A combinatorial method of tackling the problem of hierarchy collapse, and a theorem of Ajtai

A. J. Wilkie

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0. The language $L_r$ consists of the binary relation $<$, binary functions $+$ and $·$, unary function $'$ (successor) and the constant 0. The set of genuinely finite natural numbers is denoted by $\omega$.

1. Since we have no idea how to solve the problem of hierarchy collapse (without an oracle) described in the previous sections, we will mention a theorem concerning the same problem for hierarchies with an oracle. For $n \in \omega$ and a unary relation symbol $R$, let us denote by $E_R^n$ the set of formulas of the form

$$\exists \vec{x}_1 < x \forall \vec{x}_2 < x \ldots Q\vec{x}_n < x \Delta(\vec{x}_1, \ldots, \vec{x}_n, x),$$

where $\Delta$ is an open formula of the language $L_r \cup \{R\}$. Also, for $B \subseteq \omega$ let

$$E_B^n = \{ \phi(x) : \phi(x) \in E_R^R \}. $$

Here $\phi(x)^{N,B}$ is the set of $x$ for which $\phi$ is true when $L^R$ is interpreted by $N$, and $R$ by $B$. The sets $A_R^n$ and $A_B^n$ are defined analogously by letting the first quantifier be $\forall$. Their corresponding intersections are denoted $\Delta_0^R$ and $\Delta_0^B$.

2. Proposition.

For all $n \in \omega$ there exists a subset $B$ of $\omega$ such that $E_B^n \neq E_B^{n+1}$.

This proposition follows from a theorem of M. Sipser about Boolean circuits (see “Borel Sets and Circuit Complexity”, JACM 1983, pp. 61–69), whose presentation we closely follow. On the way, we study a theorem of Ajtai about the structure of classes of sets of the form $E_B^n$ that uses an analogue of the Borel hierarchy.

3. Let $M \supseteq N$. We work “inside $M$”, and it will be clear when we consider elements of $M$ as elements, and when as $M$-bounded sets or $M$-bounded functions. Moreover, whenever we use expressions like $s \subseteq M$ and $f : s \rightarrow M$, it should be understood that $s$ and $f$ are coded inside $M$ (and so $M$-bounded).

Let $s \subseteq M$. We denote by $|s|$ the size of $s$, by $2^s$ the set of Boolean functions on $s$, and by $2^{\leq s}$ the set of partial Boolean functions on $s$. The domain of a function $f$ is denoted by $\text{dom}(f)$. We denote by $B^s_f$ the set of partial functions
extending \( f \). A set of functions \( \alpha \subseteq 2^s \) is called a basic subset of \( 2^s \) if \( \alpha = \emptyset \) or \( \alpha = \bigcup_{i=0}^{n} B_i \), for some \( n \in \omega \), where all functions \( f_i \) have genuinely finite domain (i.e. \( \| \text{dom}(f_i) \| \in \omega \)). In other words, a basic subset is defined by a DNF.

The classes \( \tilde{E}_n^s \) and \( \tilde{A}_n^s \), for \( n \in \omega \), are defined by recursion on \( n \) as follows:

(i) \( \tilde{E}_0^s \) and \( \tilde{A}_0^s \) consist of all basic subsets of \( 2^s \).

(ii) \( \tilde{E}_{n+1}^s \) contains all sets of the form \( \bigcup_{i \in A} \tilde{\alpha}_i \), where the sequence \( \alpha_i \) is coded inside \( M \), all \( \alpha_i \) belong to \( \tilde{A}_n^s \), and \( A \subseteq |s|^m \) for some \( m \in \omega \).

(iii) \( \tilde{A}_{n+1}^s = \{ 2^s \setminus \alpha : \alpha \in \tilde{E}_{n+1}^s \} \).

(iv) \( A^s = \bigcup_{n \subseteq \omega} \tilde{E}_n^s \).

We mention the connection (not used in what follows) between \( A^s \) and the theory of finite models, as described in 4 and 5:

4. If \( L \) is a finite relational language (that is, \( L \) contains only a (truly) finite number of relation symbols), one denotes by \( L(R) \) the language obtained by adding to \( L \) a new unary relation symbol \( R \). If \( s \in M \) is an \( L \)-structure with domain \( s \subseteq M \), and \( f \in 2^s \), then we denote by \( (s, f) \) the resulting structure when \( R \) is interpreted by the zero-set of \( f \), i.e. \( \{ a \in s : f(a) = 0 \} \). I leave the proof of the following proposition, which isn’t difficult, as an exercise.

