Boolean constant degree functions on the slice are juntas

Yuval Filmus*, Ferdinand Ihringer†

July 23, 2019

Abstract

We show that a Boolean degree d function on the "slice" $\binom{[n]}{k} \triangleq \{(x_1, \ldots, x_n) \in \{0,1\} : \sum_{i=1}^n x_i = k\}$ is a junta (depends on a constant number $\gamma(d)$ of coordinates), assuming that $k, n-k$ are large enough. This generalizes a classical result of Nisan and Szegedy on the hypercube $\{0,1\}^n$. Moreover, we show that the maximum number of coordinates that a Boolean degree d function can depend on is the same on the slice and on the hypercube.

1 Introduction

Nisan and Szegedy [14] showed that a Boolean degree d function on the hypercube $\{0,1\}^n$ depends on at most $d2^{d-1}$ coordinates, and described a Boolean degree d function which depends on $\Omega(2^d)$ coordinates. The upper bound has recently been improved to $O(2^d)$ by Chiarelli, P. Hatami and Saks [1], who have also improved the hidden constant in the lower bound $\Omega(2^d)$; the hidden constant in the upper bound has subsequently been improved by Wellens [16]. Let us denote the optimal bound by $\gamma(d)$. The goal of this paper is to generalize this result to the *slice* (or *level*) $\binom{[n]}{k}$ or *Johnson scheme* $J(n,k)$, which consists of all points in the hypercube having Hamming weight k:

Theorem 1.1. *There exists a constant C such that the following holds. If $C^d \leq k \leq n - C^d$ and $f \colon \binom{[n]}{k} \to \{0,1\}$ has degree d, then f depends on at most $\gamma(d)$ coordinates.*

(We explain in Section 2 what degree d means for functions on the hypercube and on the slice.)

Filmus et al. [8] proved a version of Theorem 1.1 (with a non-optimal bound on the number of points) when k/n is bounded away from $0, 1$, but their bound deteriorates as k/n gets closer to $0, 1$. We use their result (which we reproduce here, to keep the proof self-contained) to bootstrap our own inductive argument.

The case $d = 1$ is much easier. The following folklore result is proved formally in [7]:

*Technion — Israel Institute of Technology, Haifa, Israel. Taub Fellow — supported by the Taub Foundations. This research was funded by ISF grant 1337/16.

†Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium. Einstein Institute of Mathematics, Hebrew University of Jerusalem, Israel. Supported by ERC advanced grant 320924. The author is supported by a postdoctoral fellowship of the Research Foundation — Flanders (FWO).

Theorem 1.2. *If $2 \leq k \leq n-2$ and $f\colon \binom{[n]}{k} \to \{0,1\}$ has degree 1, then f depends on at most one coordinate.*

The bounds on k in this theorem are optimal, since every function on $\binom{[n]}{1}$ and on $\binom{[n]}{n-1}$ has degree 1. In contrast, the bounds on k in Theorem 1.1 are probably not optimal, an issue we discuss in Section 4.

Let us close this introduction by mentioning a recent result of Keller and Klein [11], which studies Boolean functions on $\binom{[n]}{k}$ which are ϵ-close to being degree d, where distance is measured using the squared L_2 norm. Assuming that $k \leq n/2$, their result states that if $\epsilon < (k/n)^{O(d)}$ then f is $O(\epsilon)$-close to a junta.

2 Preliminaries

In this paper, we discuss Boolean functions, which are 0,1-valued functions, on two different domains: the hypercube and the slice. We will use the notation $[n] := \{1,\ldots,n\}$. A *degree d function* (in a context in which degree is defined) is a function of degree *at most d* (over the reals).

The hypercube. The n-dimensional hypercube is the domain $\{0,1\}^n$. Every function on the hypercube can be represented uniquely as a real multilinear polynomial in the n input arguments $x_1,\ldots,x_n$. The *degree* of a function on the hypercube is the degree of this polynomial. Alternatively, the degree of a function on the hypercube is the minimum degree of a real polynomial in $x_1,\ldots,x_n$ which agrees with the function on all points of the hypercube. A function on the hypercube is an *m-junta* if it depends on at most m inputs, that is, if there exists a set I of m inputs such that $f(x) = f(y)$ as long as $x_i = y_i$ for all $i \in I$; we also say that f is an I-junta. For more information on functions on the hypercube from this perspective, consult O'Donnell's monograph [15].

