Inexpressibility of Until

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We prove that the until operator $\mathcal{U}$ cannot be expressed in terms of the operators $\neg, \lor, \land, \circ, \Box, \Diamond$.

A given formula $\varphi$ contains a finite number of variables, the set of which we denote by $V = V(\varphi)$. The set of states $S = S(\varphi)$ can be identified with the Boolean algebra $\mathbb{B}[V]$. Any distribution $\mu$ on $S$ lifts to a distribution on the set of infinite sequences $S^\infty$.

For $\sigma \in S^\infty$, denote by $\sigma|_k$ the initial prefix of $\sigma$ of size $k$. Given a distribution $\mu$ on $S$, define the canonical random sequence $R$ by $R_i \sim \mu$ independently.

We say that an event $E$ is determined given an event $F$ if the probability $\Pr[E|F]$ is either zero or one. We say that a formula $\varphi$ is finitely determined with index $k$ if for any distribution $\mu$ on $S = S(\varphi)$ and any initial segment $s \in S^k$, the event $\varphi(\sigma)$ is determined given $\sigma|_k = s$.

Let us prove by induction that every formula $\varphi$ with $\neg, \lor, \land, \circ, \Box, \Diamond$ is finitely determined. The base case $\varphi = x$ for a variable $x$ is clearly finitely determined with index 1. If $\varphi$ is finitely determined with index $k$, then it’s easy to see that so is $\neg \varphi$. If $\varphi, \psi$ are finitely determined with indices $k, \ell$, respectively, then it’s easy to see that $\varphi \lor \psi$ and $\varphi \land \psi$ are finitely determined with index $\max(k, \ell)$. Moreover, if $\varphi$ is finitely determined with index $k$, then it’s easy to see that $\circ \varphi$ is finitely determined with index $k + 1$.

To complete the proof by induction, let $\psi = \Box \varphi$, where $\varphi$ is finitely determined with index $k$ (the case $\psi = \Diamond \varphi$ is completely analogous). We consider two cases: either there exists a prefix $s$ with $\mu(s) > 0$ such that $\Pr[R|R|_k = s] = 0$, or $\Pr[R] = 1$. In the first case, divide the random sequence $R \in S^\infty$ into random sequences $R[i]$ of length $k$. Notice that $\Pr[\psi(R)] \leq \prod_i \Pr[R[i] \neq s] = 0$. In the second case, the event $\psi(R)$ is the intersection of countably many events of probability 1, and so $\Pr[\psi(R)] = 1$. In both cases, $\psi$ is finitely determined with index 0.

Conversely, $\varphi = p \mathcal{U} q$ is not finitely determined. Indeed, suppose that $p, q$ are independent, with probabilities $\alpha, \beta$, respectively. The probability of $\varphi$ given a prefix in which $p$ always holds is

$$\sum_{t \geq 0} (\alpha(1-\beta))^t \beta = \frac{\beta}{1 - \alpha(1-\beta)}.$$ 

If $\alpha = \beta = 1/2$, this probability is $2/3$, and so $\varphi$ is not finitely determined.