5. Proposition.

Suppose that \( s \in M \), \( \alpha \subseteq 2^s \) and \( n \in \omega \). Then \( \alpha \) belongs to \( \tilde{E}_n^s \) (respectively, \( \tilde{A}_n^s \)) if and only if there exists a finite relational language \( L \), an \( L \)-structure \( s \in M \) with domain \( s \), an \( \exists_n \) (respectively \( \forall_n \)) formula \( \varphi(x_1, \ldots, x_k) \) of \( L(R) \), and \( a_1, \ldots, a_k \in s \) such that

\[
\alpha = \{ f \in 2^s : (s, f) \models \varphi(a_1, \ldots, a_k) \}.
\]

Moreover, if \( n \geq 1 \) then the formula \( \varphi \) can be chosen without free variables.

In order to explain Ajtai’s theorem we need the following definition:

6. Definition.

Let \( s \subseteq M \). A set \( S \subseteq 2^{\leq s} \) is called complete if

(i) For all \( f, g \in S \), \( \| \text{dom}(f) \| = \| \text{dom}(g) \| \). We denote the common value by \( \| S \| \).

(ii) For all \( f, g \in S \), if \( f \neq g \) then \( B^*_f \cap B^*_g = \emptyset \).

(iii) \( \bigcup_{f \in S} B^*_f = 2^s \). In other words, \( S \) is a collection of partial functions, all having the same domain size, such that \( \{ B^*_f : f \in S \} \) is a partition of \( 2^s \). Alternatively, \( S \) is a \( \| S \| \)-DNF tautology, all of whose clauses are mutually exclusive.

7. Theorem. (Ajtai)

Let \( s \subseteq M \) such that \( |s| \) is non-standard, and let \( \alpha \in A^s \). Then there exists a \( k \in \omega \), an \( s \)-complete set \( S \) with \( \| S \| \leq |s| - |s|^{1/k} \), and a subset \( S \) of \( S \) such that

\[
\| \alpha \triangle \bigcup_{f \in S} B^*_f \| \leq 2^{|s| - |s|^{1/k}}.
\]
Before proving 7, we deduce an important corollary.

8. **Corollary.**

Let $s \subseteq M$ such that $|s|$ is non-standard. Suppose that $\alpha \in A^s$ and $|\alpha| \geq 2^{|s|/|s|^{1/\ell}}$ for all $\ell \in \omega$. Then there exist $f \in 2^{\omega^s}$ and $m \in \omega$ such that $|\text{dom}(f)| \leq |s| - |s|^{1/m}$ and $B^s_f \subseteq \alpha$.

**Proof.** We first comment that if $S$ is $s$-complete, $t \in M$ and for all $f \in S$, $a_f \subseteq s$ is such that $\text{dom}(f) \cap a_f = \emptyset$ and $|a_f| = t$, then $S' = \{ f \cup g : f \in S, g \in 2^{\omega^t} \}$ is clearly $s$-complete with $||S'|| = ||S|| + t$. Thus one can assume that the $S$ given by 7 satisfies

$$|s| - |s|^{1/k} - 1 \leq ||S|| \leq |s| - |s|^{1/k}.$$

Let $u = \min\{|B^s_f \setminus \alpha : f \in S\}$. Using 7 and 6(ii), we have

$$2^{|s|/|s|^{1/k}} \geq |\alpha \triangle \bigcup_{f \in S} B^s_f| \geq \bigcup_{f \in S} (B^s_f \setminus \alpha) \geq u \cdot |S|.$$

Also, for all $\ell \in \omega$ we have

$$2^{|s|/|s|^{1/\ell}} \leq |\alpha| = \left| \bigcup_{f \in S} B^s_f \right| + 2^{|s|/|s|^{1/\ell}}$$

$$= |S| \cdot 2^{|s|/|S|} + 2^{|s|/|s|^{1/k}}$$

$$\leq |S| \cdot 2^{|s|/k+1} + 2^{|s|/|s|^{1/k}}.$$

Since $|s|$ is non-standard, it follows that $u \leq 2^{|s|/|s|^{1/\ell}}$ for all $\ell \in \omega$. In particular, there exists an $h \in 2^{\omega^{s^k}}$ with $|s| - |s|^{1/k} - 1 \leq |\text{dom}(h)| \leq |s| - |s|^{1/k}$ such that $|B^s_h \setminus \alpha| \leq 2^{|s|/2k}$. Let $\beta \subseteq s$ satisfy $\beta \cap \text{dom}(h) = \emptyset$ and $|s|^{1/2k} \leq |\beta| \leq |s|^{1/2k} + 1$. Then

$$2^{|s|/2k} \geq |B^s_h \setminus \alpha| = \left| \left( \bigcup_{g \in 2^\beta} B^s_{h \cup g} \right) \setminus \alpha \right| = \sum_{g \in 2^\beta} |B^s_{h \cup g} \setminus \alpha|.$$

