

Boolean constant degree functions on the slice are juntas

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Abstract

We show that a Boolean degree d function on the slice $\binom{[n]}{k} = \{(x_1, \dots, x_n) \in \{0, 1\}^n : \sum_{i=1}^n x_i = k\}$ is a junta, assuming that $k, n - k$ are large enough. This generalizes a classical result of Nisan and Szegedy on the hypercube. Moreover, we show that the maximum number of coordinates that a Boolean degree d function can depend on is the same on the slice and on the hypercube.

1 Introduction

Nisan and Szegedy [11] showed that a Boolean degree d function on the hypercube $\{0, 1\}^n$ depends on at most $d2^{d-1}$ coordinates, and described a Boolean degree d function which depends on $\Omega(2^d)$ coordinates. The upper bound has recently been improved to $O(2^d)$ by Chiarelli, P. Hatami and Saks [1], who have also improved the hidden constant in the lower bound $\Omega(2^d)$. Let us denote the optimal bound by $\gamma(d)$. The goal of this paper is to generalize this result to the *slice* $\binom{[n]}{k}$ or *Johnson scheme* $J(n, k)$, which consists of all points in the hypercube having Hamming weight k :

Theorem 1.1. *There exists a constant C such that the following holds. If $C^d \leq k \leq n - C^d$ and $f: \binom{[n]}{k} \rightarrow \{0, 1\}$ has degree d , then f depends on at most $\gamma(d)$ coordinates.*

(We explain in Section 2 what degree d means for functions on the hypercube and on the slice.)

Filmus et al. [5] proved a version of Theorem 1.1 (with a non-optimal bound on the number of points) when k/n is bounded away from 0, 1, but their bound deteriorates as k/n gets closer to 0, 1. We use their result (which we reproduce here, to keep the proof self-contained) to bootstrap our own inductive argument.

The case $d = 1$ is much easier. The following folklore result is proved formally in [?]:

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Theorem 1.2. *If $2 \leq k \leq n - 2$ and $f: \binom{[n]}{k} \rightarrow \{0, 1\}$ has degree 1, then f depends on at most one coordinate.*

The bounds on k in this theorem are optimal, since every function on $\binom{[n]}{1}$ and on $\binom{[n]}{n-1}$ has degree 1. In contrast, the bounds on k in Theorem 1.1 are probably not optimal, an issue we discuss in Section 4.

Let us close this introduction by mentioning a recent result of Keller and Klein [8], which studies Boolean functions on $\binom{[n]}{k}$ which are ϵ -close to being degree d , where distance is measured using the squared L_2 norm. Assuming that $k \leq n/2$, their result states that if $\epsilon < (k/n)^{O(d)}$ then f is $O(\epsilon)$ -close to a junta.

2 Preliminaries

In this paper, we discuss Boolean functions, which are 0,1-valued functions, on two different domains: the hypercube and the slice. We will use the notation $[n] := \{1, \dots, n\}$. A *degree d function* (in a context in which degree is defined) is a function of degree at most d .

The hypercube. The n -dimensional hypercube is the domain $\{0, 1\}^n$. Every function on the hypercube can be represented uniquely as a multilinear polynomial in the n input arguments x_1, \dots, x_n . The *degree* of a function on the hypercube is the degree of this polynomial. Alternatively, the degree of a function on the hypercube is the minimum degree of a polynomial in x_1, \dots, x_n which agrees with the function on all points of the hypercube. A function on the hypercube is an *m -junta* if it depends on at most m inputs, that is, if there exists a set I of m inputs such that $f(x) = f(y)$ as long as $x_i = y_i$ for all $i \in I$; we also say that f is an I -junta. For more information on functions on the hypercube from this perspective, consult O’Donnell’s monograph [12].