The slice. Let $0 \leq k \leq n$. The *slice* $\binom{[n]}{k}$ is the subset of $\{0,1\}^n$ consisting of all vectors having Hamming weight k, or in other words, the kth level of the Boolean lattice of subsets of $[n]$. The slice appears naturally in combinatorics, coding theory, and elsewhere, and is known to algebraic combinatorialists as the *Johnson scheme $J(n,k)$*.

Every function on the slice can be represented uniquely as a real multilinear polynomial P in the n input arguments $x_1,\ldots,x_n$ of degree at most $\min(k,n-k)$ which satisfies $\sum_{i=1}^n \frac{\partial P}{\partial x_i} = 0$ (the latter condition is known as *harmonicity*). The *degree* of a function on the slice is the degree of this polynomial. Alternatively, the degree of a function on the slice is the minimum degree of a real polynomial in $x_1,\ldots,x_n$ (not necessarily multilinear or harmonic) which agrees with the function on all points of the slice.

Note that a degree d function on the slice $\binom{[n]}{k}$ is also a degree d function on the slice $\binom{[n]}{n-k}$ and vice versa. The isomorphism between $\binom{[n]}{k}$ consists of switching zeroes and ones or, in other words, of replacing each set by its complement. Due to this we can assume that $k \leq n/2$ without loss of generality whenever it is suitable for our calculations. This also explains why all our bounds on k are symmetric throughout the document (as in Theorem 1.1).

A function f on the slice is an *m-junta* if there exist a function $g\colon \{0,1\}^m \to \{0,1\}$ and m indices $i_1 < \ldots < i_m$ such that $f(x) = g(x|_{i_1,\ldots,i_m})$, where $x|_{i_1,\ldots,i_m} = x_{i_1},\ldots,x_{i_m}$.

Alternatively, f is an m-junta if there exists a set I of m coordinates such that f is invariant under permutation of the coordinates in $[n] \setminus I$; we also say that f is an I-junta. Note that the set I is not defined uniquely (in contrast to the hypercube case): for example, $f = \sum_{i \in I} x_i$ is both an I-junta and an $[n] \setminus I$-junta.

The *pth norm* of f is given by $\|f\|_p = \mathbb{E}[|f|^p]^{1/p}$, where the expectation is over a uniform point in the slice. In particular, $\|f\|_2^2 = \mathbb{E}[f^2]$.

Let f be a Boolean function on the slice $\binom{[n]}{k}$, and let P be its unique harmonic multilinear polynomial representation. The *dth level* of f, denoted $f^{=d}$, is the homogeneous degree d part of P (the sum of all degree d monomials with their coefficients). The different levels are orthogonal: $\mathbb{E}[f^{=d} f^{=e}] = 0$ if $d \neq e$. Orthogonality of the different levels implies that

$$\|f\|_2^2 = \sum_{d=0}^{\deg f} \|f^{=d}\|_2^2.$$

Let f be a Boolean function on the slice $\binom{[n]}{k}$, and let $i, j \in [n]$. We define $f^{(i\,j)}$ to be the function given by $f^{(i\,j)}(x) = f(x^{(i\,j)})$, where $x^{(i\,j)}$ is obtained from x by switching x_i and x_j. The (i,j)*th influence* of f is $\operatorname{Inf}_{ij}[f] = \frac{1}{4}\Pr[f(x) \neq f(x^{(i\,j)})]$, where x is chosen uniformly at random over the slice. An equivalent formula is $\operatorname{Inf}_{ij}[f] = \frac{1}{4}\mathbb{E}[(f - f^{(i\,j)})^2]$. Clearly $\operatorname{Inf}_{ij}[f] = 0$ if and only if $f = f^{(i\,j)}$. The *total influence* of f is $\operatorname{Inf}[f] = \frac{1}{n}\sum_{1 \le i < j \le n} \operatorname{Inf}_{ij}[f]$. It is given by the formula [5, Lemma 27]

$$\operatorname{Inf}[f] = \sum_{d=0}^{\deg f} \frac{d(n+1-d)}{n} \|f^{=d}\|_2^2. \tag{1}$$

For a parameter $\rho \in (0,1]$, the *noise operator* T_ρ, mapping functions on the slice to functions on the slice, is defined by

$$T_\rho f = \sum_{d=0}^{\deg f} \rho^{d(1-(d-1)/n)} f^{=d}. \tag{2}$$

Alternatively [5, Lemma 27], $(T_\rho f)(x)$ is the expected value of $f(y)$, where y is chosen by applying $N \sim \operatorname{Po}(\frac{n-1}{2}\log(1/\rho))$ random transpositions to x. Lee and Yau [12] proved a log Sobolev inequality, which together with classical results of Diaconis and Saloff-Coste [2] implies that the following *hypercontractive inequality* holds for some constant $C_H > 0$:

$$\|T_\rho f\|_2 \le \|f\|_{4/3}, \qquad \rho = \left(\frac{2k(n-k)}{n(n-1)}\right)^{C_H}. \tag{3}$$

For more information on functions on the slice, consult [5, 9].

