepsilon_c \subset \mathbf{C}\times\mathbf{C}$, where $\mathbf{C} = 2^V$ for some fixed universe (V,ε), as follows:

$$\varepsilon_c = \{\langle X, Y\rangle : (X \in \mathbf{V})\&(Y \in \mathbf{C})\&(\check{X} \in Y)\} =$$

$$= \{\langle X, Y\rangle : \exists x \in V\,(\{z \in V : \langle z, x\rangle \in \varepsilon\} = X)\&(Y \in \mathbf{C})\&(x \in Y)\}.$$

Now, we construct a mapping $*_{V,\varepsilon} : \Phi_e \to \Phi_\in$, $\phi \mapsto \phi^*$. Let every atomic e-formula of the form $X\,e\,Y$ (to be understood as a statement about classes) is interpreted by the $\in$-formula $\langle X, Y\rangle \in \varepsilon_c$, and the equalities are left without changes. To interprete compound formulas we leave their connective and quantifier structure without changes but relativize all quantifiers with respect to $\mathbf{C}$. This completes the construction of $*_{v,\varepsilon}$. It is such that e^*-formulas from $\Phi_\in$ "say" about the universe of classes $\mathbf{C}_v(\mathbf{C}, \varepsilon_c)$, built up on the universe (V,ε) (particularly, $(X \text{ is a set })^* \Leftrightarrow X \in \mathbf{V}$). It is easy to see that for any axiom ϕ of the theory $M_e \subset \Phi_e$

$$Z_\in \vdash SU(V,\varepsilon) \to \phi^*(\mathbf{C}, \varepsilon_c)\,,$$

i.e. $Z_\in + SU(V,\varepsilon)$ is stronger than M.

This fact can be explained informally as follows. Indeed, the $*_{V,\varepsilon}$-interpretation of the axiom scheme of class existence from M_e postulates existence of a v-class, i.e. a subset of V, for any $\in$-formula of the form ϕ^* with $\phi \in \Phi_e$. Note, however, that the comprehension $Z_\in$-axiom-scheme guarantees the existence of v-class for an arbitrary $\in$-formula. Since $\Phi_\in$ includes $\Phi_e^* = \{\phi^* : \phi \in \Phi_e\}$, M_e can be interpreted in $Z_\in + SU(V,\varepsilon)$, but not the reverse (at least by the second Gödel incompleteness theorem: the consistency of M can be proved in $Z_\in + \exists V\,\exists\varepsilon\,SU(V,\varepsilon)$).

4. Alternative for internal universes.

4.1. The following material will be described in some informal mathematical language with a hope that the translation of statements and proofs into the corresponding $\in$-formulas and formal proofs in the $\in$-language will be a technical matter. Usual set-theoretical constructions reproduced in the universe (V, ε) will be denoted here without any additional word in the commonly accepted way with the index v, e.g., $\mathbf{P}_v(x)$, $\bigcup_v(x)$, ω_v, On_v, f_v is v-function etc. Note, however, that often these v-constructions are assumed to be introduced with some modifications similar to those that were used in order to transform the replacement axiom scheme of *ZF* into replacement v-axiom (A8) from SU. In such a way, for example, the completeness of the order relation R on x means that every $\in$-subset $\hat{x}$ has R-least element, while this should be required only for those $\in$-subsets which are expressible by $e^{\#}$-formulas. (So, v-concepts definable in SU mean something more their counterparts in *ZF*).

4.2. Theorem. *Let $v_1 = (V_1, \varepsilon_1)$ and $v_2 = (V_2, \varepsilon_2)$ be universes (within $Z_\in$). Then the following alternative holds, i.e., one and only one statement among the following three is true.*

(i) (V_1, ε_1) is isomorphic to (V_2, ε_2);
(ii) there exists a cardinal κ in V_1 (it is necessarily v-strongly inaccessible) such that $(V_{1\kappa}, \varepsilon_1)$ is isomorphic to (V_2, ε_2);
(iii) there exists a cardinal κ in V_2 (it is necessarily v-strongly inaccessible) such that (V_1, ε_1) is isomorphic to $(V_{2\kappa}, \varepsilon_2)$.