The slice. Let $0 \leq k \leq n$. The *slice* $\binom{[n]}{k}$ is the subset of $\{0, 1\}^n$ consisting of all vectors having Hamming weight k . The slice appears naturally in combinatorics, coding theory, and elsewhere, and is known to algebraic combinatorialists as the *Johnson scheme* $J(n, k)$. Every function on the slice can be represented uniquely as a multilinear polynomial P in the n input arguments x_1, \dots, x_n of degree at most $\min(k, n - k)$ which satisfies $\sum_{i=1}^n \frac{\partial P}{\partial x_i} = 0$ (the latter condition is known as *harmonicity*). The *degree* of a function on the slice is the degree of this polynomial. Alternatively, the degree of a function on the slice is the minimum degree of a polynomial in x_1, \dots, x_n (not necessarily multilinear or harmonic) which agrees with the function on all points of the slice.

A function f on the slice is an *m -junta* if there exist a function $g: \{0, 1\}^m \rightarrow \{0, 1\}$ and m indices $i_1 < \dots < i_m$ such that $f(x) = g(x|_{i_1, \dots, i_m})$, where $x|_{i_1, \dots, i_m} = x_{i_1}, \dots, x_{i_m}$. Alternatively, f is an m -junta if there exists a set I of m coordinates such that f is invariant under permutation of the coordinates in $[n] \setminus I$; we also say that f is an I -junta. Note that the set I is not defined uniquely (in contrast to the hypercube case): for example, $f = \sum_{i \in I} x_i$ is both an I -junta and an $[n] \setminus I$ -junta.

The *p th norm* of f is given by $\|f\|_p = \mathbb{E}[|f|^p]^{1/p}$, where the expectation is over a uniform point in the slice. In particular, $\|f\|_2^2 = \mathbb{E}[f^2]$.

Let f be a Boolean function on the slice $\binom{[n]}{k}$, and let P be its unique harmonic multilinear polynomial representation. The d th level of f , denoted $f^{=d}$, is the homogeneous degree d part of P (the sum of all degree d monomials with their coefficients). The different levels are orthogonal: $\mathbb{E}[f^{=d}f^{=e}] = 0$ if $d \neq e$. Orthogonality of the different levels implies that

$$\|f\|_2^2 = \sum_{d=0}^{\deg f} \|f^{=d}\|_2^2.$$

Let f be a Boolean function on the slice $\binom{[n]}{k}$, and let $i, j \in [n]$. We define $f^{(i,j)}$ to be the function given by $f^{(i,j)}(x) = f(x^{(i,j)})$, where $x^{(i,j)}$ is obtained from x by switching x_i and x_j . The (i, j) th influence of f is $\text{Inf}_{ij}[f] = \frac{1}{4} \Pr[f(x) \neq f(x^{(i,j)})]$, where x is chosen uniformly at random over the slice. An equivalent formula is $\text{Inf}_{ij}[f] = \frac{1}{4} \mathbb{E}[(f - f^{(i,j)})^2]$. Clearly $\text{Inf}_{ij}[f] = 0$ if and only if $f = f^{(i,j)}$. The total influence of f is $\text{Inf}[f] = \frac{1}{n} \sum_{1 \leq i < j \leq n} \text{Inf}_{ij}[f]$. It is given by the formula

$$\text{Inf}[f] = \sum_{d=0}^{\deg f} \frac{d(n+1-d)}{n} \|f^{=d}\|_2^2. \quad (1)$$

For a parameter $\rho \in (0, 1]$, the noise operator T_ρ , mapping functions on the slice to functions on the slice, is defined by

$$T_\rho f = \sum_{d=0}^{\deg f} \rho^{d(1-(d-1)/n)} f^{=d}. \quad (2)$$

Alternatively, $(T_\rho f)(x)$ is the expected value of $f(y)$, where y is chosen by applying $N \sim \text{Po}(\frac{n-1}{2} \log(1/\rho))$ random transpositions to x . Lee and Yau [9] proved a log Sobolev inequality, which together with classical results of Diaconis and Saloff-Coste [2] implies that the following hypercontractive inequality holds for some constant $C_H > 0$:

$$\|T_\rho f\|_2 \leq \|f\|_{4/3}, \quad \rho = \left(\frac{2k(n-k)}{n(n-1)} \right)^{C_H}. \quad (3)$$

For more information on functions on the slice, consult [6].

