A unified functorial framework for propositional-like and equational-like logics

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Appetizer. Algebraic logic (AL) is a well established discipline yet some natural questions remain out of its scope. For instance, it is well known that the Horn and universal fragments of FOL are original logics close to equational logic rather than just sublogics of FOL, and, very similarly, Gentzen's axiomatization of FOL seems has much in common with the universal equational logic: how can these phenomena be placed in the AL framework? In general, what is equational-like logic, and how are these logics related to propositional logics? Another block of questions is whether cylindric or polyadic algebraization of FOL make it possible to consider it as a kind of (complex yet) propositional logic, or there are principal differences? Has it sense to ask about the extent to which a given logic is propositional-like? And if even the questions above can be treated formally, will such an effort be helpful in logic as such or will remain a purely metalogical achievement?
A more technical but principal question is about size problems. Indeed, logics arising from semantics are not bound to be compact, moreover, there are big logics which are axiomatizable by a class rather than a set of inference rules. How can they be managed in AL?
On the other hand, categorical logic (CL) has achieved a great success in metalogical studies, and, no doubts, it also manifests the power of the algebraic approach to logic. How can one describe the difference between these two paradigms? What are their comparative advantages and disadvantages? Whether one of them subsumes the other or they are "orthogonal"?
In general, what parameters do determine the essence of one or another style of algebraizing logic, in other words, what are main axes of the space where different algebraizions of different logics

can be placed? What is the structure of this space? Does it possess some non-trivial algebra of operators mapping/building logics? For example, are there standard operators of passing from a logic to its Horn or Gentzen derivative, or from a (propositional) logic to its polyadic version?

The present paper (together with its predecessor [8] and the companion [9]) aim at demonstrating that the questions above can be consistently approached within some unexpectedly simple framework constituted by a small set of functors organized into several commutative diagrams. The functorial framework gives a concise language and states meta-studies of algebraic logic on the firm algebraic ground of category theory. The general slogan underlying the approach is that *the logic of algebraic logic is arrow logic*.

Table of contents

1 Logic in the algebraic setting: the stage and the actors

Let L be some logic. To set up metatheory of L algebraically we need the following constructs.

1.1 Definition. A *base language* is a couple $\Lambda = (\mathcal{V}, \#)$ with $\mathcal{V}$ a variety of algebras and # a functor: **Set** $\to$ **Set**. The intuition behind the components of Λ is as follows.

(a) $\mathcal{V}$ is to be thought of as arising from some equational theory of interaction between quantifiers and substitutions pertaining to L (examples are in [5, 18, 8, 16]). That is, generators of free $\mathcal{V}$-algebras are to be thought of as operation/relation symbols of the total arity while free $\mathcal{V}$-algebras themselves are algebras of L-expressions modulo α-conversion in the L-syntax.

(b) In the context of L-syntax, the functor # is to be thought of as sending a set A of L-expressions to the corresponding set of L-formulas built from elements of A. For example,if L is equational logic and A is a set of terms then $\#A = A \times A$ is the set of equations.

To preserve algebraic machinery associated with $\mathcal{V}$ working, # should possess certain properties. The minimal necessary set of them is

- preservation of Epi-JointlyMonic factorization (subsets and quotients):

$$\text{if } \iota\colon A \hookrightarrow B \text{ then } \#\iota\colon \#A \hookrightarrow \#B \tag{1}$$

$$\text{if } \varepsilon\colon A \twoheadrightarrow B \text{ then } \#\varepsilon\colon \#A \twoheadrightarrow \#B \tag{2}$$

$$\text{if } R \subset \prod_{i\in I} A_i \text{ then } \#R \subset \prod_{i\in I} \#A_i \tag{3}$$

  where $\hookrightarrow$ and $\twoheadrightarrow$ denote inclusions and surjections respectively;
- reflection of isomorphisms:

$$\text{if } \#f\colon \#A \to \# \text{ is bijection then } f\colon A \to B \text{ is bijection;} \tag{4}$$

- preservation of (direct) limits

$$\#\lim(Q_i)_{i\in I} = \lim(\#Q_i)_{i\in I}, \tag{5}$$

  of directed diagrams $Q\colon (I, \leq) \to$ **Set** consisting of surjections;

• compatibility with products

$$\#(\prod A_i) \supset \prod(\# A_i), \tag{6}$$

for any family $A\colon I \to \mathbf{Set}$.

If conditions (1) - (6) are satisfied, the functor and the language will be called *regular*. Regularity will be a default assumption throughout the paper. It follows from (ii) that each equivalence $\vartheta \subset A \times A$ gives rise to an equivalence $\vartheta^\# = \mathrm{Ker}\#\varepsilon_\vartheta$ where ε_ϑ denotes the canonical quotient map of ϑ.[1] Further we will discuss some additional conditions ensuring that $\#$ conforms with algebraic manipulations in $\mathcal{V}$. $\diamondsuit$

1.2 Theorem. If a functor $\#$ is regular then for any set A the mapping

$$(-)^\# : Eq(A) \to Eq(\# A)$$

preserves intersections and directed unions of equivalence relations and, in addition, is injective. $\diamondsuit$

1.3 Notation. $\mathcal{V}$ is a concrete category in the usual way: it is endowed with the underlying set functor $U\colon \mathcal{V} \to \mathbf{Set}$. Following notational traditions of universal algebra, we will often omit explicit mention of U and denote $\mathcal{V}$-algebras and their carrying sets with the same letters, in particular, the value of the functor $U \rhd \#$ ($\_\rhd\_$ denotes the operation of composition) on a homomorphism $h\colon A \to B$ will be denoted by $\#h\colon \#A \to \#B$ or $h^\# : \#A \to \#B$. If X is a set, $Fr^{\mathcal{V}}_X$ denotes the free $\mathcal{V}$-algebra generated by X. If $\mathcal{V}$ is clear from the context, the superscript may be omitted.

For an infinite cardinal κ, $\mathbf{Pow}_\kappa X \stackrel{\text{def}}{=} \{Y \subset X \mid Y \in \mathbf{Set}_\kappa\}$ where $Y \in \mathbf{Set}_\kappa$ means that $Card(Y) < \kappa$. We admit the value ∞ for κ which means that there are no cardinality restrictions, *eg*, the ordinary powerset of X is $\mathbf{Pow}_\infty X$. However, following the usual practice we will write $\mathbf{Pow} X$ in this case, and similarly for other constructs parameterized by cardinals: we often begin with a construct $\mathcal{X}$ without any cardinality restrictions, then

[1] Note, $\vartheta^\#$ and $\#\vartheta$ are quite different sets. For example, if $\#$ is the powerset functor then for any two subsets $X, Y \subset A$, $(X,Y) \in \vartheta^\#$ iff $\mathrm{Cn}_\vartheta X = \mathrm{Cn}_\vartheta Y$ where Cn_ϑ is the operator of closing a subset up to the least superset compatible with ϑ.

consider parameterization $\mathcal{X}_\kappa$ where κ admits the meaning ∞ and so what was formerly denoted by $\mathcal{X}$ now should be denoted by $\mathcal{X}_\infty$, thus, $\mathcal{X}$ and $\mathcal{X}_\infty$ are the same thing. Throughout the paper, κ, λ denote regular infinite "cardinals": $\omega \leq \kappa, \lambda \leq \infty$. $\diamondsuit$

1.4 Predefinition. (a) A *logic* over Λ is a couple $\mathcal{L} = (\mathcal{E}, \vdash)$ with $\mathcal{E} \subset \mathcal{V}$ and $\vdash$ a mapping which assigns to each algebra $A \in \mathcal{E}$ a consequence relation $\vdash_A$ on the set of formulas over A, $\vdash_A \subset \mathbf{Pow}\#A \times \#A$. In addition, if $h\colon A \to B$ is a homomorphism between $\mathcal{E}$-algebras, then $\Gamma \vdash_A \phi$ entails $h^\# \Gamma \vdash_B h^\# \phi$.

A class $\mathcal{E}$ of $\mathcal{V}$-algebras is to be thought of as the class of actual $\mathcal{L}$-expression algebras, that is, $\mathcal{E}$-algebras are obtained by factorizing free $\mathcal{V}$-algebras by defining relations arising from arity restrictions. A logic is called *(λ-)total* (where λ is an infinite regular cardinal) if $\mathcal{E}$ contains all $\mathcal{V}$-free algebras (generated by sets X of cardinality less than λ, $X \in \mathbf{Set}_\lambda$).

Further two additional important conditions will be added to complete this predefinition (see 3.3.4).

(b) A *morphism* of a logic $\mathcal{L}$ to a logic $\mathcal{L}'$ is a pair (F, f) with $F\colon \mathcal{V} \to \mathcal{V}'$ a functor s.t. $F \triangleright U' = U$ and $f\colon \# \to \#'$ a natural transformation. In addition, $F(\mathcal{E}) \subset \mathcal{E}'$ and if $\Gamma \vdash_A \phi$ then $f_A \Gamma \vdash'_{FA} f_A \phi$ for each $A \in \mathcal{E}$. This gives the category of logics, **Log**.

(c) Consequence relations are equivalent to closure systems (lattices) of theories, $Th A \subset \mathbf{Pow}\#A$, $A \in \mathcal{E}$, and compatibility of $\vdash$'s with homomorphisms means that $(\#h)^{-1}T \in Th A$ as soon as $T \in Th B$ for any $h\colon A \to B$. These data are displayed by commutative diagram (a) on Fig. 3 at the end of the paper where $\vdash$_**Set**, **Th-Set** are concrete categories of sets with the corresponding structure, vertical arrows going down are the underlying set functors and **CLat** is the category of complete lattices.