Syntactically, it has the form:

$$Z_\in \vdash SU(V_1, \varepsilon_1) \& SU(V_2, \varepsilon_2) \rightarrow$$

$$\rightarrow (\exists h(h \textit{ is isomorphism from } (V_{1\kappa}, \varepsilon_1) \textit{ onto } (V_2, \varepsilon_2)) \bigvee$$

$$(\exists h \exists \kappa \in On_{v1}((h \textit{ is isomorphism from } (V_{1\kappa}, \varepsilon_1 \upharpoonright V_{1\kappa}) \textit{ onto } (V_2, \varepsilon_2)) \& InA_{v1}(\kappa))) \bigvee$$

$$(\exists h \exists \kappa \in On_{v2}((h \textit{ is isomorphism from } (V_1, \varepsilon_1) \textit{ onto } (V_{2\kappa}, \varepsilon_2 \upharpoonright V_{2\kappa})) \& InA_{v2}(\kappa)))\,.$$

(Here $\bigvee$ denotes the strict disjunction.) There is an explicit construction (by transfinite recursion with respect to v-ordinals) for every isomorphism mentioned above.

Corollary. *By adding to $SU(V, \varepsilon)$ the axiom asserting nonexistence of an inaccessible cardinal, $\neg \exists \kappa \, InA_v(\kappa)$, one obtains categorical theory $\overline{SU}(V, \varepsilon)$ (in $Z_\in$):*

$$Z_\in \vdash \overline{SU}(V_1, \varepsilon_1) \& \overline{SU}(V_2, \varepsilon_2) \rightarrow (i)$$

4.3. To prove 4.2 one needs inductive arguments and recursive definitions with ordinals from (V, ε).

Theorem. *Let (V, ε) be a universe. Then*

(i) $\forall A \subset On_v((0_v \in A) \& ((\alpha \in A) \Rightarrow (\alpha' \in A)) \& ((\hat{x} \subset A) \Rightarrow$
$\Rightarrow (\bigcup_v(x) \in A)) \Rightarrow (A = On_v))$;

(ii) $\forall G \forall W \, (G \textit{ is a function}) \& (dom(G) = \mathbf{P}(V \times W)) \Rightarrow$
$\Rightarrow \exists! \, F \, ((F \textit{ is a function}) \& (dom(F) = On_v) \& (rng(F) \subset W) \&$
$\& (\forall \alpha \in On_v(F(\alpha) = G(F(\hat{\alpha}))))$.

That is, in the syntactical form:

$$Z_\in \vdash SU(V, \varepsilon) \rightarrow (i) \& (ii)$$

This theorem can be proved as its analog in the *ZF* theory (see, for innstance, [2]).

The point that F and G are out of V requires an evident modification possible because of the replacement v-axiom (A8) and comprehension $Z_{\in}$-axiom.

4.4. Let (V_1, ε_1) and (V_2, ε_2) be universes. By using theorem 4.3 we shall construct $\in$-function $F : On_1 \to On_2$ (here and furtheron, instead of indexes v_1 and v_2 we will write 1 and 2) and state its properties.

$F(0_1) = 0_2$. In the assumption that F is defined on $\hat{\alpha}$ and also that

$(*_\alpha)$ $\forall \beta \in \hat{\alpha}\,(F \upharpoonright \hat{\beta}$ is $<_\varepsilon$-isomorphism from $\hat{\beta}$ on $\widehat{F(\beta)})$,

we define $F(\alpha)$ and prove $(*_\alpha)$ for $\alpha +_1 1$.

For $\alpha = \alpha^- +_1 1$ we set $F(\alpha) = F(\alpha^-) +_1 1$, and it is obvious that $(*_\alpha)$ is valid for this α.

For a limit α, i.e. $\alpha = \bigcup_1(\alpha) = (\bigcup\{\hat{\beta} : \beta \in \hat{\alpha}\})^\vee$ we consider $\in$-set (v_2-class) $A = F''(\hat{\alpha}) = \{F(\beta) : \beta \in \hat{\alpha}\}$. ($A$ is a $\in$-set because of the comprehension axiom in the external $\in$-set theory $Z_{\in}$.) There is one and only one statement holding true among the following.