Therefore, if $|B^s_{h \cup g} \setminus \alpha| \geq 1$ for all $g \in 2^\beta$, then $2^{|s|/2k} \geq |\beta| \geq 2^{|s|/2k}$, a contradiction. Thus there exists a $g \in 2^\beta$ such that $B^s_{h \cup g} \subseteq \alpha$. Moreover,

$$|\text{dom}(h \cup g)| \leq |s| - |s|^{1/k} + |s|^{1/2k} + 1 \leq |s| - |s|^{1/2k}.$$

\[\square\]

One can use 8 to prove that certain natural sets of functions do not belong to $A^s$, and therefore are not first-order definable in the sense of 4 and 5. For example, it follows immediately from 8 that the set

$$\{ f \in 2^s : |\{ a \in s : f(a) = 0 \} | \text{ is even} \}$$
doesn’t belong to $A^*$ (for non-standard $s$).

9. Remark.

Ajtai proved a theorem stronger than 7, where “there exists a $k \in \omega$” is replaced by “for each standard rational number $\eta$ such that $0 < \eta < 1$”. However, the proof that I give of 7 is much simpler than Ajtai’s, and 7 is sufficient for most applications.

10. In order to prove 7, we shall need the following definitions and lemmas. We fix $s \subseteq M$ such that $|s|$ is non-standard.

If $\sigma$ is a non-trivial basic subset of $2^s$ (i.e. not $\emptyset$ or $2^s$), then clearly there exists a unique minimal subset $X \subseteq s$ which is genuinely finite such that $\sigma = \bigcup_{i<n} B_{s^f_i}$, where $n \in \omega \setminus \{0\}$ and $f_i \in 2^X$ for all $i < n$. We denote this $X$ by $\text{supp}(\sigma)$, and we write $\|\sigma\|$ for $|X|$. If $\sigma$ is trivial then we use the convention $\text{supp}(\sigma) = \emptyset$ and $\|\sigma\| = 0$. Thus we have

11. $|\sigma| \leq \left( 1 - \frac{1}{2^{\|\sigma\|}} \right) 2^{|s|}$ if $\sigma \neq 2^s$.

If $\alpha \in \tilde{A}^*_s$, we can write $\alpha = \bigcap_{i<C} \sigma_i$, where for certain $n, m \in \omega$ we have that $|C| < |s|^m$, that for all $i < C$, $\sigma_i$ is a basic subset of $2^s$ with $0 < \|\sigma_i\| \leq n$, and that for all different $i, j < C$ we have $\text{supp}(\sigma_i) \neq \text{supp}(\sigma_j)$. Note that $n \in \omega$ since $\|\sigma_i\| \in \omega$ for all $i < C$ and the sequence $\langle \sigma_i : i < C \rangle$ is $M$-coded. We denote by $\|\alpha\|$ the smallest value of $n$. If $\alpha$ is trivial, we put $\|\alpha\| = 0$. Let us choose now a subset $D_\alpha$ of $\{0, 1, \ldots, C - 1\}$ which is maximal under the property that for all different $i, j \in D_\alpha$, $\text{supp}(\sigma_i) \cap \text{supp}(\sigma_j) = \emptyset$. Using 11, we get that

12. $|\alpha| \leq \left( 1 - \frac{1}{2^{\|\alpha\|}} \right)^{|D_\alpha|} 2^{|s|}$.

Moreover, since $D_\alpha$ is maximal we have

13. For all $f \in 2^\subseteq s$ such that $\bigcup_{i \in D_\alpha} \text{supp}(\sigma_i) \subseteq \text{dom}(f)$, we have that $B^*_f \cap \alpha = B^*_f \cap \alpha_f$ for some $\alpha_f \in \tilde{A}^*_s$ with $\|\alpha_f\| \leq \|\alpha\| - 1$.