3 Main theorem

For the rest of this section, we fix an integer $d \ge 1$. Our goal is to prove Theorem 1.1 for this value of d. We will use the phrase *universal constant* to refer to a constant independent of d.

The strategy of the proof is to proceed in three steps:

1. Bootstrapping: Every Boolean degree d function on $\binom{[2n]}{n}$ is a K^d-junta.
2. Induction: If every Boolean degree d function on $\binom{[n]}{k}$ is an M-junta, then the same holds for $\binom{[n+1]}{k}$ and $\binom{[n+1]}{k+1}$ (under certain conditions).
3. Culmination: If a Boolean degree d function on $\binom{[n]}{k}$ is an L-junta but not an $(L-1)$-junta, then (under certain conditions) there exists a Boolean degree d function on the hypercube depending on L coordinates.

We also show a converse to the last step: given a Boolean degree d function on the hypercube depending on L coordinates, we show how to construct Boolean degree d functions on large enough slices that are L-juntas but not $(L-1)$-juntas.

3.1 Bootstrapping

We bootstrap our approach by proving that every Boolean degree d function on $\binom{[2n]}{n}$ is a junta. The proof is a simple application of hypercontractivity, and already appears in [8]. We reproduce a simplified version here in order to make the paper self-contained.

The main idea behind the proof is to obtain a dichotomy on the influences of the function.

Lemma 3.1. *There exists a universal constant α such that all non-zero influences of a Boolean degree d function on $\binom{[2n]}{n}$ are at least α^d.*

Proof. Let f be a Boolean degree d function on $\binom{[2n]}{n}$. Given $i, j \in [n]$, consider the function $f_{ij} = (f - f^{(i\ j)})/2$, related to the (i,j)th influence of f by $\operatorname{Inf}_{ij}[f] = \|f_{ij}\|_2^2$. Since f is $0,1$-valued, f_{ij} is $0, \pm 1$-valued. Since f has degree d, it follows that f_{ij} can be written as a degree d polynomial, and so has degree at most d. Hypercontractivity (3) implies that for some universal constant ρ, we have

$$\|T_\rho f_{ij}\|_2 \le \|f_{ij}\|_{4/3}. \tag{4}$$

We can estimate the left-hand side of (4) using (2):

$$\|T_\rho f_{ij}\|_2^2 = \sum_{e=0}^{d} \rho^{2e(1-(e-1)/n)} \|f_{ij}^{=e}\|_2^2 \ge \rho^{2d} \sum_{e=0}^{d} \|f_{ij}^{=e}\|_2^2 = \rho^{2d} \|f_{ij}\|_2^2 = \rho^{2d} \operatorname{Inf}_{ij}[f].$$

We can calculate the right-hand side of (4) using the fact that f_{ij} is $0, \pm 1$-valued:

$$\|f_{ij}\|_{4/3}^{4/3} = \mathbb{E}[|f_{ij}|^{4/3}] = \mathbb{E}[|f_{ij}|^2] = \|f_{ij}\|_2^2 = \operatorname{Inf}_{ij}[f].$$

Combining the estimates on both sides of (4), we conclude that

$$\rho^{2d} \operatorname{Inf}_{ij}[f] \le \operatorname{Inf}_{ij}[f]^{3/2}.$$

Hence either $\operatorname{Inf}_{ij}[f] = 0$ or $\operatorname{Inf}_{ij}[f] \ge \rho^{4d}$. □

The next step is to prove a degenerate triangle inequality (for the non-degenerate version, see Wimmer [17, Lemma 5.4] and Filmus [5, Lemma 5.1]).

Lemma 3.2. *Let f be a function on a slice. If* $\mathrm{Inf}_{ik}[f] = \mathrm{Inf}_{jk}[f] = 0$ *then* $\mathrm{Inf}_{ij}[f] = 0$.

