Inexpressibility of Until

Yuval Filmus

October 26, 2009

We prove that the until operator $\mathcal{U}$ cannot be expressed in terms of the operators $\neg, \vee, \wedge, \circ, \Box, \Diamond$.

A given formula φ contains a finite number of variables, the set of which we denote by $V = V(\varphi)$. The set of states $\mathcal{S} = \mathcal{S}(\varphi)$ can be identified with the Boolean algebra $\mathbb{B}[V]$. Any distribution μ on $\mathcal{S}$ lifts to a distribution on the set of infinite sequences $\mathcal{S}^\infty$.

For $\sigma \in \mathcal{S}^\infty$, denote by $\sigma|_k$ the initial prefix of σ of size k. Given a distribution μ on $\mathcal{S}$, define the canonical random sequence $\mathcal{R}$ by $\mathcal{R}_i \sim \mu$ independently.

We say that an event E is *determined* given an event F if the probability $\Pr[E|F]$ is either zero or one. We say that a formula φ is *finitely determined* with index k if for any distribution μ on $\mathcal{S} = \mathcal{S}(\varphi)$ and any initial segment $s \in \mathcal{S}^k$, the event $\varphi(\sigma)$ is determined given $\sigma|_k = s$.

Let us prove by induction that every formula φ with $\neg, \vee, \wedge, \circ, \Box, \Diamond$ is finitely determined. The base case $\varphi = x$ for a variable x is clearly finitely determined with index 1. If φ is finitely determined with index k, then it's easy to see that so is $\neg\varphi$. If φ, ψ are finitely determined with indices k, ℓ, respectively, then it's easy to see that $\varphi \vee \psi$ and $\varphi \wedge \psi$ are finitely determined with index $\max(k, \ell)$. Moreover, if φ is finitely determined with index k, then it's easy to see that $\circ\varphi$ is finitely determined with index $k+1$.

To complete the proof by induction, let $\psi = \Box\varphi$, where φ is finitely determined with index k (the case $\psi = \Diamond\varphi$ is completely analogous). We consider two cases: either there exists a prefix s with $\mu(s) > 0$ such that $\Pr[\mathcal{R}|\mathcal{R}|_k = s] = 0$, or $\Pr[\mathcal{R}] = 1$. In the first case, divide the random sequence $\mathcal{R} \in \mathcal{S}^\infty$ into random sequences $\mathcal{R}_{[i]}$ of length k. Notice that $\Pr[\psi(\mathcal{R})] \leq \prod_i \Pr[\mathcal{R}_{[i]} \neq s] = 0$. In the second case, the event $\psi(\mathcal{R})$ is the intersection of countably many events of probability 1, and so $\Pr[\psi(\mathcal{R})] = 1$. In both cases, ψ is finitely determined with index 0.

Conversely, $\varphi = p\mathcal{U}q$ is not finitely determined. Indeed, suppose that p, q are independent, with probabilities α, β, respectively. The probability of φ given a prefix in which p always holds is

$$\sum_{t \geq 0} (\alpha(1-\beta))^i \beta = \frac{\beta}{1 - \alpha(1-\beta)}.$$

If $\alpha = \beta = 1/2$, this probability is $2/3$, and so φ is not finitely determined.