A Question from The Probabilistic Method

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April 2008

1 Introduction

This note concerns question 8 in chapter 2 of the well-known textbook “The Probabilistic Method”. This question was given in a take-home exam by Nati Linial in April 2008. We present a complete solution, as well as several partial ones which are of interest.

The question is as follows:

Given integers $n \geq k \geq 1$ and an orthogonal $n \times n$ matrix A, show that $\max_c \sum_{r=0}^{k-1} A_{rc}^2 \geq k/n$, and similarly $\min_c \sum_{r=0}^{k-1} A_{rc}^2 \geq k/n$. Moreover, produce an instance A with equality.

The inequality is easily proved by noting that the squared sum of the first k rows is k (since each row is a unit vector), and so the largest column has squared sum at least k/n, and the smallest one at most k/n. The rest of this note concerns instances where the inequalities are tight.

2 Solution with Hadamard Matrices

Hadamard matrices can be defined as follows:

$$H_0 = \begin{bmatrix} 1 \end{bmatrix},$$

$$H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}.$$

The matrix H_n is a $2^n \times 2^n$ symmetric matrix and satisfies $H_n^2 = 2^n I_{2^n}$. This is clear for H_0 and follows for H_{n+1} since

$$\begin{aligned} H_{n+1}^2 &= \begin{bmatrix} H_n^2 + H_n^2 & H_n^2 - H_n^2 \\ H_n^2 - H_n^2 & H_n^2 + H_n^2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 2^n I_{2^n} & 0_{2^n} \\ 0_{2^n} & 2 \cdot 2^n I_{2^n} \end{bmatrix} = 2^{n+1} I_{2^{n+1}}. \end{aligned}$$

It follows that $2^{-n/2} H_n$ is orthogonal. Moreover, since $\left(\pm 2^{-n/2}\right)^2 = 2^{-n}$ it trivially follows that the squared sum of the first (in fact any) k elements of any column is $k/2^n$.

3 Solution with Complex Vandermonde Matrices

Let $\omega = e^{2\pi i/n}$ be a primitive n-th root of unity, and define the complex matrix $A_{ij} = \omega^{ij}$. This matrix is unitary up to a constant n:

$$(A^* A)_{ij} = \sum_{k=0}^{n-1} \omega^{k(j-i)} = \begin{cases} n \cdot 1 = n & i = j, \\ \frac{\omega^n - 1}{\omega - 1} = 0 & i \neq j. \end{cases}$$

Moreover, each entry in the matrix has unit norm, and so if we normalize the matrix by $1/\sqrt{n}$ we obtain a unitary matrix which is a solution to our problem, apart from the fact that this matrix is complex instead of real. We continue by presenting two attempts to remedy this problem.

4 Realization of Vandermonde Matrix using a Representation of $\mathbb{C}$

Our first attempt to realize the Vandermonde matrix as a real matrix with similar properties is through the two-dimensional representation of $\mathbb{C}$ over $\mathbb{R}$. The representation is as follows:

$$M(x + yi) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

It is now easy to check that for $z, w \in \mathbb{C}$ we have $M(z)^{\mathrm{T}} = M(\overline{z})$, $M(z) + M(w) = M(z + w)$ and $M(z)M(w) = M(zw)$.

For any complex matrix A, denote by $M(A)$ the real matrix obtained by replacing each element z by $M(z)$. Thus $M(A)$ is double the size of A. It is easy to see that $M(A^*) = M(A)^{\mathrm{T}}$. Moreover, it easily follows from linearity that $M(AB) = M(A)M(B)$. Thus if A is unitary, $M(A)$ is orthogonal.

Taking now as A the Vandermonde matrix described in the previous section, we obtain an orthogonal matrix $M(A)$. Moreover, it is easy to see that $M(A)$ satisfies the conditions of our problem for $2n$ and $2k$. We thus obtain a solution for the problem in case n, k are both even.