Proof. If $\mathrm{Inf}_{ik}[f] = \mathrm{Inf}_{jk}[f] = 0$ then $f(x) = f(x^{(i\,k)}) = f(x^{(j\,k)})$, and so $f(x) = f(x^{(i\,k)(j\,k)(i\,k)}) = f(x^{(i\,j)})$. It follows that $\mathrm{Inf}_{ij}[f] = 0$. □

To complete the proof, we use formula (1), which implies that $\mathrm{Inf}[f] \leq d$ for any Boolean degree d function f, together with an idea of Wimmer [17, Proposition 5.3].

Lemma 3.3. *There exists a universal constant $K > 1$ such that for $n \geq 2$, every Boolean degree d function on $\binom{[2n]}{n}$ is a K^d-junta.*

Proof. Let f be a Boolean degree d function on $\binom{[2n]}{n}$. Construct a graph G on the vertex set $[2n]$ by connecting two vertices i, j if $\mathrm{Inf}_{ij}[f] \geq \alpha^d$, where α is the constant from Lemma 3.1. Let M be a maximal matching in G. It is well-known that the $2|M|$ vertices of M form a vertex cover V, that is, any edge of G touches one of these vertices. Therefore if $i, j \notin V$ then $\mathrm{Inf}_{ij}[f] < \alpha^d$, and so $\mathrm{Inf}_{ij}[f] = 0$ according to Lemma 3.1. In other words, f is a V-junta. It remains to bound the size of V.

Let (i, j) be any edge of M, and let k be any other vertex. Lemma 3.2 shows that either $\mathrm{Inf}_{ik}[f] \neq 0$ or $\mathrm{Inf}_{jk}[f] \neq 0$, and so either (i, k) or (j, k) is an edge of G, according to Lemma 3.1. It follows that G contains at least $|M|(2n - 2)/2$ edges (we divided by two since some edges could be counted twice). Therefore $n\,\mathrm{Inf}[f] = \sum_{1 \leq i < j \leq n} \mathrm{Inf}_{ij}[f] \geq \alpha^d |M|(n - 1)$. On the other hand, (1) shows that

$$\mathrm{Inf}[f] = \sum_{e=0}^{d} \frac{e(n+1-e)}{n} \|f^{=e}\|_2^2 \leq d \sum_{e=0}^{d} \|f^{=e}\|_2^2 = d\|f\|_2^2 \leq d.$$

It follows that

$$|V| = 2|M| \leq 2\frac{n}{n-1} d(1/\alpha)^d \leq (8/\alpha)^d.$$

□

3.2 Induction

The heart of the proof is an inductive argument which shows that if Theorem 1.1 holds (with a non-optimal bound on the size of the junta) for the slice $\binom{[n]}{k}$, then it also holds for the slices $\binom{[n+1]}{k}$ and $\binom{[n+1]}{k+1}$, assuming that n is large enough and that k is not too close to 0 or n. Given a Boolean degree d function f on $\binom{[n+1]}{k}$ or $\binom{[n+1]}{k+1}$, the idea is to consider restrictions of f obtained by fixing one of the coordinates.

Lemma 3.4. *Suppose that every Boolean degree d function on $\binom{[n]}{k}$ is an M-junta, where $M \geq 1$.*

Let f be a Boolean degree d function on $\binom{[n+1]}{k+b}$, where $b \in \{0, 1\}$. For each $i \in [n+1]$, let f_i be the restriction of f to vectors satisfying $x_i = b$. Then for each $i \in [n+1]$ there exists a set $S_i \subseteq [n+1] \setminus \{i\}$ of size at most M and a function $g_i \colon \{0,1\}^{S_i} \to \{0, 1\}$, depending on all inputs, such that $f_i(x) = g_i(x|_{S_i})$.

Proof. Choose $i \in [n+1]$. The domain of f_i is isomorphic to $\binom{[n]}{k}$. Moreover, since f can be represented as a polynomial of degree d, so can f_i, hence $\deg f_i \leq d$. By assumption, f_i is an M-junta, and so $f_i(x) = h_i(x|_{T_i})$ for some set T_i of M indices and some function $h_i \colon \{0,1\}^{T_i} \to \{0,1\}$. Let $S_i \subseteq T_i$ be the set of inputs that h_i depends on. Then there exists a function $g_i \colon \{0,1\}^{S_i} \to \{0,1\}$ such that $g_i(x|_{S_i}) = h_i(x|_{T_i})$, completing the proof. □

Each of the sets S_i individually contains at most M indices. We now show that in fact they contain at most M indices *in total*.