3 Main theorem

For the rest of this section, we fix an integer $d \geq 1$. Our goal is to prove Theorem 1.1 for this value of d . We will use the phrase *universal constant* to refer to a constant independent of d .

The strategy of the proof is to proceed in three steps:

1. Bootstrapping: Every Boolean degree d function on $\binom{[2n]}{n}$ is a K^d -junta.
2. Induction: If every Boolean degree d function on $\binom{[n]}{k}$ is an M -junta, then the same holds for $\binom{[n+1]}{k}$ and $\binom{[n+1]}{k+1}$ (under certain conditions).

3. Culmination: If a Boolean degree d function on $\binom{[n]}{k}$ is an L -junta but not an $(L-1)$ -junta, then (under certain conditions) there exists a Boolean degree d function on the hypercube depending on L coordinates.

We also show a converse to the last step: given a Boolean degree d function on the hypercube depending on L coordinates, we show how to construct Boolean degree d functions on large enough slices that are L -juntas but not $(L-1)$ -juntas.

3.1 Bootstrapping

We bootstrap our approach by proving that every Boolean degree d function on $\binom{[2n]}{n}$ is a junta. The proof is a simple application of hypercontractivity, and already appears in [5]. We reproduce a simplified version here in order to make the paper self-contained.

The main idea behind the proof is to obtain a dichotomy on the influences of the function.

Lemma 3.1. *There exists a universal constant α such that all non-zero influences of a Boolean degree d function on $\binom{[2n]}{n}$ are at least α^d .*

Proof. Let f be a Boolean degree d function on $\binom{[2n]}{n}$. Given $i, j \in [n]$, consider the function $f_{ij} = (f - f^{(i,j)})/2$, related to the (i, j) th influence of f by $\text{Inf}_{ij}[f] = \|f_{ij}\|_2^2$. Since f is 0, 1-valued, f_{ij} is 0, ± 1 -valued. Since f has degree d , it follows that f_{ij} can be written as a degree d polynomial, and so has degree at most d . Hypercontractivity (3) implies that for some universal constant ρ , we have

$$\|T_\rho f_{ij}\|_2 \leq \|f_{ij}\|_{4/3}. \quad (4)$$

We can estimate the left-hand side of (4) using (2):

$$\|T_\rho f_{ij}\|_2^2 = \sum_{e=0}^d \rho^{2e(1-(e-1)/n)} \|f_{ij}^{e=e}\|_2^2 \geq \rho^{2d} \sum_{e=0}^d \|f_{ij}^{e=e}\|_2^2 = \rho^{2d} \|f_{ij}\|_2^2 = \rho^{2d} \text{Inf}_{ij}[f].$$

We can calculate the right-hand side of (4) using the fact that f_{ij} is 0, ± 1 -valued:

$$\|f_{ij}\|_{4/3}^{4/3} = \mathbb{E}[|f_{ij}|^{4/3}] = \mathbb{E}[|f_{ij}|^2] = \|f_{ij}\|_2^2 = \text{Inf}_{ij}[f].$$

Combining the estimates on both sides of (4), we conclude that

$$\rho^{2d} \text{Inf}_{ij}[f] \leq \text{Inf}_{ij}[f]^{3/2}.$$

Hence either $\text{Inf}_{ij}[f] = 0$ or $\text{Inf}_{ij}[f] \geq \rho^{4d}$. □

The next step is to prove a degenerate triangle inequality (for the non-degenerate version, see Wimmer [13, Lemma 5.4] and Filmus [4, Lemma 5.1]).