Thus, an algebraic counterpart of L appears as a quadruple $\mathcal{L} = (\mathcal{V}, \mathcal{E}, \#, \vdash)$ or, equivalently, $(\mathcal{V}, \mathcal{E}, \#, Th)$ with components as above. It is reasonable to make the pair $(\mathcal{V}, \#)$ into a separate construct since further we will built different constructions over the same base language Λ. If Λ is fixed, a logic can be considered a pair $\mathcal{L} = (\mathcal{E}, \vdash)$ or $(\mathcal{E}, Th)$. $\diamondsuit$

There are two principally different ways of setting logics: proof-theoretic via deduction with a system of inference rules, and model-theoretic via validity in a class of models. Their algebraic versions are as follows.

1.5 Construction. A *inference system* over a language Λ is a family $\mathcal{I} = \langle I_X \subset \mathbf{Pow}\#Fr_X \times \#Fr_X \mid X \in \mathbf{Set} \rangle$, where Fr_X denotes the free $\mathcal{V}$-algebra generated by a set X (of metavariables) and elements $(\Gamma, \phi) \in I_X$ are called *inference rules* over X. $\mathcal{I}$ is called $\kappa\lambda$*-compact* if $I_X \neq \emptyset$ entails $X \in \mathbf{Set}_\lambda$ and $I_X \subset \mathbf{Pow}_\kappa\#Fr_X \times \#Fr_X$. If one closes $\mathcal{I}$ with all instances of inference rules via mappings $h: X \to Y$, then $\mathcal{I}$ becomes a functor $\iota: \mathcal{F} \to \mathbf{I\text{-}Set}$ commuting diagram (inf) on Fig. 1(a) ($\mathcal{F}$ is the class of all free algebras in $\mathcal{V}$ and $\mathbf{I\text{-}Set}$ is a category whose objects are pairs (X, I) with X a set and $I \subset \mathbf{Pow}(X) \times X$, and morphisms from (X, I) to (X', I') are mappings $f: X \to X'$ s.t. $f(R) \in I'$ for any $R \in I$; do not consider the right square for a while).

An inference system generates on each formula set $\#A$, $A \in \mathcal{V}$ a consequence relation $\Vdash_A^{(\mathcal{I})} \subset \mathbf{Pow}\#A \times \#A$ in the standard way by substitutions of A-expressions for metavariables, that is, by homomorphisms $h: Fr_X \to A$, $X \in \mathbf{Set}$. Thus, a proof-theoretic logic in algebraic setting is a couple of a language Λ and a pair $\mathcal{L}_{\text{inf}} = (\mathcal{E}, \mathcal{I})$ over Λ.

A logic $\mathcal{L} = (\mathcal{E}, \vdash)$ is said to be *axiomatizable* if $\vdash_A = \Vdash_A^{(\mathcal{I})}$ for all $A \in \mathcal{E}$. In this case we write $\vdash = \Vdash^{(\mathcal{I})}$. $\diamondsuit$

1.6 Construction. (a) A *(logical) matrix* over a language Λ is a pair (M, D) with M a $\mathcal{V}$-algebra (to be thought of as arising from a model) and $D \subset \#M$ a designated subset of *true* items in $\#M$. A class of matrices over Λ is called a *matrix semantics*. It can be presented in a functorial way as shown by diagram (true) on Fig. 1(b) where $\mathcal{M}$ is a matrix class and $\mathbf{D\text{-}Set}$ is the category of pairs (X, D), $D \subset X \in \mathbf{Set}$ as objects and functions preserving membership in D's as morphisms.

A *congruence* on (M, D) is a congruence on algebra M s.t. $\vartheta^\#$ is compatible with D in the usual sense. A matrix (M, D) is called *reduced* if there are no congruences other than identity. A class of matrices is called reduced if each matrix in it is reduced.

(b) Given an algebra A, a clause $(\Gamma, \phi) \in \mathbf{Pow}\#A \times \#A$ is *valid* in a matrix (M, D) if, for any homomorphism $h: A \to M$, $h^\#\phi \in D$ as soon as $h^\#\Gamma \subset D$. Given a matrix class (matrix semantics) $\mathcal{M}$, the validity relation gives rise to a logic $(\models_A^{(\mathcal{M})}, A \in \mathcal{E})$ in the usual way via homomorphisms $h: A \to M$, $A \in \mathcal{E}$, $M \in \mathcal{M}$.

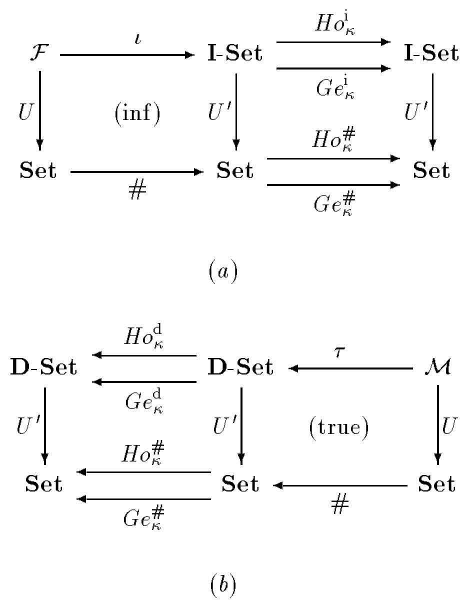

Fig. 1. Horn's and Gentzen's constructions via functors

Thus, a model-theoretic logic is a language Λ and a pair $\mathcal{L}_{\mathrm{mod}} = (\mathcal{E}, \mathcal{M})$ over this language. A logic is called *m-free* if $\mathcal{M} = \mathcal{V}$, that is, any $\mathcal{V}$-algebra can be considered as semantic. This is dual to the totalness condition, $\mathcal{E} \supset \{Fr_X \mid X \in \mathbf{Set}\}$, and total logics might be called *e-free*. $\diamondsuit$

The notion of matrix semantics is model-theoretic rather than algebraic since there can well be the case when there are different matrices, say, (M, D) and (M', D'), with the same carrier algebra: $M = M'$ but $D \neq D'$.

1.7 Definition. A *prealgebraic semantics* is a pair $(\mathcal{M}, \tau)$ with $\mathcal{M} \subset \mathcal{V}$ and $\tau : \mathcal{M} \to \mathbf{D\text{-}Set}$ a (*true*) functor commuting the diagram (true) on Fig. 1 (note, we have changed the meaning of $\mathcal{M}$ in the diagram). τ is called *reduced* if each matrix (M, D_M^τ) is reduced where D_M^τ denotes the first component of the pair $\tau(M)$, further we will omit the index τ and write D_M.

A prealgebraic semantics is called *algebraic* if (the language Λ is regular and) the true functor preserves subobjects and products: if $M \subset M'$ then

$D_M = M \cap D_{M'}$ and $D_{\prod_{i\in I} M_i} = \prod_{i\in I} D_i$. $\diamondsuit$

1.8 Conjecture. Let $\#$ preserves equalizers: $\#Eq(f,g) = Eq(\#f, \#g)$ for any two functions $f, g\colon A \to B$, where $Eq(f,g) \stackrel{\text{def}}{=} \{a \in A \mid fa = ga\}$. Then a prealgebraic semantics is algebraic iff τ is equationally defined, that is, there is a set $\delta(x) \approx \epsilon(x) = \{\delta_i(x) \approx \epsilon_i(x) \mid i \in I\}$ of couples of $\mathcal{V}$-terms in a single variable s.t. $D_M = \{a \in \#M \mid (\delta_i^M)^\#(a) = (\epsilon_i^M)^\#(a) \text{ for all } i \in I\}$ for any $M \in \mathcal{M}$. $\diamondsuit$

2 Examples: packing the diversity into the framework

2.1 Propositional vs congruental logics

We begin with *propositional* logics for which (i) the functor $\#$ is the identity functor **Id**, that is, formulas coincide with expressions, and (ii) $\mathcal{E}$-algebras are free.

2.1.1 Propositional logics.

(i) *The classical propositional calculus.* The language Λ is $(\mathbf{PreBA}, \mathbf{Id})$ where **PreBA** is the variety of all algebras of some Boolean signature, say, $\wedge, \vee, \sim$, but no axioms are imposed. $\mathcal{E}$ is $\{Fr_X \mid X \in \mathbf{Set}_\omega\}$ and $\vdash$ is determined by the standard inference rules.

(ii) *Kripke semantics for intuitionistic propositional logic.* $\Lambda = (\mathbf{PreHA}, \mathbf{Id})$ where **PreHA** is the variety of algebras in the Heyting signature $(\wedge, \vee, \Rightarrow, \bot)$. $\mathcal{E} = \{Fr_X \mid X \in \mathbf{Set}_\omega\}$ and $\vdash$ is determined by validity in some Kripke semantics $\mathcal{K}$, *ie*, $\Gamma \vdash_X \phi$ iff $\Gamma \models_X^{(\mathcal{K})} \phi$. $\models^{(\mathcal{K})}$ can be presented via matrix validity as $\models^{(\mathcal{M})}$ if $\mathcal{M}$ is taken to be the class $\{(M, D) \mid M = M(K), K \in \mathcal{K}, D = \{1_M\}\}$ where $M(K)$ is a **PreHA**-algebra (in fact, of course, **HA**) extracted from a Kripke structure K in the standard way, $\mathcal{K}$ is the class of Kripke structures. Evidently, $\mathcal{M}$ is algebraic. $\diamondsuit$

We continue with *propositional-like* logics for which the functor $\#$ is still identity but expression algebras are not free.

2.1.2 First order logic via polyadic Boolean algebras.

$\Lambda = (\mathbf{PrePBA}_\omega, \mathbf{Id})$ where $\mathbf{PrePBA}_\omega$ is the variety of ω-dimensional pre-polyadic Boolean algebras, that is, algebras in the polyadic Boolean signature but without any logical axioms – only axioms regulating interaction of quantifiers and substitutions are required to be satisfied.