Lemma 3.5. *Under the assumptions of Lemma 3.4, suppose further that $n \geq (M+1)^3$ and $M+2 \leq k \leq n+1-(M+2)$.*

The union of any $M+1$ of the sets S_i contains at most M indices.

Proof. We can assume, without loss of generality, that the sets in question are $S_1, \ldots, S_{M+1}$. Denote their union by A, and let $B = A \cup \{1, \ldots, M+1\}$. Since $|B| \leq (M+1)^2 \leq n$, there exists a point $r \in [n+1] \setminus B$. We proceed by bounding the number of unordered pairs of distinct indices $i, j \in [n+1] \setminus \{r\}$ such that $f_r = f_r^{(i\,j)}$, which we denote by N. Since f_r is an M-junta, we know that $N \geq \binom{n-M}{2}$. We will now obtain an upper bound on N in terms of $|A|$ and $|B|$.

Let $1 \leq \ell \leq M+1$, and suppose that $i \in S_\ell$ and $j \notin S_\ell \cup \{\ell, r\}$. We claim that $f_r^{(i\,j)} \neq f_r$. Indeed, since g_ℓ depends on all inputs, there are two inputs y, z to g_ℓ, differing only on the ith coordinate, say $y_i = b$ and $z_i = 1-b$, such that $g_\ell(y) \neq g_\ell(z)$. Since $M+2 \leq k \leq n+1-(M+2)$, we can extend y to an input x to f satisfying additionally the constraints $x_\ell = x_r = b$ and $x_j = 1-b$. Since $x_\ell = x_r = b$, the input x is in the common domain of f_ℓ and f_r. Notice that $f_r(x) = f_\ell(x) = g_\ell(y)$, whereas $f_r(x^{(i\,j)}) = f_\ell(x^{(i\,j)}) = g_\ell(z)$, since $x_i = y_i = b$ whereas $x_j = 1-b$. By construction $g_\ell(y) \neq g_\ell(z)$, and so $f_r \neq f_r^{(i\,j)}$.

The preceding argument shows that if $i \in A$ and $j \notin B \cup \{r\}$ then $f_r \neq f_r^{(i\,j)}$. Therefore $\binom{n}{2} - N \geq |A|(n-|B|) \geq |A|(n-(M+1)^2)$. Combining this with the lower bound $N \geq \binom{n-M}{2}$, we deduce that

$$|A|(n-(M+1)^2) \leq \binom{n}{2} - \binom{n-M}{2} = \frac{M(2n-M-1)}{2}.$$

Rearrangement shows that

$$|A| \leq \frac{M(2n-M-1)}{2(n-(M+1)^2)} = \left(1 + \frac{(M+1)(2M+1)}{2n-2(M+1)^2}\right) M.$$

When $n > (M^2 + (3/2)M + 1)(M+1)$, we have $\frac{(M+1)(2M+1)}{2n-2(M+1)^2} < \frac{1}{M}$, and so $|A| < M+1$. We conclude that when $n \geq (M+1)^3$, we have $|A| \leq M$. □

Corollary 3.6. *Under the assumptions of the preceding lemma, the union of $S_1, \ldots, S_{n+1}$ contains at most M indices.*

Proof. Suppose that the union contained at least $M+1$ indices $i_1, \ldots, i_{M+1}$. Each index i_t is contained in some set S_{j_t}, and in particular the union of $S_{j_1}, \ldots, S_{j_{M+1}}$ contains at least $M+1$ indices, contradicting the lemma. □

Denoting the union of all S_i by S, it remains to show that f is an S-junta.

Lemma 3.7. *Suppose that every Boolean degree d function on $\binom{[n]}{k}$ is an M-junta, where $M \geq 1$; that $n \geq (M+1)^3$; and that $M+2 \leq k \leq n+1-(M+2)$.*

Any Boolean degree d function on $\binom{[n+1]}{k}$ or on $\binom{[n+1]}{k+1}$ is an M-junta.

Proof. Let b, f_i, g_i, S_i be defined as in Lemma 3.4, and let S denote the union of $S_1, \ldots, S_{n+1}$. Corollary 3.6 shows that $|S| \leq M$. Since $n \geq (M+1)^3$, it follows that there exists an index $r \in [n+1] \setminus S$. We will show that $f(x) = g_r(x|_{S_r})$, and so f is an M-junta.