We define $\text{supp}(\alpha) = \bigcup_{i \in D_\alpha} \text{supp}(\sigma_i)$, so that

14. $|\text{supp}(\alpha)| \leq \|\alpha\| \cdot |D_\alpha|$.

We need the following combinatorial lemma:

15. Lemma.

Let $\beta_1, \beta_2, \ldots, \beta_t \subseteq s$, $m \in \omega$ and $p, q$ be standard natural numbers such that $p, q > 0$ and $p + q < 1$. Suppose that $t \leq |s|^m$ and that $|\beta_i| \leq |s|^p$ for $i = 1, \ldots, t$. Then there exist $H \subseteq s$ and $\ell \in \omega$ such that $|H| \geq |s|^q$ and $|H \cap \beta_i| < \ell$ for $i = 1, \ldots, t$. 

4
Proof. Suppose, for the sake of contradiction, that for all \( \ell \in \omega \) and \( H \subseteq s \) with \( |H| = |s|^q + 1 \) there exists an \( i \), where \( 1 \leq i \leq t \), such that \( |H \cap \beta_i| \geq \ell \). Then for all \( \ell \in \omega \),
\[
\left( \frac{|s|}{u} \right) \leq \sum_{i=1}^{t} \left( \frac{|\beta_i|}{\ell} \right) \left( \frac{|s| - \ell}{u - \ell} \right).
\]
(We suppose, of course, that the function \( i \mapsto \beta_i \) is coded inside \( M \).)

Thus for all \( \ell \in \omega \),
\[
\left( \frac{|s|}{u} \right) \leq |s|^m \left( \frac{|\sigma|}{\ell} \right) \left( \frac{|s|}{u - \ell} \right).
\]
It follows that \( |s|^\ell \leq |s|^m |s|^p u^\ell \) for all \( \ell \in \omega \). Taking \( \ell \) big enough, this contradicts the facts that \( u = |s|^q + 1 \), \( p + q < 1 \), \( p + q \) is standard and \( s \) is non-standard.

We will now prove the following theorem, from which 7 clearly follows.

16. Theorem.

Let \( N, m \in \omega \) and \( \langle \alpha_i : i < A \rangle \) be an \( M \)-coded sequence such that \( A < |S|^m \) and either all \( \alpha_i \) belong to \( 2^N \), or all \( \alpha_i \) belong to \( E_N^s \). Then there exist \( k \in \omega \) and an \( s \)-complete set \( S \) with \( ||S|| \leq |s| - |s|^{1/k} \) such that for all \( i \in A \) the following property holds:
\[
||\alpha_i \Delta \bigcup_{f \in S} (B_f^s \cap \sigma_{f,i})|| \leq 2^{|s| - |s|^{1/k}}, \quad (*)
\]
where for all \( i < A \) and for all \( f \in S \), \( \sigma_{f,i} \) is a basic subset of \( 2^s \) (and the function \( (f,i) \mapsto \sigma_{f,i} \) is coded inside \( M \)).

Proof. First of all, let us consider the case that \( \alpha_i \in \bar{A}_1^s \) for all \( i < A \). We comment that \( \max_{i < A} ||\alpha_i|| \in \omega \) since \( ||\alpha_i|| \in \omega \) for all \( i \in A \) and the sequence \( \langle \alpha_i : i < A \rangle \) is \( M \)-coded. We proceed by induction on \( \max_{i < A} ||\alpha_i|| \).

If \( \max_{i < A} ||\alpha_i|| = 0 \), then \( \alpha_i \in \{ \emptyset, 2^s \} \) for all \( i < A \). Thus, we can define \( S = \{ \emptyset \} \), \( k = 1 \) and \( \sigma_{\emptyset,i} = \alpha_i \), which satisfies (*).

Suppose now that \( \max_{i < A} ||\alpha_i|| = n + 1 \), where \( n \in \omega \). Define
\[
E = \{ i < A : |D_{\alpha_i}| \leq \sqrt{|s|} \}
\]
and \( t_i = \text{supp}(\alpha_i) \) for \( i \in E \). Using 15, we get \( H \subseteq s \) and \( \ell \in \omega \) so that

17. \( |H| \geq \sqrt{|s|} \) and

18. For all \( i \in E \), \( |H \cap t_i| \leq \ell \).

Let \( u_i = H \cap t_i \) for all \( i \in E \). For every \( h \in 2^s \setminus H \), \( i \in E \) and \( h(i) \in 2^{u_i} \) we clearly have:

19. \( \text{dom}(h) \cap \text{dom}(h(i)) = \emptyset \) and \( t_i \subseteq \text{dom}(h) \cup \text{dom}(h(i)) \), and so, using 13:
20. \(B^*_{h,i}(\alpha) \cap \alpha_i = B^*_{hi,\cup}(\alpha) \cap \alpha_i \) for some \(\alpha_{h,\cup}(\alpha) \in \tilde{A}_1 \) with \(\|\alpha_{h,\cup}(\alpha)\| \leq n\).