(a_α) $\exists \lambda \in On_2(\lambda > A)$, i.e. $\exists \lambda \in On_2(A \subset \hat{\lambda})$;

(b_α) $\neg\exists \lambda \in On_2(\lambda > A)$, i.e. $\forall \lambda \in On_2 \exists \gamma \in A\,(\lambda < \gamma)$.

In the first case by comprehension axiom (A8') for (V_2, ε_2) $A \in \mathbf{V}_2$ and, hence, there is $\check{A} \in V_2$. We set $F(\alpha) = \bigcup_2(\check{A}) = (\bigcup\{\widehat{F(\beta)} : \beta \in \hat{\alpha}\})^\vee$. Now, the statement $(*_{\alpha+_1 1})$ can be verified directly.
In the case (b_α) we should have $A = On_2$, since otherwise there is the contradiction with $(*_\alpha)$, i.e., F defined on $\hat{\alpha}$ is really an $<_\varepsilon$-isomorphism from $\hat{\alpha}$ onto On_2.
As a result, by recursion on v_1-ordinals, we obtain the following alternative.

(i) There exists such $\kappa \in On_1$ that (b_κ) holds, then a $<_\varepsilon$-isomorphic bijection $F : \hat{\kappa} \to On_2$ can be constructed.

(ii) (a_α) holds for all $\alpha \in On_1$; then we can use recursion on all the ordinals to get an $<_\varepsilon$-isomorphic injection $F : On_1 \to On_2$, and in addition the following alternative holds here:

(ii') there is $\lambda > F''(On_1)$ and then there exists the least $\kappa > F''(On_1)$ and F is an $<_\varepsilon$-isomorphic bijection from On_1 onto $\hat{\kappa}$;

(ii") there is not $\lambda > F''(On_1)$, then F is a bijection from On_1 onto On_2.

Summarizing the above consideration we obtain the following

Theorem. *Let (V_1, ε_1) and (V_2, ε_2) be universes (within $Z_{\in}$). Then one and only one statement among the following three is valid.*

(i) On_1 is $<_\varepsilon$-isomorphic to On_2.

(ii) There exists such $\kappa \in On_1$ that $\hat{\kappa}$ is $<_\varepsilon$-isomorphic to On_2.

(iii) There exists such $\kappa \in On_2$ that On_1 is $<_\varepsilon$-isomorphic to $\hat{\kappa}$.

There are explicit constructions for all isomorphisms mentioned here.

4.5. Now we turn to a proof of theorem 4.2. Let (V_1, ε_1) and (V_2, ε_2) be universes. $<_\varepsilon$-isomorphism stated by the theorem is denoted by F. We shall prove that for any limit $\lambda \in dom(F)$ there is an isomorphism from $(V_{1\lambda}, \varepsilon_1)$ onto $(V_{2F(\lambda)}, \varepsilon_2)$.

By the recursion up to λ we shall define $\in$-mapping $H : \hat{\lambda} \to {V_2}^{V_1}$ below. (Simultaneously, we shall formulate its properties.)
$H(0_1) = H_0 = \emptyset$. Assuming that H is already defined on $\hat{\alpha}$ (for $\alpha \in \hat{\lambda}$) with:

$(*_\alpha)\quad \forall\beta \in \hat{\alpha}\,(dom(H_\beta) = V_{1\beta}$ and H_β is an ε-isomorphism from $V_{1\beta}$ on $V_{2F(\beta)}$,

we define H_α and prove that $(*_\alpha)$ is valid for $\alpha = \alpha^- +_1 1$ too.