Lemma 3.2. *Let f be a function on a slice. If $\text{Inf}_{ik}[f] = \text{Inf}_{jk}[f] = 0$ then $\text{Inf}_{ij}[f] = 0$.*

Proof. If $\text{Inf}_{ik}[f] = \text{Inf}_{jk}[f] = 0$ then $f(x) = f(x^{(i,k)}) = f(x^{(j,k)})$, and so $f(x) = f(x^{(i,k)(j,k)(i,k)}) = f(x^{(i,j)})$. It follows that $\text{Inf}_{ij}[f] = 0$. □

To complete the proof, we use formula (1), which implies that $\text{Inf}[f] \leq d$ for any Boolean degree d function f , together with an idea of Wimmer [13, Proposition 5.3].

Lemma 3.3. *There exists a universal constant $K > 1$ such that for $n \geq 2$, every Boolean degree d function on $\binom{[2n]}{n}$ is a K^d -junta.*

Proof. Let f be a Boolean degree d function on $\binom{[2n]}{n}$. Construct a graph G on the vertex set $[2n]$ by connecting two vertices i, j if $\text{Inf}_{ij}[f] \geq \alpha^d$, where α is the constant from Lemma 3.1. Let M be a maximal matching in G . It is well-known that the $2|M|$ vertices of M form a vertex cover V , that is, any edge of G touches one of these vertices. Therefore if $i, j \notin V$ then $\text{Inf}_{ij}[f] < \alpha^d$, and so $\text{Inf}_{ij}[f] = 0$ according to Lemma 3.1. In other words, f is a V -junta. It remains to bound the size of V .

Let (i, j) be any edge of M , and let k be any other vertex. Lemma 3.2 shows that either $\text{Inf}_{ik}[f] \neq 0$ or $\text{Inf}_{jk}[f] \neq 0$, and so either (i, k) or (j, k) is an edge of G , according to Lemma 3.1. It follows that G contains at least $|M|(2n - 2)/2$ edges (we divided by two since some edges could be counted twice). Therefore $n \text{Inf}[f] = \sum_{1 \leq i < j \leq n} \text{Inf}_{ij}[f] \geq \alpha^d |M|(n - 1)$. On the other hand, (1) shows that

$$\text{Inf}[f] = \sum_{e=0}^d \frac{e(n+1-e)}{n} \|f^{=e}\|_2^2 \leq d \sum_{e=0}^d \|f^{=e}\|_2^2 = d \|f\|_2^2 \leq d.$$

It follows that

$$|V| = 2|M| \leq 2 \frac{n}{n-1} d (1/\alpha)^d \leq (8/\alpha)^d. \quad \square$$

3.2 Induction

The heart of the proof is an inductive argument which shows that if Theorem 1.1 holds (with a non-optimal bound on the size of the junta) for the slice $\binom{[n]}{k}$, then it also holds for the slices $\binom{[n+1]}{k}$ and $\binom{[n+1]}{k+1}$, assuming that n is large enough and that k is not too close to 0 or n . Given a Boolean degree d function f on $\binom{[n+1]}{k}$ or $\binom{[n+1]}{k+1}$, the idea is to consider restrictions of f obtained by fixing one of the coordinates.

Lemma 3.4. *Suppose that every Boolean degree d function on $\binom{[n]}{k}$ is an M -junta, where $M \geq 1$.*

Let f be a Boolean degree d function on $\binom{[n+1]}{k+b}$, where $b \in \{0, 1\}$. For each $i \in [n+1]$, let f_i be the restriction of f to vectors satisfying $x_i = b$. Then for each $i \in [n+1]$ there exists a set $S_i \subseteq [n+1] \setminus \{i\}$ of size at most M and a function $g_i: \{0, 1\}^{S_i} \rightarrow \{0, 1\}$, depending on all inputs, such that $f_i(x) = g_i(x|_{S_i})$.