We can suppose that the projections of \(\alpha_{h,\cup}(\alpha) \) to \(2^H\), which we denote \(\alpha^*_{h,\cup}(\alpha) \in \tilde{A}_1^H\), exist. They also satisfy \(\|\alpha^*_{h,\cup}(\alpha)\| \leq n\).

Thus, for fixed \(h \in 2^\land H\), we can apply the induction hypothesis for the sequence \((\alpha^*_{h,\cup}(\alpha) : i \in E, h^{(i)} \in 2^{\land H}\) (since \(\alpha^*_{h,\cup}(\alpha) \in \tilde{A}_1^H\) and its size is at most \(2^{|E|} \leq 2^{|A|} < |s|^{m+1} \leq |H|^{4(m+1)}\), using 17) to obtain \(k_h \in \omega\) and an \(H\)-complete set \(S_h\) satisfying \(|S_h| \leq |H| - |H|^{1/k_h}\) such that for all \(i \in E\) and \(h^{(i)} \in 2^u\):

21. 

\[
\left|\alpha^*_{h,\cup}(\alpha) \triangle \bigcup_{f \in S_h} (B_f^H \cap \tau_{f,i,h^{(i)}})\right| \leq 2^{|H| - |H/1/k_h|},
\]

where the \(\tau_{f,i,h^{(i)}}\) are basic subsets of \(2^H\).

We can suppose that the function \(h \mapsto k_h\) is coded within \(M\), so that \(k^* \triangleq \max\{k_h : h \in 2^\land H\}\) belongs to \(\omega\). Moreover, we can clearly suppose that for all \(h \in 2^\land H\), \(k_h = k^*\) and \(|S_h| = |H| - |H|^{1/k^*}\) (see the proof of 8) while 21 remains true.

Let \(S = \{h \cup f : h \in 2^\land H, f \in S_h\}\). Thus \(S\) is \(s\)-complete, and by 17:

22. 

\(|S| \leq |s| - |H| + (|H| - |H/1/k^*|) = |s| - |H|^{1/k^*} \leq |s| - |s|^{1/k^*}\).

We remark that if \(g \in S\) then \(g = h \cup f\), where \(h \in 2^\land H\) and \(f \in S_h\), and this representation is unique (since \(S_h\) is \(H\)-complete), and so we can define, for \(i \in E\) and \(g \in S\),

\[
\sigma_{g,i} = \bigcup_{h^{(i)} \in 2^u} \left( B^*_{h,i}(\alpha) \cap \tau_{f,i,h^{(i)}}^\dag \right),
\]

where \(\tau_{f,i,h^{(i)}}^\dag\) is the lifting of \(\tau_{f,i,h^{(i)}}\) (given by 21) to \(2^s\). Thus \(\sigma_{g,i}\) is a basic subset of \(2^s\), since \(u_i\) is genuinely finite.

For all \(i \in E\) we clearly get, using 20:

23. 

\[
\alpha_i = \bigcup_{h \in 2^\land H} \bigcup_{h^{(i)} \in 2^u} \left( B^*_{h,i}(\alpha) \cap \alpha_{h,\cup}(\alpha) \right)
\]

and

24. 

\[
\bigcup_{g \in S} (B^*_{g} \cap \sigma_{g,i}) = \bigcup_{h \in 2^\land H} \bigcup_{h^{(i)} \in 2^u} \left( B^*_{h,i}(\alpha) \cap \left( \bigcup_{f \in S_h} (B_f^H \cap \tau_{f,i,h^{(i)}}^\dag) \right) \right).
\]

Thus
25. \[
|\alpha_i \triangle \bigcup_{g \in S} (B^*_g \cap \sigma_{g,i})| = \bigg| \bigcup_{h \in 2^{s\setminus H}} \bigg( \bigcup_{h^{(i)} \in 2^{u_i}} \left( \alpha_{h \cup h^{(i)}} \triangle \bigcup_{f \in S_h} \left( B^*_f \cap \tau_{f,i,h^{(i)}}^{(i)} \right) \right) \right) \bigg|.
\]