If $\alpha = \alpha^- +_1 1$ then $x \in V_{1\alpha}$ implies the inclusion $\hat{x} \subset V_{1\alpha}$ and, therefore, $\in$-set (v_2-class) $H''_{\alpha^-}(\hat{x}) \subset V_{2F(\alpha)}$ is defined. By the comprehension axiom for (V_2, ε_2), the class is contained in $\mathbf{V}_2$ and we can set $H_\alpha(x) = (H''_{\alpha^-}(\hat{x}))^\vee$; thus, H_α is defined on all $V_{1\alpha}$. Now we prove that H_α is an isomorphism:

$$\forall x, z \in dom(H_\alpha)\,(x \neq z \Rightarrow \hat{x} \neq \hat{z}) \Rightarrow (H''_{\alpha^-}(\hat{x}) \neq H''_{\alpha^-}(\hat{z}))\,.$$

Because of inductive assumption, if H_α is an isomorphism then $H_\alpha(x) \neq H_\alpha(z)$ (since $\check{(\cdot)}$ is injection from $\mathbf{V}$ into V). So, H_α is injective and actually as it can be verified, is a ε-homomorphism. Let us show that H_α is a bijection from $V_{1\alpha}$ on $V_{2F(\alpha)}$. Since F is an $<_\varepsilon$-isomorphism then $F(\alpha) = F(\alpha^-) +_1 1$ implies $\hat{y} \subset V_{2F(\alpha)}$, therefore, $\in$-set (v_1-class) $H^{-1}_{\alpha^-}(\hat{y})$ is defined and, hence, there exists $x = (H^{-1}_{\alpha^-}(\hat{y}))^\vee \in V_{1\alpha}$, moreover, $H_\alpha(x) = y$. It can be verified immediately that H^{-1}_α is an ε-isomorphism too.

If α is a limit ordinal, i.e. $\alpha = \bigcup_1(\alpha) = (\bigcup\{\hat{\beta} : \beta \in \hat{\alpha}\})^\vee$, then we define: $H_\alpha = \bigcup\{H_\beta : \beta \in \hat{\alpha}\}$. H_α maps $V_{1\alpha}$ onto $V_{2F(\alpha)}$ because $F(\alpha) = (\bigcup\{F(\hat{\beta}) : \beta \in \hat{\alpha}\})^\vee$. Its bijectivity and ε-isomorphicy are obvious because of $(*_\alpha)$.

Finally, by setting $H = \bigcup\{H_\alpha : \alpha \in \hat{\lambda}\}$ we get an isomorphism from $V_{1\lambda}$ onto $V_{2F(\lambda)}$, required in the theorem. In accordance with which one among three possibilities of the theorem is chosen, owing to v-axiom of foundation the construction H, provided above allows to obtain an isomorphism either from V_1 onto V_2 or from $V_{1\kappa}$ onto V_2 or from V_1 onto $V_{2\kappa}$.

Besides, since in the two last cases $V_{1\kappa}$ and $V_{2\kappa}$ are the models of SU, then κ is strongly inaccessable either in V_1 or in V_2 (in the v-sense and, moreover, in the sense of ZF theory being interpreted in SU). Theorem 4.2 is proved.

5. *ZF* as the external set theory.

Though taking ZF for the external set theory do not fit well to methodological considerations provided in section 3, nevertheless, it makes possible to explain the position of SU among the commonly accepted set theories. Namely, SU can be interpreted in $ZF + \exists\kappa\, InA(\kappa)$. In more detail,

5.1. Theorem. *The following formulas are provable in* $ZF_\in$.

(i) $InA(\kappa) \rightarrow SU(R_\kappa, \in\upharpoonright R_\kappa)\,,$

(ii) $SU(V,\varepsilon) \rightarrow \exists\kappa\,\exists h\,(InA(\kappa)\&$
$\&(h$ *is an isomorphism from* (V,ε) *onto* $(R_\kappa, \in\upharpoonright R_\kappa)))$.

(R_κ denotes the κ-level of the von Neumann universe in $ZF_\in$).