Proof. Choose $i \in [n+1]$. The domain of f_i is isomorphic to $\binom{[n]}{k}$. Moreover, since f can be represented as a polynomial of degree d , so can f_i , hence $\deg f_i \leq d$. By assumption, f_i is an M -junta, and so $f_i(x) = h_i(x|_{T_i})$ for some set T_i of M indices and some function $h_i: \{0, 1\}^{T_i} \rightarrow \{0, 1\}$. Let $S_i \subseteq T_i$ be the set of inputs that h_i depends on. Then there exists a function $g_i: \{0, 1\}^{S_i} \rightarrow \{0, 1\}$ such that $g_i(x|_{S_i}) = h_i(x|_{T_i})$, completing the proof. \square

Each of the sets S_i individually contains at most M indices. We now show that in fact they contain at most M indices *in total*.

Lemma 3.5. *Under the assumptions of Lemma 3.4, suppose further that $n \geq (M + 1)^3$ and $M + 2 \leq k \leq n + 1 - (M + 2)$.*

The union of any $M + 1$ of the sets S_i contains at most M indices.

Proof. We can assume, without loss of generality, that the sets in question are S_1, \dots, S_{M+1} . Denote their union by A , and let $B = A \cup \{1, \dots, M + 1\}$. Since $|B| \leq (M + 1)^2 \leq n$, there exists a point $r \in [n + 1] \setminus B$. We proceed by bounding the number of unordered pairs of distinct indices $i, j \in [n + 1] \setminus \{r\}$ such that $f_r = f_r^{(i,j)}$, which we denote by N . Since f_r is an M -junta, we know that $N \geq \binom{n-M}{2}$. We will now obtain an upper bound on N in terms of $|A|$ and $|B|$.

Let $1 \leq \ell \leq M + 1$, and suppose that $i \in S_\ell$ and $j \notin S_\ell \cup \{\ell, r\}$. We claim that $f_r^{(i,j)} \neq f_r$. Indeed, since g_ℓ depends on all inputs, there are two inputs y, z to g_ℓ , differing only on the i th coordinate, say $y_i = b$ and $z_i = 1 - b$, such that $g_\ell(y) \neq g_\ell(z)$. Since $M + 2 \leq k \leq n + 1 - (M + 2)$, we can extend y to an input x to f satisfying additionally the constraints $x_\ell = x_r = b$ and $x_j = 1 - b$. Since $x_\ell = x_r = b$, the input x is in the common domain of f_ℓ and f_r . Notice that $f_r(x) = f_\ell(x) = g_\ell(y)$, whereas $f_r(x^{(i,j)}) = f_\ell(x^{(i,j)}) = g_\ell(z)$, since $x_i = y_i = b$ whereas $x_j = 1 - b$. By construction $g_\ell(y) \neq g_\ell(z)$, and so $f_r \neq f_r^{(i,j)}$.

The preceding argument shows that if $i \in A$ and $j \notin B \cup \{r\}$ then $f_r \neq f_r^{(i,j)}$. Therefore $\binom{n}{2} - N \geq |A|(n - |B|) \geq |A|(n - (M + 1)^2)$. Combining this with the lower bound $N \geq \binom{n-M}{2}$, we deduce that

$$|A|(n - (M + 1)^2) \leq \binom{n}{2} - \binom{n-M}{2} = \frac{M(2n - M - 1)}{2}.$$

Rearrangement shows that

$$|A| \leq \frac{M(2n - M - 1)}{2(n - (M + 1)^2)} = \left(1 + \frac{(M + 1)(2M + 1)}{2n - 2(M + 1)^2}\right) M.$$

When $n > (M^2 + (3/2)M + 1)(M + 1)$, we have $\frac{(M+1)(2M+1)}{2n-2(M+1)^2} < \frac{1}{M}$, and so $|A| < M + 1$. We conclude that when $n \geq (M + 1)^3$, we have $|A| \leq M$. \square

Corollary 3.6. *Under the assumptions of the preceding lemma, the union of S_1, \dots, S_{n+1} contains at most M indices.*

Proof. Suppose that the union contained at least $M + 1$ indices i_1, \dots, i_{M+1} . Each index i_t is contained in some set S_{j_t} , and in particular the union of $S_{j_1}, \dots, S_{j_{M+1}}$ contains at least $M + 1$ indices, contradicting the lemma. \square

Denoting the union of all S_i by S , it remains to show that f is an S -junta.