Now, for all \( h \in 2^{s\setminus H} \) and \( h^{(i)} \in 2^{u_i} \),
\[
|B^*_h \cap \left( \alpha_{h \cup h^{(i)}} \triangle \bigcup_{f \in S_h} \left( B^*_f \cap \tau_{f,i,h^{(i)}}^{(i)} \right) \right)| = \left| \alpha_{h \cup h^{(i)}} \triangle \bigcup_{f \in S_h} \left( B^*_f \cap \tau_{f,i,h^{(i)}}^{(i)} \right) \right| \leq 2^{\|H\| - |H|^{1/k^*}}
\]
(by 21), and so
26. For all \( i \in E \),
\[
|\alpha_i \triangle \bigcup_{g \in S} (B^*_g \cap \sigma_{g,i})| \leq 2^{|s| - |H|} \cdot 2^f \cdot 2^{\|H\| - |H|^{1/k^*}} \leq 2^{\|s\| - |s|^{1/k^*}},
\]
where the first inequality follows from 25 and 18, and the second one from 17.

Now, let us put \( \sigma_{g,i} = \emptyset \) for all \( g \in S \) if \( i \notin E \). Thus for all \( i \notin E \),
\[
|\alpha_i \triangle \bigcup_{g \in S} (B^*_g \cap \sigma_{g,i})| = |\alpha_i| \leq \left( 1 - \frac{1}{2^{n+1}} \right) \sqrt{|s|} \cdot 2^{|s|}
\]
(using 12 and the definition of \( E \)), and so
27. For all \( i \notin E \),
\[
|\alpha_i \triangle \bigcup_{g \in S} (B^*_g \cap \sigma_{g,i})| \leq 2^{\|s\| - |s|^{1/k^*}}.
\]

The induction is now complete (see 22, 26 and 27), so that we have proved the case \( \alpha_i \in A^*_i \) of the theorem.

In order to prove the theorem in general, we remark that if it holds for a sequence \( \langle \alpha_i : i < A \rangle \), then it also holds for its complement \( \langle 2^s \setminus \alpha_i : i < A \rangle \), since the class of basic subsets is closed under complementation. Thus it suffices to prove that if the theorem holds for the sequence \( \langle \alpha_{i,j} : i, j < A \rangle \) then it also holds for the sequence \( \langle \bigcup_{j < A} \alpha_{i,j} : i < A \rangle \).

Let us therefore choose \( k \in \omega \) and an \( s \)-complete set \( S \) with \( \|S\| \leq |s| - |s|^{1/k} \) such that for all \( i, j < A \),
\[
|\alpha_{i,j} \triangle \bigcup_{f \in S} (B^*_f \cap \sigma_{f,i,j})| \leq 2^{\|s\| - |s|^{1/k}},
\]
where the $\sigma_{f,i,j}$ are basic subsets of $2^s$. (We comment that the sequence $\langle \alpha_{i,j} : i,j < A \rangle$ has length at most $A^2 \leq |s|^{2m}$.) Let $\beta_{f,i} = \bigcap_{j < A} \sigma_{f,i,j}$ for $f \in S$ and $i < A$, so that $\beta_{f,i} \in \tilde{A}_1^s$. Thus for all $i < A$,

$$\left( \bigcap_{j < A} \alpha_{i,j} \right) \setdiff \bigcup_{f \in S} (B_s^f \cap \beta_{f,i}) \subseteq \bigcup_{j < A} \left( \alpha_{i,j} \setdiff \bigcup_{f \in S} (B_s^f \cap \sigma_{f,i,j}) \right),$$

and so

28. For all $i < A$,

$$\left| \left( \bigcap_{j < A} \alpha_{i,j} \right) \setdiff \bigcup_{f \in S} (B_s^f \cap \beta_{f,i}) \right| \leq A \cdot 2^{s - |s|^{1/k}}.$$

Clearly, for all $i < A$ and $f \in S$ the projection $\beta_{f,i}^*$ of $\beta_{f,i}$ into $s \setminus \text{dom}(f)$ belongs to $\tilde{A}_1^{(s \setminus \text{dom}(f))}$. The already proven case of the theorem implies that for all $f \in S$ there exist $k' \in \omega$ and an $(s \setminus \text{dom}(f))$-complete set $S_f$, with $||S_f|| \leq |s \setminus \text{dom}(f)| - |s \setminus \text{dom}(f)|^{1/k'} \leq |s| - ||S|| - (|s| - ||S||)^{1/k'}$, such that