$\mathcal{E} = \{A_\Pi \mid \Pi$ is a signature of finitary predicate symbols$\}$ where $A_\Pi = Fr_{|\Pi|}/R(\Pi)$, $|\Pi|$ denotes the underlying set of the signature Π and $R(\Pi)$ is the corresponding set of defining relations arising from arities so that A_Π becomes a locally finite (lf) $\mathbf{PrePBA}$. It is easy to see that any $\mathcal{E}$-algebra A_Π is isomorphic to the corresponding formula algebra $Fm\Pi$ modulo renaming bound variables.

For the proof-theoretic FOL, $\vdash$ is determined by a system $\mathcal{I}$ of ordinary inference rules of the predicate calculus, which must be written without any reservations about occurrence of variables in quantified formulas, this is indeed possible (see, *eg*, [3]). Thus, $\vdash\ =\ \Vdash^{(\mathcal{I})}$. For the model-theoretic FOL, $\vdash$ is determined by means of models: $\Gamma \vdash_{A_\Pi} \phi$ iff $\Gamma \models^{(Mod\Pi)}_{Fm\Pi} \phi$. The right-hand-side relation can be presented by validity in matrices, $\models^{(Mod\Pi)}_{Fm\Pi} = \models^{(\mathcal{M})}_{A_\Pi}$, in a usual way:

$$\mathcal{M} = \{(M_X, D) \mid M_X = \mathbf{Rel}_\omega X, X \in \mathbf{Set}, D = \{X^\omega\} \subset M_X\}$$

where $\mathbf{Rel}_\omega X$ is the set of ω-dimensional relations on X.

This logic will be denoted by $\mathbf{LR}_\omega$. If Π is a signature and α is an ordinal then let $\mathbf{LR}^\Pi_\alpha$ denotes the logic with $\Lambda = \Lambda_\alpha = (\mathbf{PrePBA}_\alpha, \mathbf{Id})$ and $\mathcal{E} = \{A_{\Pi'} \mid \Pi' \subset \Pi\}$. It is easy to see that in this way the logic in question becomes a functor

$$\mathbf{L}^{(-)}_{-} \colon \mathbf{Ord}^{\mathrm{op}} \times \mathbf{PSig}_\omega \to \mathbf{Log}$$

where $\mathbf{PSig}_\omega$ denotes the category of all relational signatures of finitary arity. Note, the size problems of big logics can be treated only by considering a family of logics since the variety $\mathbf{PrePBA}_\alpha$ of any arbitrarily big but fixed dimension α is not sufficient. $\diamondsuit$

We further continue with *equational* logics, EqLs, for which the functor # is the Cartesian product functor $\_ \times \_$ sending a set X to $X \times X$ and a function $f \colon X \to Y$ to $f \times f \colon X \times X \to Y \times Y$.

2.1.3 Equational logics.

(i) *Birkhoff's (invariant) equational logic via substitution algebras.*

The language $\Lambda = (\mathbf{SA}_\omega, \_ \times \_)$ where $\mathbf{SA}_\omega$ is the variety of ω-dimensional substitution algebras introduced by Feldman ([11]) and equationally axiomatized by Cīrulis ([5]).

$\mathcal{E} = \{A_\Phi \mid \Phi$ is a signature of finitary operation symbols$\}$ where $A_\Phi = Fr_{(|\Phi|)}/R(\Phi)$ is obtained similarly to 2.1.2. Then each $\mathcal{E}$-algebra A_Φ is isomorphic to the corresponding free Φ-term algebra, $Tm\Phi$.

Deductively, $\vdash$ is $\Vdash^{(\mathcal{I})}$ where $\mathcal{I}$ is the set of Birkhoff's inference rules. Semantically, $\vdash$ is determined by universal validity in algebras, $\models^{(Alg\Phi)}$, as follows. Any Φ-algebra is nothing but a mapping $\mu \colon |\Phi| \to \mathbf{Op}_\omega X$ into the clone of ω-dimensional operations over the carrying set X of the algebra, in addition, μ must satisfy the arity restrictions of Φ. Then it can be expanded to an $\mathbf{SA}$-homomorphism $\bar{\mu} \colon A_\Phi \to \mathbf{Op}_\omega X$ and for any $a, b \in A_\Phi$,

$$\mu \models (a, b) \text{ iff } (\bar{\mu}a, \bar{\mu}b) \in \Delta_{\mathbf{Op}_\omega X} \stackrel{\text{def}}{=} \{(f, f) \mid f \in \mathbf{Op}_\omega X\} \qquad (7)$$

It is easy to see that $\models^{(Alg\Phi)} = \models^{(\mathcal{M})}$ with

$$\mathcal{M} = \left\{(M_X, D) \,\middle|\, M_X = \mathbf{Op}_\omega X, D = \Delta_{Op_\omega X}\right\}.$$

Filters of this logic are themselves $\mathbf{SA}$-congruences. In addition, for a fixed algebra $A_\Phi \in \mathcal{E}$, filters (*ie*, theories) turned out to be fully invariant Φ-congruences on A_Φ considered as a Φ-algebra. This explains the term "invariant EqL". This logic will be denoted by $\mathbf{LE}_\omega$. Again, it gives rise to a functor $\mathbf{LE}_{-}^{(-)} \colon \mathbf{Ord}^{\mathrm{op}} \times \mathbf{FSig}_\omega \to \mathbf{Log}$ where $\mathbf{FSig}_\omega$ denotes the category of finitary functional signatures.

(ii) λ-*calculus*. The language $\Lambda = (\mathbf{Pre}\lambda\mathbf{SA}_\omega, \_ \times \_)$ where $\mathbf{Pre}\lambda\mathbf{SA}_\omega$ is is the variety of pre-lambda substitution algebras satisfying all axioms of lambda substitution algebras ([10, 14]) besides $\beta(\eta)$ conversion: the latter are logical axioms while α-conversion is a syntactical one. $\mathcal{E}$ is defined similarly to the example (i). $\vdash$ is determined by β-conversion (together with η-conversion for the case of extensional lambda calculus), or, equivalently, via validity in environment models (reflexive domains) which again can be presented by a matrix semantics ([10, 14]).

(iii) *Congruental equational logic.* In contrast to Birkhoff's EqL, this logic is indexed by operation signatures, that is, a signature Φ gives rise to a logic $\mathcal{L}_{\Phi}$ as follows. $\Lambda_{\Phi} = (\mathbf{Alg}\Phi, \_ \times \_)$ with $\mathbf{Alg}\Phi$ the variety of all Φ-algebras and $\mathcal{E}_{\Phi} = \{T_X \mid X \in \mathbf{Set}\}$ where T_X is the free Φ-algebra Fr_X generated by X. $\vdash$ is $\models^{(Alg\Phi)}$ defined by equality of terms under a given interpretation of variables in a Φ-algebra: a model is a mapping $h\colon X \to M$, $M \in \mathbf{Alg}\Phi$, which can be expanded to a homomorphism $\bar{h}\colon Fr_X \to M$ and for any $t, s \in T_X$,

$$h \models (t, s) \text{ iff } (\bar{h}t, \bar{h}s) \in \Delta_M \stackrel{\text{def}}{=} \{(a, a) \mid a \in M\} \tag{8}$$

(compare with (7)). This can be expressed by matrices, $\models^{(Alg\Phi)} = \models^{(\mathcal{M})}$, for the matrix semantics $\mathcal{M} = \mathcal{M}_{\Phi} = \{(M, D) \mid M \in \mathbf{Alg}, \Phi D = \Delta_M\}$. Note, in contrast to the invariant $\mathbf{LE}_{\omega}$, this logic is *immediately* m-free: any Φ-algebra can be at once considered a model whereas for $\mathbf{LE}_{\omega}$ this is also true but only by virtue of the corresponding representation theorem.

Filters of the logic we are considering are nothing but ordinary congruences which gives the name of the logic and denotation $\mathbf{CE}_{\Phi}^{(\infty)}$; the ∞ subindex shows that there are no restrictions on cardinalities of sets generating $\mathcal{E}$-algebras. If such restrictions are imposed, $\mathcal{E}_{\Phi} = \{T_X \mid X \in \mathbf{Set}_{\lambda}\}$, then the logic is denoted by $\mathbf{CE}_{\Phi}^{(\lambda)}$. When Φ ranges over the category of operation signatures $\mathbf{FSig}_{\omega}$ and λ ranges over the class of cardinals, the construction we have just considered becomes a functor

$$\mathbf{CE}_{-}^{(-)}\colon {\mathbf{FSig}_{\omega}}^{\mathrm{op}} \times \mathbf{Card} \to \mathbf{Log}$$

(note the duality with Birkhoff's logic, we remind that categories **Ord** and **Card** are equivalent). $\diamondsuit$

Finally, we turn to the (strict) Horn and universal fragments of FOL, which we prefer to consider as special equational-like logics rather than sublogics of $\mathbf{L}_{\alpha}$. This consideration is in a special section 2.2 below but first we need to generalize the equational congruental logic for signatures containing relational symbols.

2.1.4 Relational congruental logic. Similarly to the congruental EqL above, this logic is indexed by signatures. Let $\Sigma = (\Phi, \Pi)$ be a signature of finitary operation and relation symbols. The language is $\Lambda = (\mathbf{Alg}\Phi, \coprod_{\Pi})$ where the functor $\coprod_{\Pi}$ sends a set X into the set $\coprod\{X^{n(P)} \mid P \in \Pi_{\approx}\}$, $\Pi_{\approx} = \Pi \cup \{\approx\}$, $n(P)$ is the arity of P and $n(\approx) = 2$. Note, the set

$\coprod\{X^{n(P)} \mid P \in \Pi_{\approx}\}$ is isomorphic to the set $\bigcup\left\{X_P^{n(P)} \mid P \in \Pi_{\approx}\right\}$ where $X_P^{n(P)} = \left\{P(x_1, \ldots, x_{n(P)}) \mid x_i \in X\right\}$. For example, if Π consists of a binary symbol P and a ternary Q then $\coprod_\Pi(X) = (X^2 \sqcup X^2 \sqcup X^3) \cong (X^2_{\approx} \cup X^2_P \cup X^3_Q)$. Correspondingly, if $f\colon X \to Y$ then $\coprod_\Pi(f) = \coprod\{f^{n(P)} \mid P \in \Pi_{\approx}\}$.