Corollary. $ZF_\in \vdash \exists V\,\exists\varepsilon\, SU(V,\varepsilon) \leftrightarrow \exists\kappa\, InA(\kappa)\,.$

5.2. To prove the theorem we have already prepared almost all necessary tools. First of all, the statement (ii) of 4.3 can be formulated in $ZF_\in$ in the following form (we denote $ZF_\in$-classes, i.e. $\in$-formulas by letters $\mathbf{G}$, $\mathbf{F}$; the universe of all $\in$-sets will be denoted by $\mathbf{V}$). For any arbitrary function $\mathbf{G}$ defined on $\mathbf{V}$ there is a unique function $\mathbf{F}$ defined on On_v such that

$$\forall\alpha \in On_v(\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}\upharpoonright\hat{\alpha}))\,.$$

Owing to the replacement $ZF_{\in}$-axiom $\mathbf{F}$ is $\in$-set, and, thus, the last statement is essentially equivalent to 4.3(ii). When (V_2, ε_2) is substituted for $(\mathbf{V}, \in)$ we can realize the constructions F and H as in proofs 4.4, 4.5 by using the replacement $ZF_{\in}$-axiom. Since the possibilities (i) and (ii) in theorems 4.4, 4.5 are obviously not valid here, we obtain 5.1(ii). (i) can be verified immediately (the fact that $(R_{\kappa+1}, \in\upharpoonright R_{\kappa+1})$ is the model of M is well-known, however, as we can see, $InA(\kappa)$ means something more).

6. Categoricity and independence.

To avoid misunderstanding that may arise in this question, we note that categoricity of the theory $\overline{SU}(V,\varepsilon) = SU(V,\varepsilon)\&\neg\exists\kappa\, InA_v(\kappa)$ means neither deducibility nor refutability in $\overline{SU}(V,\varepsilon)$ (within $Z_{\in}$) of any $\in$-formula $\phi(V,\varepsilon)$ "speaking" about (V,ε) but implies the following scheme of $\in$-theorems.

$$Z_{\in} \vdash \overline{SU}(V,\varepsilon)\&\phi(V,\varepsilon) \rightarrow \forall V'\, \forall\varepsilon'\, (\overline{SU}(V',\varepsilon') \Rightarrow \phi(V',\varepsilon'))\,.$$

Thus, for example, in respect to the continuum-hypothesis CH 5.1*(ii)* implies:

$$ZF_{\in} + CH_{\in} \vdash SU(V,\varepsilon) \rightarrow CH_v(V,\varepsilon), \quad ZF_{\in} + \neg CH_{\in} \vdash SU(V,\varepsilon) \rightarrow \neg CH_v(V,\varepsilon)\,.$$

If $ZF + \exists V\, \exists\varepsilon\, SU(V,\varepsilon)$ is consistent, then, owing to the fact that $CH_{\in}$ is independent of $\exists\kappa\, InA_{\in}(\kappa)$, the theories

$$ZF_{\in} + \exists V\, \exists\varepsilon\, (SU(V,\varepsilon)\&CH_v(V,\varepsilon))\,, \quad ZF_{\in} + \exists V\, \exists\varepsilon\, (SU(V,\varepsilon)\&\neg CH_v(V,\varepsilon))$$

are consistent too.

7. Concluding remarks: "Mathematical structures vs. formal theories".

It appears that immersion of the (V,ε)-universe theory described above into some external formal set theory is not an obligatory step dictated by considerations of rigour. Indeed, as a matter of fact, the very concept of formal system is introduced in usual intuitive mathematical language within some intuitive set-theoretical and logical basis (denoted further by IB). Of course, the chain of formalisms can be developed as far as desired: theories, metatheories, metametatheories, ..., but the usual mathematical language and its IB, in which all the chain is considered, remains anyway on the top. The only thing we can do is to reduce IB as is possible to make it more reliable.

Setting set theories in formal logical languages (going back to the Hilbert formalism) allows to remove transfinite constructions from IB by considering them as symbolic, i.e. syntactical objects. At the same time we can introduce the universe of sets as a Bourbaki structure (section 2) which allows to present transfinite constructions as formal objects in a nonsyntactical way. It seems that this approach possesses certain advantages. At any rate, it is more natural for a mathematician to work with a set universe rather as a Bourbaki structure than as in model theory terms. It seems that the introduction of higher order axioms in such a way can bring more light on some problems of the set theory.

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