5 Realization of Vandermonde Matrix using Folding

Our second attempt to realize the Vandermonde matrix stems from the similarities between the Hadamard and Vandermonde solutions for small n. As a typical example, consider $n = 4$. The two (un-normalized) solutions H_2 and V_4 are as follows:

$$H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \qquad V_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

If we put $i = 1$ in V_4 that we obtain H_2 up to switching the second and third rows and columns! Putting $i = -1$ will also work but requires a different permutation of H_2.

Let us see if this strange coincidence holds water. For definiteness we choose the assignment $i = 1$, although (as the reader can check) $i = -1$ will also work. Starting with the Vandermonde matrix, we obtain the following symmetric matrix:

$$A_{ij} = \cos\frac{2\pi ij}{n} + \sin\frac{2\pi ij}{n}.$$

Let us calculate A^2 (we shall consider indices modulo n):

$$\begin{aligned}
\left(A^2\right)_{ij} &= \sum_{k=0}^{n-1} A_{ik}A_{jk} = \sum_{k=0}^{n-1} \frac{1}{2}\left(A_{ik}A_{jk} + A_{i(-k)}A_{j(-k)}\right) \\
&= \sum_{k=0}^{n-1} \frac{1}{2}\left[\left(\cos\frac{2\pi ik}{n} + \sin\frac{2\pi ik}{n}\right)\left(\cos\frac{2\pi jk}{n} + \sin\frac{2\pi jk}{n}\right) + \right. \\
&\qquad \left.\left(\cos\frac{2\pi ik}{n} - \sin\frac{2\pi ik}{n}\right)\left(\cos\frac{2\pi jk}{n} - \sin\frac{2\pi jk}{n}\right)\right] \\
&= \sum_{k=0}^{n-1}\left(\cos\frac{2\pi ik}{n}\cos\frac{2\pi jk}{n} + \sin\frac{2\pi ik}{n}\sin\frac{2\pi jk}{n}\right) \\
&= \sum_{k=0}^{n-1}\frac{1}{2}\left(\cos\frac{2\pi(i+j)k}{n} + \cos\frac{2\pi(i-j)k}{n} - \cos\frac{2\pi(i+j)k}{n} + \cos\frac{2\pi(i-j)k}{n}\right) \\
&= \sum_{k=0}^{n-1}\cos\frac{2\pi(i-j)k}{n} = \operatorname{Re}\sum_{k=0}^{n-1}\omega^{(i-j)k} = \begin{cases} n & i=j, \\ 0 & i\neq j. \end{cases}
\end{aligned}$$

Thus A is (up to normalization) orthogonal! The first row is composed of 1s and so has the right sum of squares, but unfortunately the other entries are not necessarily of unit norm. It thus seems that we have found a solution only for $k = 1$. This shortcoming can be amended by noting the following:

$$\begin{aligned}
A_{ij}^2 + A_{(-i)j}^2 &= \left(\cos\frac{2\pi ij}{n} + \sin\frac{2\pi ij}{n}\right)^2 + \left(\cos\frac{2\pi ij}{n} - \sin\frac{2\pi ij}{n}\right)^2 \\
&= 2\cos^2\frac{2\pi ij}{n} + 2\sin^2\frac{2\pi ij}{n} = 2.
\end{aligned}$$

It follows that a proper rearranging of the rows will lead to a solution for any k. Indeed, putting row i together with row $-i$ produces a partition of the set of rows into $\lfloor\frac{n-1}{2}\rfloor$ pairs and 1 or 2 singletons, depending on the parity. If $k = 2l$ is even then l of the pairs should be put as the first k rows (if $k = n$ then the pairs are supplemented by the remaining rows). If $k = 2l + 1$ then the first k rows should consist of l pairs and one of the singletons. Thus, we have obtained a solution of the problem for any n, k. Moreover, by permuting the columns to match the permutation of the rows we get a symmetric solution.