Expression algebras $\mathcal{E}$ are $\{T_X \mid X \in \mathbf{Set}\}$ where $T_X = Fr_X^{(\Phi)}$ is the free Φ-algebra generated by X. Thus, any formula $\phi \in \coprod_\Pi T_X$ is an element of $(T_X)_P^{n(P)}$ for some $P \in \Pi_{\approx}$, which can be considered an expression $P(\phi_1, \ldots, \phi_{n(P)})$ with $\phi_i \in T_X$.

At last, $\vdash$ is $\models^{(Mod\,\Sigma)}$ determined by validity in Σ-models as follows. Any Σ-model is a Φ-algebra M endowed with relations $P^M \subset M^{n(P)}$, $P \in \Pi$. Validity is considered w.r.t. homomorphisms $h\colon X \to M$: given a formula $\phi = P(x_1, \ldots, x_{n(P)})$ with $P \in \Pi_{\approx}$, $h \models \phi$ iff $(\bar{h}x_1, \ldots, \bar{h}x_{n(P)}) \in P^M$ where, of course, $\approx^M = \Delta_M$. Now it is easy to see that this definition can be reproduced in the matrix style: given a "formal" formula $\phi \in \coprod_\Pi(T_X)$, say, $\phi \in (T_X)_P^{n(P)}$ for some $P \in \Pi_{\approx}$, the interpreted formula $(\bar{h}x_1, \ldots, \bar{h}x_{n(P)})$ is nothing but $\phi \triangleright \bar{h}$ while a Σ-model is nothing but a matrix (M, D) with $M \in \mathbf{Alg}\Phi, D = \Delta_M \sqcup \coprod\{P^M \mid P \in \Pi\} \subset \coprod_\Pi M$. This gives a matrix class $\mathcal{M} = \mathcal{M}_\Sigma$ s.t. $\models^{(Mod\,\Sigma)} = \models^{(\mathcal{M})}$.

The logic will be denoted by $\mathbf{C}_\Sigma^{(\infty)}$. It is principally distinct from FOL $\mathbf{L}_\omega$ in that it is both e-free and m-free. In this sense, the logic is very much like the congruental equational logic $\mathbf{CE}_\Phi^{(\infty)}$ and, indeed, a $\mathbf{C}_\Sigma^{(\infty)}$-filter on a Σ-model M is a Φ-congruence $\vartheta \subset M \times M$ compatible with relations: if $(a_1, \ldots, a_n) \in P^M$ and all $(a_i, b_i) \in \vartheta$ then $(b_1, \ldots, b_n) \in P_M$ for each $P \in \Pi$. Size problems for this logic can be managed within a single logic but its inference rules must be fibred over $\mathbf{Set}$: any big but fixed cardinal λ is not sufficient.

When Σ ranges over the category of signatures $\mathbf{Sig}$, and $\mathcal{E} = \{T_X \mid X \in \mathbf{Set}_\lambda\}$ where λ ranges over $\mathbf{Card}$, the construction becomes a functor

$$\mathbf{C}_{-}^{(-)}\colon \mathbf{Sig}_\omega{}^{\mathrm{op}} \times \mathbf{Card} \to \mathbf{Log}.$$

In fact, the congruental equational logic $\mathbf{CE}_{-}^{(-)}$ considered in 2.1.3(iii) is the restriction of this functor to the subcategory $\mathbf{FSig}_\omega$ of $\mathbf{Sig}_\omega$. ♢

The following tables (a),(b),(c) present the logics we have considered in a concise way:

Logic	$\mathcal{V}$	$\#$	$\mathcal{E}$	$(\mathcal{M}, \tau)$
PropCalculus	$\mathbf{PreBA}$	$\mathbf{Id}$	$\mathbf{FrPreBA}$	$(\mathbf{SetPreBA},\mathbf{1})$
RelFOL	$\mathbf{PrePBA}_\alpha$	$\mathbf{Id}$	$\{F_\Pi/R_\Pi : \Pi \in \mathbf{PSig}_\omega\}$	$(\mathbf{SetPrePBA}_\alpha,\mathbf{1})$
Birkhoff's EqL	$\mathbf{SA}_\alpha$	$(-)^2$	$\{F_\Phi/R_\Phi : \Phi \in \mathbf{FSig}_\omega\}$	$(\mathbf{SetSA}_\alpha,\Delta)$
General FOL	$\mathbf{PrePTBA}_\alpha$	$\langle(-)^2, \mathbf{Id}\rangle$	$\{F_\Sigma/R_\Sigma : \Sigma \in \mathbf{Sig}_\omega\}$	$(\mathbf{SetPrePTBA}_\alpha,\langle\Delta, \mathbf{1}\rangle)$
Congr. EqL	$\mathbf{Alg}\Phi$	$(-)^2$	$\mathbf{Fr}_\lambda\mathbf{Alg}\Phi$	$(\mathbf{Alg}\Phi,\Delta)$
Congr.RelFOL	$\mathbf{Alg}\Phi$	$\coprod_\Pi$	$\mathbf{Fr}_\lambda\mathbf{Alg}\Phi$	$(\mathbf{Alg}\Phi,\Pi^{(-)})$
Congr. FOL	$\mathbf{Alg}\Phi$	$(-)^2 \sqcup \coprod_\Pi$	$\mathbf{Fr}_\lambda\mathbf{Alg}\Phi$	$(\mathbf{Alg}\Phi,\Delta \sqcup \Pi^{(-)})$

(a)

Logic	Functorial presentation
Birkhoff's EqL	$\mathbf{LE}_-^{(-)} : \mathbf{Ord}^{\mathrm{op}} \times \mathbf{FSig}_\omega \to \mathbf{Log}$
RelFOL	$\mathbf{LR}_-^{(-)} : \mathbf{Ord}^{\mathrm{op}} \times \mathbf{PSig}_\omega \to \mathbf{Log}$
General FOL	$\mathbf{L}_-^{(-)} : \mathbf{Ord}^{\mathrm{op}} \times \mathbf{Sig}_\omega \to \mathbf{Log}$
Congr. EqL	$\mathbf{CE}_-^{(-)} : \mathbf{FSig}_\omega{}^{\mathrm{op}} \times \mathbf{Card} \to \mathbf{Log}$
Congr. RelFOL	$\mathbf{CR}_-^{(-)} : \mathbf{PSig}_\omega{}^{\mathrm{op}} \times \mathbf{Card} \to \mathbf{Log}$
Congr. FOL	$\mathbf{C}_-^{(-)} : \mathbf{Sig}_\omega{}^{\mathrm{op}} \times \mathbf{Card} \to \mathbf{Log}$

(b)

Logic	Functorial presentation in a notational arrangement
Birkhoff's EqL	$\mathbf{E}_-^{(-)} : \mathbf{Ord}^{\mathrm{op}} \times \mathbf{FSig}_\omega \to \mathbf{Log}$
RelFOL	$\mathbf{R}_-^{(-)} : \mathbf{Ord}^{\mathrm{op}} \times \mathbf{PSig}_\omega \to \mathbf{Log}$
General FOL	$\mathbf{L}_-^{(-)} : \mathbf{Ord}^{\mathrm{op}} \times \mathbf{Sig}_\omega \to \mathbf{Log}$
Congr. FOL	$\mathbf{C}_-^{(-)} : \mathbf{Sig}_\omega{}^{\mathrm{op}} \times \mathbf{Card} \to \mathbf{Log}$

(c)

Note, logics **CE** and **CR** (being considered as functors) are merely restrictions of the logic **C** to the corresponding domain subcategory. It is reasonable to denote them also by **C** and thus there is one congruental logic. In contrast, logics **LE** and **LR** are reducts of the logic **L**, that is, in fact, are quite different from **L** and so can be well denoted by **E** and **R**. (So, for example, for an ordinal α and an operation signature $\Phi \in \mathbf{FSig}_\omega \subset \mathbf{Sig}_\omega$, logics $\mathbf{E}^\alpha_\Phi$ and $\mathbf{L}^\alpha_\Phi$ are essentially different). The table (c) summarizes our notational efforts.

2.2 Building logics from logics

Horn's and Gentzen's constructions via functors

It is intuitively evident that conditional and universal equational logics (see, *eg*, [17]) as well as (strict) Horn and universal FOL have much in common with the Gentzen-style axiomatization of FOL. In our framework this observation can be made precise.