Lemma 3.7. *Suppose that every Boolean degree d function on $\binom{[n]}{k}$ is an M -junta, where $M \geq 1$; that $n \geq (M + 1)^3$; and that $M + 2 \leq k \leq n + 1 - (M + 2)$.*

Any Boolean degree d function on $\binom{[n+1]}{k}$ or on $\binom{[n+1]}{k+1}$ is an M -junta.

Proof. Let b, f_i, g_i, S_i be defined as in Lemma 3.4, and let S denote the union of S_1, \dots, S_{n+1} . Corollary 3.6 shows that $|S| \leq M$. Since $n \geq (M + 1)^3$, it follows that there exists an index $r \in [n + 1] \setminus S$. We will show that $f(x) = g_r(x|_{S_r})$, and so f is an M -junta.

Consider any input x to f . If $x_r = b$ then x is in the domain of f_r , and so clearly $f(x) = f_r(x) = g_r(x|_{S_r})$. Suppose therefore that $x_r = 1 - b$. Since $M + 2 \leq k \leq n + 1 - (M + 2)$, there exists a coordinate $s \in [n + 1] \setminus S$ such that $x_s = b$, putting x in the domain of f_s . Again since $M + 2 \leq k \leq n + 1 - (M + 2)$, there exists a coordinate $t \in [n + 1] \setminus (S \cup \{s\})$ such that $x_t = b$. Since $x^{(r\ t)}$ is in the domain of f_r , we have

$$f(x) = f_s(x) = g_s(x|_{S_s}) = g_s(x_{S_s}^{(r\ t)}) = f_s(x^{(r\ t)}) = f_r(x^{(r\ t)}) = g_r(x^{(r\ t)}|_{S_r}) = g_r(x|_{S_r}). \quad \square$$

3.3 Culmination

Combining Lemma 3.3 and Lemma 3.7, we obtain a version of Theorem 1.1 with a suboptimal upper bound on the size of the junta.

Lemma 3.8. *There exists a universal constant $C > 1$ such that whenever $C^d \leq k \leq n - C^d$, every Boolean degree d function on $\binom{[n]}{k}$ is a C^d -junta.*

Proof. Let K be the constant from Lemma 3.3, and let $M = K^d$. We choose $C := (K + 2)^3$.

Let us assume that $k \leq n/2$ (the proof for $k \geq n/2$ is very similar). Lemma 3.3 shows that every Boolean degree d function on $\binom{[2k]}{k}$ is an M -junta. If $m \geq 2k$ then $m \geq 2k \geq (M + 1)^3$ and $M + 2 \leq k \leq m - (M + 2)$. Therefore Lemma 3.7 shows that if every Boolean degree d function on $\binom{[m]}{k}$ is an M -junta, then the same holds for $\binom{[m+1]}{k}$. Applying the lemma $n - 2k$ times, we conclude that every Boolean degree d function on $\binom{[n]}{k}$ is a C^d -junta. \square

To complete the proof of the theorem, we show how to convert a Boolean degree d function on the slice depending on many coordinates to a Boolean degree d function on the hypercube depending on the same number of coordinates.

Lemma 3.9. *Suppose that f is a Boolean degree d function on $\binom{[n]}{k}$ which is an L -junta but not an $(L - 1)$ -junta, where $L \leq k, n - k$. Then there exists a Boolean degree d function g on $\{0, 1\}^L$ which depends on all coordinates.*

Proof. Without loss of generality, we can assume that $f(x) = g(x_1, \dots, x_L)$ for some Boolean function g on the L -dimensional hypercube. Since f is not an $(L - 1)$ -junta, the function g depends on all coordinates. Since $L \leq k, n - k$, as x goes over all points in $\binom{[n]}{k}$, the vector x_1, \dots, x_L goes over all points in $\{0, 1\}^L$. It remains to show that there is a degree d polynomial agreeing with g on $\{0, 1\}^L$.