29. For all $i < A$ and $f \in S$,

$$\left| \beta_{f,i}^* \setdiff \bigcup_{g \in S_f} (B_s^g \setminus \text{dom}(f) \cap \tau_{g,i}) \right| \leq 2^{s - |s| - ||S||^{1/k'}} \leq 2^{s - ||S|| - (|s| - ||S||)^{1/k'}}.$$

where the $\tau_{g,i}$ are basic subsets of $2^{s \setminus \text{dom}(f)}$. Like above, we can suppose that $k'$ does not depend on $f$, and that for all $f, g \in S$, $||S_f|| = ||S_g||$.

Let $S^* = \{ f \cup g : f \in S, g \in S_f \}$, and for all $f \in S, g \in S_f$ and $i < A$, let $\sigma_{f \cup g,i} = \tau_{g,i}^\dagger$, where $\tau_{g,i}^\dagger$ is the lifting of $\tau_{g,i}$ to $2^s$. We remark that

30. $||S^*|| \leq |s| - (|s| - ||S||)^{1/k'} \leq |s| - |s|^{1/k'}$.

Using 29 we have that for all $i < A$,

$$\left| \bigcup_{f \in S} \left( (B_s^f \cap \beta_{f,i}) \setdiff \left( B_s^f \cap \bigcup_{g \in S_f} (B_s^g \cap \tau_{g,i}^\dagger) \right) \right) \right| \leq |S| \cdot 2^{s - ||S|| - (|s| - ||S||)^{1/k'}} = 2^{s - (|s| - ||S||)^{1/k'}} \leq 2^{s - |s|^{1/k'}}.$$

(The fact that $|S| = 2||S||$ follows easily from the definition of an $s$-complete set.)
But
\[
\bigcup_{f \in S} \left( B^*_f \cap \bigcup_{g \in S_f} (B^*_g \cap \tau^l_{g,i}) \right) = \bigcup_{h \in S^*} (B^*_h \cap \sigma_{h,i}),
\]
and so
\[
\left| \bigcup_{f \in S} (B^*_f \cap \beta_f) \triangle \bigcup_{h \in S^*} (B^*_h \cap \sigma_{h,i}) \right| \leq 2^{s_1 - |s_1|/k^*}.
\]
This, along with 28 and 30, implies that if we set \( k^* = kk' + 1 \) then for all \( i < A \),
\[
\left| \bigcap_{j < A} \alpha_{i,j} \triangle \bigcup_{h \in S^*} (B^*_h \cap \sigma_{h,i}) \right| \leq 2^{s_1 - |s_1|/k^*}
\]
and \( ||S^*|| \leq |s| - |s_1|/k^* \), which is what we wanted to prove. \( \Box \)

31. The results of 7 and 8 can be used in the study of sets of the form \( \Delta^B_0 \) (see 1). For example, it’s an open question whether the class of \( \Delta_0 \) sets (without an oracle) is closed under “counting modulo 2”, that is: does \( B \in \Delta_0 \) (where \( B \subseteq \omega \)) imply that \( B^{even} \in \Delta_0 \), where \( B^{even} = \{ n \in \omega : |m : m \leq n : n \in B| \} \) is even? On the other hand, using the remark mentioned after the proof of 8 and an enumeration of the formulas in \( \bigcup_{n \in \omega} E_n^B \), it is easy to construct a set \( B \subseteq \omega \) such that \( B^{even} \notin \Delta^B_0 \) (see Paris-Wilkie below).

Closing, I’d like to mention an amusing result that can be proven using the same method. I leave the details as an exercise.

32. Proposition. (Ajtai)

Suppose that \( M \) is denumerable. Let \( a \in M \) be non-standard, and \( \tilde{a} \) be the substructure of \( M \) with domain \( \{ x \in M : x < a \} \) (recall that the language of \( M \) is relational). Then there exist \( A, B \subseteq \{ x \in M : x < a \} \), coded inside \( M \), such that \( (\tilde{a}, A) \cong (\tilde{a}, B) \) but, according to \( M \), \( |A| \) is even while \( |B| \) is odd!

33. Bibliography.