Given an infinite regular cardinal κ, we introduce functors

$$\begin{aligned} &Ho^\#_\kappa, Ge^\#_\kappa : \mathbf{Set} \to \mathbf{Set},\\ &Ho^{\mathrm{i}}_\kappa, Ge^{\mathrm{i}}_\kappa : \mathbf{I\text{-}Set} \to \mathbf{I\text{-}Set},\\ &Ho^{\mathrm{d}}_\kappa, Ge^{\mathrm{d}}_\kappa : \mathbf{D\text{-}Set} \to \mathbf{D\text{-}Set}\end{aligned}$$

which form commutative diagrams (a),(b) on Fig. 1 in the following way (only the values the functors take on objects $X \in \mathbf{Set}$, $(X, D) \in \mathbf{D\text{-}Set}$ and $(X, I) \in \mathbf{I\text{-}Set}$ are pointed, those on morphisms are then clear):

$$Ho^\#_\kappa(X) = \mathbf{Pow}_\kappa X \times X, \qquad Ge^\#_\kappa(X) = \mathbf{Pow}_\kappa X \times \mathbf{Pow}_\kappa X$$

$$Ho^{\mathrm{d}}_\kappa(X, D) = \left\langle Ho^\#_\kappa(X), \left\{(P, q) \in Ho^\#_\kappa(X) \mid \text{ if } P \subset D \text{ then } q \in D\right\}\right\rangle$$

$$Ge^{\mathrm{d}}_\kappa(X, D) = \left\langle Ge^\#_\kappa(X), \left\{(P, Q) \in Ge^\#_\kappa(X) \mid \text{ if } P \subset D \text{ then } Q \cap D \neq \emptyset\right\}\right\rangle$$

$$Ho^{\mathrm{i}}_\kappa(X, I) = \left\langle Ho^\#_\kappa(X), \left\{(\emptyset, \Gamma \to \phi) \mid (\Gamma, \phi) \in I_X\right\} \cup HoStruct_X\right\rangle$$

$$Ge^{\mathrm{i}}_\kappa(X, I) = \left\langle Ge^\#_\kappa(X), \left\{(\emptyset, \Gamma \to \{\phi\}) \mid (\Gamma, \phi) \in I_X\right\} \cup GeStruct_X\right\rangle$$

where we write $\Gamma \to \phi$, $\Gamma \to \Phi, \psi$ for, respectively, $(\Gamma, \phi) \in \mathbf{Pow}_\kappa(X) \times X$, $(\Gamma, \Phi \cup \{\psi\}) \in \mathbf{Pow}_\kappa(X) \times \mathbf{Pow}_\kappa X$ and $HoStruct_X$, $GeStruct_X$ are, respectively, the collections of inference rules (i),(ii) and (i)-(iv):

(i) for all $\phi \in \Gamma$ infer $\Gamma \to \phi$;
(ii) From $\Gamma \to \phi$ for all $\phi \in \Phi$ and $\Phi \to \Psi$ infer $\Gamma \to \Psi$;
(iii) From $\Gamma \to \Phi, \psi$ and for $\Gamma' \supset \Gamma$ and $\Phi' \supset \Phi$ infer $\Gamma' \to \Phi', \psi$;
(iv) From $\Gamma \to \Phi, \psi_1, \psi_2$ and $\Gamma, \psi_1 \to \Phi, \psi_2$ infer $\Gamma \to \Phi, \psi_2$.

Warning: when (i) is used for $GeStruct_X$, $\Gamma \to \phi$ denotes $\Gamma \to \{\phi\}$ whereas when (ii) is used for $HoStruct_X$, $\Gamma \to \Phi$ denotes the set $\{\Gamma \to \phi \mid \phi \in \Phi\}$.

The first two functors induce operators on languages: given $\Lambda = (\mathcal{V}, \#)$, define

$$Ho_\kappa^{\mathrm{lng}}(\Lambda) = (\mathcal{V}, \# \rhd Ho_\kappa^{\#}) \text{ and } Ge_\kappa^{\mathrm{lng}}(\Lambda) = (\mathcal{V}, \# \rhd Ge_\kappa^{\#}).$$

The middle two functors induce operators on matrix classes: given a matrix class $\mathcal{M}$,

$$Ho_\kappa^{\mathrm{mtr}}(\mathcal{M}) = \left\{ \langle M, Ho_\kappa^{\mathrm{d}}(D) \rangle \mid (M, D) \in \mathcal{M} \right\},$$
$$Ge_\kappa^{\mathrm{mtr}}(\mathcal{M}) = \left\{ \langle M, Ge_\kappa^{\mathrm{d}}(D) \rangle \mid (M, D) \in \mathcal{M} \right\}$$

to be two new classes over languages $Ho_\kappa^{\mathrm{lng}}(\Lambda)$ and $Ge_\kappa^{\mathrm{lng}}(\Lambda)$ respectively[2].

The last two functors induce the following two operators on inference systems: given a system $\mathcal{I} = \langle I_X \subset \mathbf{Pow}\#Fr_X \times \#Fr_X \mid X \in \mathbf{Set} \rangle$, define

$$Ho_\kappa^{\mathrm{inf}}(\mathcal{I}) = \left\{ Ho_\kappa^{\mathrm{i}}(R) \mid R \in I_X, X \in \mathbf{Set} \right\} \cup \left\{ (\mathrm{Str})_h \mid h \in \mathbf{ArrSet} \right\},$$
$$Ge_\kappa^{\mathrm{inf}}(\mathcal{I}) = \left\{ Ge_\kappa^{\mathrm{i}}(R) \mid R \in I_X, X \in \mathbf{Set} \right\} \cup \left\{ (\mathrm{Str})_h \mid h \in \mathbf{ArrSet} \right\}$$

where $(\mathrm{Str})_h$ is the family of inference rules indexed by $h\colon X \to Y$:
$(\mathrm{Str})_h$ from $\Gamma \to \Phi$ infer $\bar{h}^{\#}\Gamma \to \bar{h}^{\#}\Phi$
where $\bar{h}\colon Fr(X) \to Fr(Y)$ is the free extension of h.

A routine check should show that all our definitions are coordinated[3]:

2.2.1 Theorem[?]**.** Given a language Λ, if $\mathcal{M}$ is axiomatizable by an inference system $\mathcal{I}$, $\mathcal{I} = \mathrm{Ax}[\mathcal{M}]$, then

$$Ho_\kappa^{\mathrm{inf}}\mathcal{I} = \mathrm{Ax}[Ho_\kappa^{\mathrm{mtr}}\mathcal{M}] \text{ and } Ge_\kappa^{\mathrm{inf}}\mathcal{I} = \mathrm{Ax}[Ge_\kappa^{\mathrm{mtr}}\mathcal{M}].$$

[2] types of functors and of their arguments/values are slightly confused here to simplify notation

[3] The paper was prepared under the pressure of the approaching deadline and the result below is "partly proved": it is stated by virtue of general reasons and verifying some essential details but a whole proof was not completed. Here and further this is marked by the ?-index

In other words, if $\models^{(\mathcal{M})} = \Vdash^{(\mathcal{I})}$ then

$$\models^{(Ho_\kappa^{\mathrm{mtr}}\mathcal{M})} = \Vdash^{(Ho_\kappa^{\mathrm{inf}}\mathcal{I})} \text{ and } \models^{(Ge_\kappa^{\mathrm{mtr}}\mathcal{M})} = \Vdash^{(Ge_\kappa^{\mathrm{inf}}\mathcal{I})}$$

$\diamondsuit$

Now an evident observation is that the Horn and universal formulas with less than λ variables and κ-bounded conjunctions/disjunctions are formulas in the languages $Ho_\kappa^{\mathrm{lng}}(\Lambda)$ and $Ge_\kappa^{\mathrm{lng}}(\Lambda)$ respectively, and the Horn and universal logics are logics over these languages defined as

$$\mathbf{Horn}_{\kappa\Sigma}^{(\lambda)} = \left(\models_A^{(Ho_\kappa^{\mathrm{mtr}}\mathcal{M}_\Sigma)}, A \in \mathbf{Alg}_\lambda\Phi\right)$$
$$\mathbf{Univ}_{\kappa\Sigma}^{(\lambda)} = \left(\models_A^{(Ge_\kappa^{\mathrm{mtr}}\mathcal{M}_\Sigma)}, A \in \mathbf{Alg}_\lambda\Phi\right).$$

By combining this simple observation with simple ?-theorem 2.2.1 we obtain

2.2.2 Corollary? Given a signature $\Sigma = (\Phi, \Pi)$, for any $\omega \le \kappa, \lambda \le \infty$ the following holds:

$$\mathrm{Ax}[\mathbf{Horn}_{\kappa\Sigma}^{(\lambda)}] = Ho_\kappa^{\mathrm{inf}}\mathrm{Ax}[\mathbf{C}_\Sigma^{(\lambda)}],$$
$$\mathrm{Ax}[\mathbf{Univ}_{\kappa\Sigma}^{(\lambda)}] = Ge_\kappa^{\mathrm{inf}}\mathrm{Ax}[\mathbf{C}_\Sigma^{(\lambda)}],$$
$$\mathrm{Ax}[\mathbf{Gentz}_{\kappa\Sigma}^{(\lambda)}] = Ge_\kappa^{\mathrm{inf}}\mathrm{Ax}[\mathbf{L}_{\lambda\kappa}^{(\Sigma)}].$$

$\diamondsuit$

Thus, inference rules for all the three logics in question can be obtained in a uniform and transparent way by a kind of homomorphism/congruence chasing.[4]

The corollary can be written via equalities

$$\mathbf{Horn}_{\kappa\Sigma}^{(\lambda)} = Ho_\kappa^{\mathrm{log}}[\mathbf{C}_\Sigma^{(\lambda)}],$$
$$\mathbf{Univ}_{\kappa\Sigma}^{(\lambda)} = Ge_\kappa^{\mathrm{log}}[\mathbf{C}_\Sigma^{(\lambda)}],$$
$$\mathbf{Gentz}_{\kappa\lambda}^{(\Sigma)} = Ge_\kappa^{\mathrm{log}}[\mathbf{L}_\lambda^{(\Sigma)}].$$

[4] This can be compared with obtaining these rules in [17, 12]. Getting implicational/universal inference rules via congruences was the motivating intention of [17] but Quackenbush's proof is only for algebras and, in addition, mixes structural aspects of the congruental logic and Horn's/Gentzen's functors

which give a precise description of the intuition that Horn's and universal fragments of FOL are equational-like logics whereas the Gentzen's versions of FOL is an equational-like superstructure over FOL.