Since f has degree at most d , there is a degree d multilinear polynomial P such that $f = P$ for every point in $\binom{[n]}{k}$. If π is any permutation of $\{L + 1, \dots, n\}$ then $f = f^\pi$, where $f^\pi(x) = f(x^\pi)$. Denoting the set of all such permutations by Π , if we define $Q := \mathbb{E}_{\pi \in \Pi}[P^\pi]$ then $f = Q$ for every point in $\binom{[n]}{k}$. The polynomial Q is a degree d polynomial which is invariant under permutations from Π .

For $a_1, \dots, a_L \in \{0, 1\}^L$ summing to $a \leq d$, let $Q_{a_1, \dots, a_L}(x_{L+1}, \dots, x_n)$ be the coefficient of $\prod_{i=1}^L x_i^{a_i}$ in Q . This is a degree $d - a$ symmetric polynomial in x_{L+1}, \dots, x_n , and so a classical result of Minsky and Papert [10] (see also [11, Lemma 3.2]) implies that there exists a degree $d - a$ univariate polynomial R_{a_1, \dots, a_L} such that $Q_{a_1, \dots, a_L}(x_{L+1}, \dots, x_n) =$

$R_{a_1, \dots, a_L}(x_{L+1} + \dots + x_n)$ for all $x_{L+1}, \dots, x_n \in \{0, 1\}^{n-L}$. Since $x_{L+1} + \dots + x_n = k - x_1 - \dots - x_L$, it follows that for inputs x in $\binom{[n]}{k}$, we have

$$f(x_1, \dots, x_n) = \sum_{\substack{a_1, \dots, a_L \in \{0, 1\}^L \\ a_1 + \dots + a_L \leq d}} \prod_{i=1}^L x_i^{a_i} \cdot R_{a_1, \dots, a_L}(k - x_1 - \dots - x_L).$$

The right-hand side is a degree d polynomial in x_1, \dots, x_L which agrees with g on $\{0, 1\}^L$. \square

Theorem 1.1 immediately follows from combining Lemma 3.8 and Lemma 3.9.

We conclude this section by proving a converse of Lemma 3.10.

Lemma 3.10. *Suppose that g is a Boolean degree d function on $\{0, 1\}^L$ depending on all coordinates. Then for all n, k satisfying $L \leq k \leq n - L$ there exists a Boolean degree d function f on $\binom{[n]}{k}$ which is an L -junta but not an $(L - 1)$ -junta.*

Proof. We define $f(x_1, \dots, x_n) = g(x_1, \dots, x_L)$. Clearly, f is an L -junta. Since g has degree at most d , there is a polynomial P which agrees with g on all points of $\{0, 1\}^L$. The same polynomial also agrees with f on all points of $\binom{[n]}{k}$, and so f also has degree at most d . It remains to show that f is not an $(L - 1)$ -junta.

Suppose, for the sake of contradiction, that f were an $(L - 1)$ -junta. Then there exists a set S of size at most $L - 1$ and a Boolean function $h: \{0, 1\}^S \rightarrow \{0, 1\}$ such that $f(x) = h(x|_S)$. Since $|S| < L$, there exists some coordinate $i \in \{1, \dots, L\} \setminus S$. Since g depends on all coordinates, there are two inputs y, z to g differing only in the i th coordinate, say $y_i = 0$ and $z_i = 1$, such that $g(y) \neq g(z)$. Since $n \geq 2L$, there exists a coordinate $j \in [n] \setminus (\{1, \dots, L\} \cup S)$. Since $L \leq k \leq n - L$, we can extend y to an input \tilde{y} to f such that $x_j = 1$. The input $\tilde{z} = \tilde{y}^{(i,j)}$ extends z . Since $i, j \notin S$, the inputs \tilde{y}, \tilde{z} agree on all coordinates in S , and so $f(\tilde{y}) = h(\tilde{y}|_S) = h(\tilde{z}|_S) = f(\tilde{z})$. On the other hand, $f(\tilde{y}) = g(y) \neq g(z) = f(\tilde{z})$. This contradiction shows that f cannot be an $(L - 1)$ -junta. \square