Halmos' construction via functors

To be completed (see [15, 6]

3 Inference vs validity

3.1 Model-theoretic setting

The validity relation between sequences (Γ, ϕ) and matrices induces a Galois correspondence: a matrix class generates a logic $(\models_A^{(\mathcal{M})}, A \in \mathcal{E})$ and a logic $\mathcal{L} = (\mathcal{E}, \vdash)$ generates a matrix class of its models, $Mod\mathcal{L}$. Operators of closing $\mathcal{L}$ up to $(\mathcal{E}, \models^{(Mod\mathcal{L})})$ and closing $\mathcal{M}$ up to $Mod(\mathcal{E}, \models^{(\mathcal{M})})$ are described in the following two theorems.

3.1.1 Construction. (a) The notions of matrix homomorphism and strict matrix homomorphism are defined in the usual way and, thus, expansion and reduction operators, $\overleftarrow{\mathbf{H}}_s$ and $\mathbf{H}_s$ respectively, are defined on matrix classes. Further, owing to (ii),(v) in 1.1, the notions of submatrix of a given matrix and the product of a matrix class are definable in a evident way. The corresponding operators on classes of matrices are denoted by respectively $\mathbf{S}$ and $\mathbf{P}$.
(b) Given a matrix class $\mathcal{M}$ and a class of algebras $\mathcal{E}$, expression $\mathcal{M}\lceil_{\mathcal{E}}$ denotes $\{(M, D) \in \mathcal{M} \mid M \in \mathcal{E}\}$ It can be shown that if $\mathcal{E}$ is closed under subalgebras then $\mathbf{SP}\mathcal{M}\Big\lceil_{\mathbf{Q}\mathcal{E}}$ is also closed under subalgebras, here $\mathbf{Q}\mathcal{E}$ is the class of all isomorphic images of the quotients of $\mathcal{E}$-algebras. $\diamondsuit$

3.1.2 Theorem. Let $\Lambda = (\mathcal{V}, \#)$ be a regular language, For any matrix class $\mathcal{M}$ over Λ and any total $\mathcal{E} \subset \mathcal{V}$, the following holds:

$$Mod(\models_A^{(\mathcal{M})}, A \in \mathcal{E}) = \overleftarrow{\mathbf{H}}_s \mathbf{H}_s \mathbf{SP}\mathcal{M}$$

$\diamondsuit$

3.1.3 Theorem. Let $\Lambda = (\mathcal{V}, \#)$ be a regular language, $\mathcal{E} \subset \mathcal{V}$ is total and $\vdash\; = \langle \vdash_A \mid A \in \mathcal{E} \rangle$ is a family of consequence relations. Then the following conditions are equivalent.

(a) $\vdash\; = \;\Vdash^{(\mathcal{I})}$ for some inferential system $\mathcal{I}$;

(b) $\vdash\; = \;\models^{(\mathcal{M})}$ for some matrix semantics $\mathcal{M}$;

(c) for any homomorphism $h\colon A \to B$ and any epimorphism $\varepsilon\colon A \to B$ the following two *structurality conditions* hold:

$$(\mathrm{Str})_h \qquad \Gamma \vdash_A \phi \Longrightarrow h^{\#}\Gamma \vdash_B h^{\#}\phi,$$

$$(\mathrm{Str})_\varepsilon \qquad \Delta \vdash_B \psi \Longrightarrow (\#\varepsilon)^{-1}\Delta \vdash_A^{(\mathrm{Ker}\varepsilon)} (\#\varepsilon)^{-1}\psi$$

where for any congruence ϑ on A, $\vdash_A^{(\vartheta)}$ denotes the intersection of all consequence relation $\vdash_A'$ including $\vdash_A$ and compatible with ϑ: if $(\phi, \psi) \in \vartheta^{\#}$ then $\Gamma \vdash_A' \phi$ iff $\Gamma \vdash_A' \psi$ for any $\Gamma \cup \{\phi, \psi\} \subset \#A$.

(c)' for any $T \subset \#B$,

$$(\mathrm{Str})_h' \qquad T \in Th B \Rightarrow (\#h)^{-1} T \in Th A \text{ for any } h \in Hom(A, B),$$

$$(\mathrm{Str})_\varepsilon' \qquad (\#\varepsilon)^{-1} T \in Th A \text{ for } \varepsilon \in Epi(A, B) \Rightarrow T \in Th B.$$

$\diamondsuit$

Thus, to pass from a logic $(\vdash_A, A \in \mathcal{E})$ to the logic $(\models_A^{(Mod\mathcal{L})}, A \in \mathcal{E})$ one has to close $\vdash$ up to the least $\mathcal{E}$-indexed family of consequence relations including $\vdash$ componentwise and satisfying structurality conditions (3.1.3),(3.1.3) or, equivalently, (3.1.3),(3.1.3). Totalness of $\mathcal{E}$ is important here, otherwise $\mathcal{E}$ may contain essential gaps so that closing up to structural consequence is not sufficient to generate all theories induced by $Mod\mathcal{L}$. If $\mathcal{E}$ is not total, certain closure properties must be required to preserve the theorem. A conjecture is that closedness under subalgebras is sufficient.

3.2 Lindenbaum-Tarski correspondence via the Grothendik construction

We describe the construction for the case when # is the identity functor. It is trivially generalized for when # is the Cartesian product functor but the general situation of arbitrary regular # is an open problem.

3.2.1 From theories to congruences. If one deals with deductively defined logics, $\mathcal{L} = (\mathcal{E}, \vdash)$, $\vdash = \Vdash^{\mathcal{I}}$, then the theory lattice is the initial construct from which congruences are derived by the family of Leibniz operators, $\Omega^{(\mathcal{I})} = \left\langle \Omega_A^{(\mathcal{I})} \mid A \in \mathcal{E} \right\rangle$. It is easy to see ([9]) that for any protoalgebraic logic $\mathcal{L}$, each Leibniz operator $\Omega_A^{(\mathcal{I})}$ has the lower (left) adjoint

$$\mathrm{H}_A^{(\mathcal{I})} \colon CoA \to Th^{(\mathcal{I})}A,\ \mathrm{H}_A^{(\mathcal{I})}\vartheta \stackrel{\text{def}}{=} \bigcap \left\{ T \in Th^{(\mathcal{I}}A \mid T \text{ is compatible with } \vartheta \right\}.$$

If, in addition, the Leibniz operators commute with homomorphisms (that is, the logic $\mathcal{L}$ is weakly congruental [4]) then the family of Leibniz adjunctions

$$(\mathrm{H}_A^{(\mathcal{I})} \backslash \Omega_A^{(\mathcal{I})}, A \in \mathcal{V})$$

becomes a fibred adjunction between functors as shown on diagram (a1) on Fig. 2. This local adjunction can be globalized via the Grothendik construction into an adjunction between the category of theories and the variety $\mathcal{V}$ (diagram (a2)). This latter adjunction generates a reflective subcategory of $\mathcal{V}$ which is a quasivariety via well known arguments of categorical algebra. This quasivariety is nothing but the quasivariety semantics for $\mathcal{L}$ and, thus, any weakly congruental total logic $\mathcal{L}$ gives rise to an adjunctive embedding between the category of $\mathcal{L}$-theories and some quasivariety of $\mathcal{V}$-algebras ([9]). If $\mathcal{L}$ is algebraizable, the adjunction becomes an equivalence of categories. Thus,

Lindenbaum-Tarski corr. = Grothendik constr. (Leibniz adj.).

Connections between properties of $\mathcal{L}$ and its Leibniz adjunction were carefully studied by Blok and Pigozzi in the setting when $\mathcal{E}$ consists of a single countably-generated free algebra and # is the Cartesian product functor $X \mapsto X^k$. The generalization for arbitrary (but regular) set functors # and arbitrary (but full in a certain sense) classes $\mathcal{E}$ is an open problem. As for #, the very existence of the Leibniz operator depends on the properties of # and it is an open problem whether regularity of # is sufficient. Due to this obstacle, further we will call the functor # regular if it also satisfies some unknown additional conditions providing the existence of Leibniz operators. $\diamondsuit$

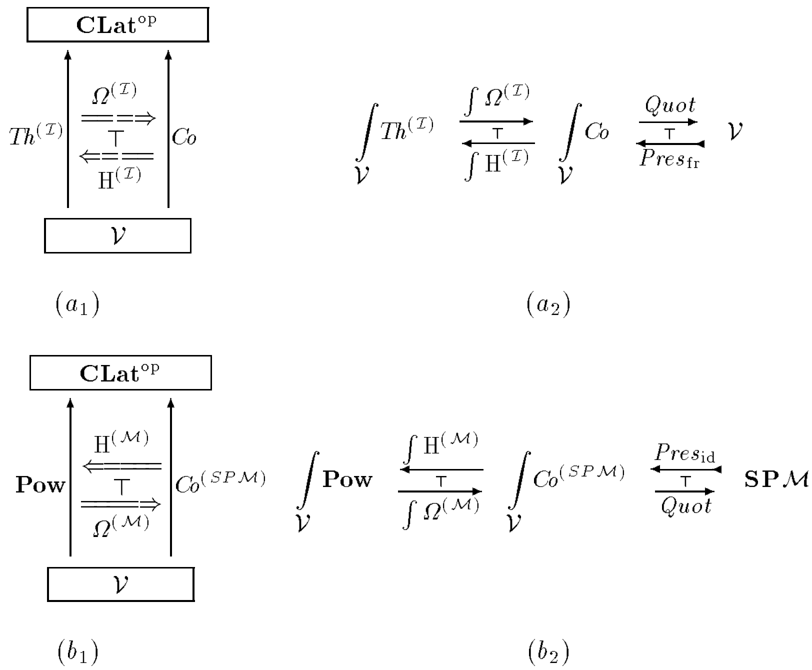

Fig. 2. Lindenbaum-Tarski (co)correspondence via the Grothendik construction

3.2.2 From algebraic semantics to logic. Dually, if one deals with model-theoretically defined logics and begins with a prealgebraic semantics $\tau \colon \mathcal{M} \to \mathbf{D\text{-}Set}$ then the family of congruences whose quotients are in $\mathbf{SP}\mathcal{M}$ are the initial construct from which theories are derived by the family of *coLeibniz operator*

$$\mathrm{H}_A^{(\mathcal{M})} \colon Co^{(\mathrm{SP}\mathcal{M})} A \to \mathbf{Pow}(A), A \in \mathcal{V}$$

as follows:

$$\mathrm{H}_A^{(\mathcal{M})}(\vartheta) \stackrel{\mathrm{def}}{=} (\varepsilon_\vartheta)^{-1} D_{A/\vartheta}$$

where ε_ϑ is the canonical map : $A \to A/\vartheta$. It is shown in [9] that if τ is compatible with subobjects and products then each $\mathrm{H}_A^{(\mathcal{M})}$ preserves intersections and the entire family $(\mathrm{H}_A^{(\mathcal{M})}, A \in \mathcal{V})$ is compatible with (inverses) of homomorphisms. Thus, the family becomes a fibred adjunction between two functors as shown on diagram (b1) Fig. 2. Locally, each fiber adjunction

$\Omega_A^{(\mathcal{M})} \backslash \mathrm{H}_A^{(\mathcal{M})}$, $A \in \mathcal{V}$, gives a closure operator, hence, a closure system on A, elements of this system are nothing but $\models_A^{(\mathcal{M})}$-theories. By applying the Grothendik construction, the fibred coLeibniz adjunction is globalized into an adjunction between $\mathbf{SP}\mathcal{M}$ and the category of subsets of $\mathcal{V}$-algebras, in fact, the matrix class $\mathcal{M}_{\mathcal{V}} = \{(A, D \mid A \in \mathcal{V}, D \subset \#A\}$ (note the difference in the origin of embeddings of algebra classes into the congruence fibration which are used in the diagrams (a2) and (b2)). This latter adjunction generates a reflective subcategory of $\mathcal{M}_{\mathcal{V}}$ which is nothing but the theory category of $(\models_A^{(\mathcal{M})}, A \in \mathcal{V})$. Thus, any algebraic semantics gives rise to an adjunctive embedding between its $\mathbf{SP}$-closure and its theory category (that is, the category of all theories in this semantics). If semantics is reduced, the adjunction becomes an equivalence:

Lindenbaum-Tarski cocorr. = Grothendik constr. (coLeibniz adj.).

It is an open problem how this machinery works in the case of non-total classes $\mathcal{E}$. A conjecture is that if $\mathcal{E}$ is close under $\mathbf{S}$ and $\mathbf{L}_\omega^{\hookrightarrow}$ then results need only the evident modification by restricting classes on $\mathcal{E}$-quotients. $\diamondsuit$

3.3 Compactness

3.3.1 Managing compactness is a crucial issue when we deal with model-theoretic logics because they cannot be assumed to be compact *a priori*. Moreover, in general, there are big logics when a class of models is not axiomatizable by any set of formulas but is axiomatizable by a proper class of formulas, *eg*, this is the case for the unbounded Horn logic and $\mathbf{SP}$-closed classes of models, and for the unbounded universal FOL and $\mathbf{S}$-closed classes of models (see [13]). Thus, to axiomatize these logics one needs to deal with classes of inference rules which we prefer to write as fibrations (see 1.5), correspondingly, the logic is considered over a proper class $\mathcal{E}$ of expression algebras. Incidentally, if a general set-theoretical axiom – Vopěnka's principle– holds, these big logics are proven to be *set-compact*, that is, axiomatizable by a set of formulas. However, even in this case, fibrations are essential if we wish to specify all axiomatizable classes of models, *eg*, all

$\mathbf{SP}$-closed classes, within a single framework. Indeed, it was shown in [7] (by using one Mal'cev's example) that for any infinite regular cardinal λ there are two different quasivarieties of groups, $\mathbf{SP}\mathcal{K}_1 = \mathcal{K}_1 \neq \mathcal{K}_2 = \mathbf{SP}\mathcal{K}_2$ whose implicational theories with less than λ variables and arbitrary number of premises coincide. Thus, dealing with classes of expression algebras generated by sets $X \in \mathbf{Set}_\lambda$ with $\lambda = \infty$ is essential for managing size problems of the Horn and universal logics.

By the same reasons, size problems of FOL in the framework of its polyadic style algebraization can be treated properly only via fibrations of logics over $\Lambda_\alpha, \alpha \in \mathbf{Ord}$. $\diamondsuit$

3.3.2 Let κ, λ be infinite regular cardinal. The principal actors in the compactness play are the following properties.

- A regular set functor $\#\colon \mathbf{Set} \to \mathbf{Set}$ is called $\kappa\lambda$-*cocontinuous* if
  $$\#\lim(X_i)_{i\in I} = \lim(\#X_i)_{i\in I}, \quad \#\bigcup_{i\in I} Y_i \supset \bigcup_{i\in I} \#Y_i$$
  for any κ-directed diagram $X\colon (I,\leq) \to \mathbf{Set}$ consisting of surjections and any λ-directed diagram $Y\colon (I,\leq) \to \mathbf{Set}$ consisting of inclusions. This property of the language Λ will be a default assumption in the next properties of algebraic semantics and logics over Λ.
- A structural logic $\mathcal{L} = (\vdash_A, A \in \mathcal{E})$ is called $\kappa\lambda$-*compact* if each $\vdash_A$ is κ-compact, $\mathcal{E} = \mathbf{S}_\lambda\mathcal{E}$ and the entire family of consequence relations is λ-*definable*, that is, $\Gamma \vdash_A \phi$ implies $\Gamma \cap B \vdash_B \phi$ for some $B \in \mathbf{S}_\lambda A$ with $\phi \in B$. Note, κ-compactness of $\vdash_A$ is equivalent to closedness of $Th A$ under κ-directed unions which, in turn, is equivalent to closedness under κ-filtered intersections. λ-definability of $(\vdash_A, A \in \mathcal{E})$ is equivalent to the following condition: for any $T \subset A$, if $T \cap B \in Th B$ for all $B \in \mathbf{S}_\lambda A$ then $T \in Th A$ (the converse is always true due to structurality).
- The Leibniz operators family of a logic $\mathcal{L}$ is called $\kappa\lambda$-*continuous* if each $\Omega_A^{(\mathcal{L})}$ preserves κ-directed unions and $\Omega_B^{(\mathcal{L})}(T \cap \#B) = \Omega_A^{(\mathcal{L})}(T) \cap (\#B)^2$ for any for any $T \in Th^{(\mathcal{L})}A$, $B \in \mathbf{S}_\lambda A$.
- An algebraic semantics $(\mathcal{M}, \tau)$ is called $\kappa\lambda$-*compact* if $\mathcal{M} = \mathbf{L}_\lambda^{\hookrightarrow}\mathbf{L}_\kappa^{\twoheadrightarrow}\mathbf{SP}\mathcal{M}$, or, equivalently, $\mathcal{M} = \mathbf{L}_\lambda^{\hookrightarrow}\mathbf{SP}_\kappa{}^{\mathrm{r}}\mathcal{M}$. The functor $\tau\colon \mathcal{M} \to \mathbf{D}\text{-}\mathbf{Set}$ is called $\kappa\lambda$-*cocontinuous* if it preserves κ-directed quotient limits and λ-directed unions.

$\diamondsuit$

3.3.3 Theorem$^?$**.** [5] Let A be a regular $\kappa\lambda$-cocontinuous-language.

(i) If $(\mathcal{M}, \tau)$ is an algebraic semantics with τ $\kappa\lambda$-cocontinuous and reduced, then, if $\mathcal{E} \subset \mathcal{V}$ is total, the logic $(\models_A^{(Mi,\tau)}, A \in \mathcal{E})$ is $\kappa\lambda$-compact iff $\mathcal{M}$ is $\kappa\lambda$-compact.

(ii) If $\mathcal{L} = (\vdash_A, A \in \mathcal{E})$ is a structural logic with $\kappa\lambda$-continuous and bijective Leibniz operators then, if $\mathcal{E}$ is total, the quasivariety of algebras carrying $Mod(\mathcal{L})$ is $\kappa\lambda$-compact iff $\mathcal{L}$ is $\kappa\lambda$-compact. $\diamondsuit$

How this theorem can be strengthen for non-total $\mathcal{E}$ is an open problem. A conjecture is that all remain valid if $\mathcal{E}$ is closed under λ-generated subalgebras and κ-directed unions of subalgebras, $\mathcal{E} = \mathbf{L}_\lambda^{\hookrightarrow}\mathbf{S}_\lambda\mathcal{E}$.