4 Discussion

Optimality. Lemma 3.10 shows that the size of the junta in Theorem 1.1 is optimal. However, it is not clear whether the bounds on k are optimal. The theorem fails when $k \leq d$ or $k \geq n - d$, since in these cases every function has degree d . This prompts us to define the following two related quantities:

1. $\zeta(d)$ is the minimal value such that every Boolean degree d function on $\binom{[n]}{k}$ is an $O(1)$ -junta whenever $\zeta(d) \leq k \leq n - \zeta(d)$.
2. $\xi(d)$ is the minimal value such that every Boolean degree d function on $\binom{[n]}{k}$ is a $\gamma(d)$ -junta whenever $\xi(d) \leq k \leq n - \xi(d)$.

Clearly $d < \zeta(d) \leq \xi(d)$. We can improve this to $\zeta(d) \geq \eta(d) \geq 2^{\lceil \frac{d+1}{2} \rceil}$, where $\eta(d)$ is the maximum integer such that there exists a non-constant univariate degree d polynomial P_d

satisfying $P_d(0), \dots, P_d(\eta(d) - 1) \in \{0, 1\}$. Given such a polynomial P_d , we can construct a degree d function f_d on $\binom{[n]}{k}$ which is not a junta:

$$f_d(x_1, \dots, x_n) = P_d(x_1 + \dots + x_{\lfloor n/2 \rfloor}).$$

When $k < \eta(d)$, the possible values of $x_1 + \dots + x_{\lfloor n/2 \rfloor}$ are such that f_d is Boolean. One can check that f_d is not an L -junta unless $L \geq \lfloor n/2 \rfloor$. The following polynomial shows that $\eta(d) \geq 2^{\lceil \frac{d+1}{2} \rceil}$:

$$P_d(\sigma) = \sum_{e=0}^d (-1)^e \binom{\sigma}{e},$$

where $\binom{\sigma}{0} = 1$. While this construction can be improved for specific d (for example, $\eta(7) = 9$ and $\eta(12) = 16$), the upper bound $\eta(d) \leq 2d$ shows that this kind of construction cannot give an exponential lower bound on $\zeta(d)$.

Curiously, essentially the same function appears in [3, Section 7] as an example of a degree d function on the biased hypercube which is almost Boolean but somewhat far from being constant.

Extensions. It would be interesting to extend Theorem 1.1 to other domains. In recent work [?], we explored Boolean degree 1 functions on various domains, including various association schemes and finite groups, and the multislice (consult the work for the appropriate definitions). Inspired by these results, we make the following conjectures:

1. If f is a Boolean degree d function on the Grassmann scheme then there are $O(1)$ points and hyperplanes such that $f(S)$ depends only on which of the points is contained in S , and which of the hyperplanes contain S .
2. If f is a Boolean degree d function on the multislice $M(k_1, \dots, k_m)$ for $k_1, \dots, k_m \geq \exp(d)$ then f is a $\gamma_m(d)$ -junta, where $\gamma_m(d)$ is the maximum number of coordinates that a Boolean degree d function on the Hamming scheme $H(n, m)$ can depend on. This conjecture has subsequently been proven in [7].
3. If f is a Boolean degree d function on the symmetric group then it is computed by a constant depth n -way decision tree whose internal nodes correspond to queries of the form $\pi(i)$ or $\pi^{-1}(J)$.

We leave it to the reader to show that $\gamma_m(d)$ exists. In fact, simple arguments show that $m^{d-1} \leq \gamma_m(d) \leq \gamma(\lceil \log_2 m \rceil d)$.

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