3.3.4 Definition (*predefinition 1.4 refined***).** A logic $\mathcal{L} = (\mathcal{V}, \#, \mathcal{E}, \vdash)$ is called *$\kappa\lambda$-regular* if (i) the functor $\#$ is regular and $\kappa\lambda$-cocontinuous, (ii) the class $\mathcal{E}$ is closed under $\mathbf{L}_\lambda^{\hookrightarrow}\mathbf{S}_\lambda$, (iii) the family $(\vdash_A, A \in \mathcal{E})$ is structural (see structurality conditions in theorem 3.1.3) and (iv) the Leibniz operators are order-preserving, $\kappa\lambda$-continuous and bijective. $\diamondsuit$

3.3.5 Conjecture. Let $\mathcal{L}$ be a $\kappa\lambda$-regular logic. Then the following conditions are equivalent.

(a) $\vdash = \Vdash^{(\mathcal{I})}$ for some $\kappa\lambda$-compact inferential system $\mathcal{I}$;

(b) $\vdash = \models^{(\mathcal{M})}$ for some $\kappa\lambda$-compact matrix semantics $\mathcal{M}$;

(c) $\mathcal{L}$ is $\kappa\lambda$-compact. $\diamondsuit$

4 General discussion

Let L be a logic one wishes to algebraize. To this end one should find a quadruple $\mathcal{L} = (\mathcal{V}, \#, \mathcal{E}, \mathcal{I})$ (with $\mathcal{I}$ an inference system fibration) when L is defined proof-theoretically, or a quadruple $\mathcal{L} = (\mathcal{V}, \#, \mathcal{E}, \langle\mathcal{M}, \tau\rangle)$ (with $\langle\mathcal{M}, \tau\rangle$ an algebraic semantics) when L is defined model-theoretically, s.t. L-constructs could be encoded by $\mathcal{L}$-components or their derivatives. The

[5] see footnote on page 13

paper hopefully demonstrates that these components are the major actors in playing algebraic logic.

4.1 The functor # plays a principal role and captures the very nature of the logic in question: whether it is propositional-like when formulas coincide with expressions, or equational-like when one first build expressions (terms) and only then formulas by some non-trivial **Set**-construction. In addition, some ways of building logics are determined pure structurally in terms of manipulating with # (like, *eg*, Horn's or Gentzen's constructions). Introducing general **Set**-functors sending expressions to formulas is the main novelty of the paper. ♢

4.2 The choice of variety $\mathcal{V}$ can seem a rather technical step, similar to, say, the choice between polyadic or cylindric algebra styles of algebraizing FOL. However, the context behind $\mathcal{V}$ becomes much deeper if we admit algebras over carrying structures different from sets, *eg*, many-sorted sets with variable number of sorts. In particular, the choice of $\mathcal{V}$ determines algebraic properties of the class $\mathcal{E}$ of actual L-expression algebras. For example, in either polyadic or cylindric style of algebraizing FOL, formula algebras are locally finite and hence not free, moreover, they cannot be made free in either of languages since the class of locally finite algebras is not definable by (even conditional) equations. This is a great disadvantage of these styles of algebraization since one loses the the projectivity property of free algebras and homomorphism chasing becomes much more complicated. In contrast, in categorical logic style of algebraizing logics one makes contexts of expressions explicit so that each context becomes a separate sort and the carrying structures are then variable many-sorted sets or graphs. This complicates the algebraic machinery associated with $\mathcal{V}$ but the benefit is that expression algebras are free. In many situations this advantage of categorical logic greatly exceeds complications of dealing with algebras over graphs. ♢

Anyway, after a choice of language $\Lambda = (\mathcal{V}, \#)$ is made, one obtains the possibility of algebraic presentation of (i) L-expressions by a class $\mathcal{E} \subset \mathcal{V}$, (ii) L-inference rules by an inferential system $\mathcal{I}$ over Λ, and the inference procedure by homomorphisms $h\colon Fr_X \to A$, $X \in \mathbf{Set}, A \in \mathcal{E}$, (iii) semantics of L by a matrix class $\mathcal{M}$ over A and homomorphisms $h: A \to M$, $A \in \mathcal{E}, (M, D) \in \mathcal{M}$. Each of the constructs (ii),(iii) determines a family of consequence relations, $(\Vdash_A^{(\mathcal{I})}, A \in \mathcal{V})$ and $(\models_A^{(\mathcal{M})}, A \in \mathcal{V})$, and thus one comes to an algebraic counterpart of L, the logic $\mathcal{L} = (\mathcal{V}, \#, \mathcal{E}, \vdash)$ or, equivalently, $(\mathcal{V}, \#, \mathcal{E}, Th)$ as shown on diagram (a) Fig. 3.

4.3 The essence of algebraizing logic *a la* Lindenbaum-Tarski is in stating a correspondence between theories and congruences and then algebras. For equational-like logics the correspondence is trivial since theories are themselves congruences. For propositional-like logics the theory-congruence correspondence is the chief actor as it was convincingly demonstrated by Blok and Pigozzi. A crucial for algebraic logic fact is that the theory-congruence correspondence is a fibred adjunction between functors Th and Co into the category of complete lattices as is shown on diagram (b) Fig. 3. With the Grothendik construction (section 3.2) it gives rise to a global adjunction between the category of theories of the logic and its quasivariety semantics.

The accuracy of correspondence between logic and algebra is determined by the properties of the local adjunctions which were discovered by Blok and Pigozzi; they state the $(Th \rightleftharpoons Co)$-axis in the picture (c). $\diamondsuit$

4.4 The closure properties of the class $\mathcal{E}$ play an essential role in stating correspondence between logic and algebra – we have touched upon this in section 3.3. The issue was stressed in works of the Hungarian school (see [2] and references therein) but it seems its value is underestimated in the Polish-like approaches to algebraizing FOL. The problems generated by non-total classes $\mathcal{E}$ were mentioned above. $\diamondsuit$

Thus, we obtain a general 4-dimensional space of logics presented on Fig. 3(c) where also the Polish and (the structural projection of) the Hungarian directions are traced. The subspace defined by conditions $(Th \rightleftharpoons Co) = $ 1-1 or **Id** and $\mathcal{E} = \mathcal{F}_\infty$ or $\mathcal{F}_\lambda$ for some cardinal λ is the paradise of (λ-)total algebraizable logics. Their class is easily manageable by algebraic means yet is sufficiently wide and includes, in particular, (i) ordinary algebraizable propositional logics ($\# = \mathbf{Id}$, $(Th \rightleftharpoons Co) = $ 1-1 and $\mathcal{E} = \mathcal{F}_\omega$), (ii) congruental logics ($(Th \rightleftharpoons Co) = \mathbf{Id}$ and $\# = \_ \times \_$ or $\coprod$, see 2.1.4) (iii) Horn/Gentzen-derivatives of congruental logics (when $\#$ factors through

$\_ \times \_$ or $\coprod$). Unfortunately, polyadic/cylindric style algebraization of FOL drive it out of the paradise. In contrast, in categorical logic where $\mathcal{V}$ is not bounded to be an algebraic theory over **Set**, the class of total algebraizable logics is much wider.

Propositional-like logics live on the plane $\# = \mathbf{Id}$ under the level $(Th \rightleftharpoons Co)$ = 1-1 (inclusive). In this fragment of the space *the truth is algebraic*, that is, is definable by terms and equations whereas structural aspects are poor. In contrast, equational-like logics live on the orthogonal plane $(Th \rightleftharpoons Co) = \mathbf{Id}$ where *the truth is structural*, that is, is defined by some functor $\sigma\colon \mathbf{Set} \to \mathbf{D}\text{-}\mathbf{Set}$ (*eg*, the diagonal functor) so that the true functor τ on Fig. 1 is reduced to the composition $U \triangleright \sigma$. These logics are, so to say, algebraic-logically poor, but it would be better to say that they are algebraic-logically simpler. In this respect benefits of Gentzen-style axiomatization of FOL are instructive; also, it is worth noting the statement in [1] that unexpectedly large part of algebraic FOL is structural rather than logical. It is also remarkable the great interest to structurally rich logics (Girard's linear logic and the like) recently emerged in theoretical computer science.

In general, different combinations of parameters determine different cells in the space. A lesson we can learn from the experience of algebraic logic is that for any reasonable logic an appropriate cells in the algebraic logic space can be always found.

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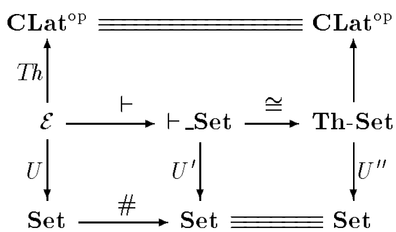
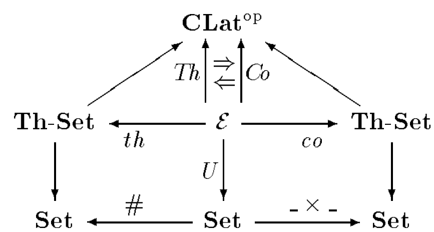
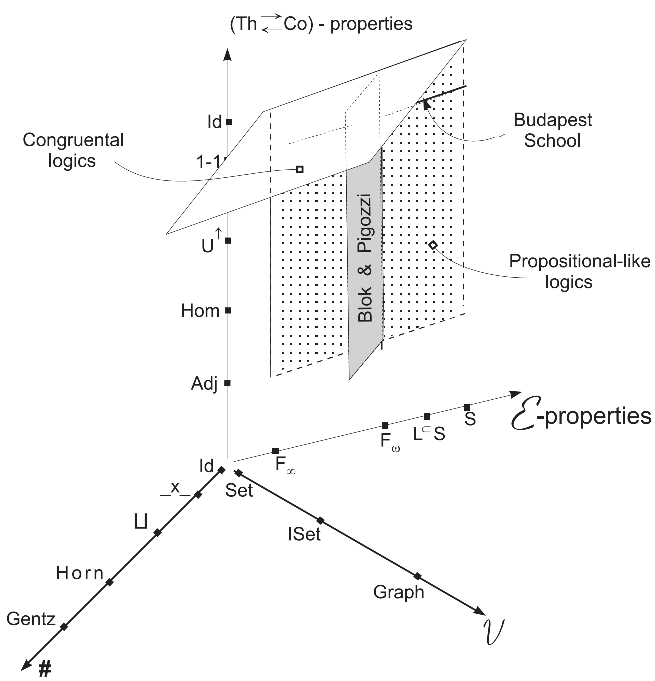


(a) Logic



(b) Logic and algebra



(c) The space of algebraic logics

Fig. 3. What matters in algebraic logic