Consider any input x to f. If $x_r = b$ then x is in the domain of f_r, and so clearly $f(x) = f_r(x) = g_r(x|_{S_r})$. Suppose therefore that $x_r = 1 - b$. Since $M + 2 \leq k \leq n + 1 - (M+2)$, there exists a coordinate $s \in [n+1] \setminus S$ such that $x_s = b$, putting x in the domain of f_s. Again since $M + 2 \leq k \leq n + 1 - (M+2)$, there exists a coordinate $t \in [n+1] \setminus (S \cup \{s\})$ such that $x_t = b$. Since $x^{(r\ t)}$ is in the domain of f_r, we have

$$f(x) = f_s(x) = g_s(x|_{S_s}) = g_s(x^{(r\ t)}_{S_s}) = f_s(x^{(r\ t)}) = f_r(x^{(r\ t)}) = g_r(x^{(r\ t)}|_{S_r}) = g_r(x|_{S_r}). \quad \square$$

3.3 Culmination

Combining Lemma 3.3 and Lemma 3.7, we obtain a version of Theorem 1.1 with a suboptimal upper bound on the size of the junta.

Lemma 3.8. *There exists a universal constant $C > 1$ such that whenever $C^d \leq k \leq n - C^d$, every Boolean degree d function on $\binom{[n]}{k}$ is a C^d-junta.*

Proof. Let K be the constant from Lemma 3.3, and let $M = K^d$. We choose $C := (K+2)^3$.

Let us assume that $k \leq n/2$ (the proof for $k \geq n/2$ is very similar). Lemma 3.3 shows that every Boolean degree d function on $\binom{[2k]}{k}$ is an M-junta. If $m \geq 2k$ then $m \geq 2k \geq (M+1)^3$ and $M + 2 \leq k \leq m - (M+2)$. Therefore Lemma 3.7 shows that if every Boolean degree d function on $\binom{[m]}{k}$ is an M-junta, then the same holds for $\binom{[m+1]}{k}$. Applying the lemma $n - 2k$ times, we conclude that every Boolean degree d function on $\binom{[n]}{k}$ is a C^d-junta. $\square$

To complete the proof of the theorem, we show how to convert a Boolean degree d function on the slice depending on many coordinates to a Boolean degree d function on the hypercube depending on the same number of coordinates.

Lemma 3.9. *Suppose that f is a Boolean degree d function on $\binom{[n]}{k}$ which is an L-junta but not an $(L-1)$-junta, where $L \leq k, n-k$. Then there exists a Boolean degree d function g on $\{0,1\}^L$ which depends on all coordinates.*

Proof. Without loss of generality, we can assume that $f(x) = g(x_1, \ldots, x_L)$ for some Boolean function g on the L-dimensional hypercube. Since f is not an $(L-1)$-junta, the function g depends on all coordinates. Since $L \leq k, n-k$, as x goes over all points in $\binom{[n]}{k}$, the vector $x_1, \ldots, x_L$ goes over all points in $\{0,1\}^L$. It remains to show that there is a degree d polynomial agreeing with g on $\{0,1\}^L$.

Since f has degree at most d, there is a degree d multilinear polynomial P such that $f = P$ for every point in $\binom{[n]}{k}$. If π is any permutation of $\{L+1, \ldots, n\}$ then $f = f^\pi$, where $f^\pi(x) = f(x^\pi)$. Denoting the set of all such permutations by Π, if we define $Q := \mathbb{E}_{\pi \in \Pi}[P^\pi]$ then $f = Q$ for every point in $\binom{[n]}{k}$. The polynomial Q is a degree d polynomial which is invariant under permutations from Π.

For $a_1, \ldots, a_L \in \{0,1\}^L$ summing to $a \leq d$, let $Q_{a_1,\ldots,a_L}(x_{L+1}, \ldots, x_n)$ be the coefficient of $\prod_{i=1}^L x_i^{a_i}$ in Q. This is a degree $d - a$ symmetric polynomial in $x_{L+1}, \ldots, x_n$, and so a classical result of Minsky and Papert [13] (see also [14, Lemma 3.2]) implies that there exists a degree $d-a$ univariate polynomial $R_{a_1,\ldots,a_L}$ such that $Q_{a_1,\ldots,a_L}(x_{L+1},\ldots,x_n) = R_{a_1,\ldots,a_L}(x_{L+1}+\cdots+x_n)$ for all $x_{L+1},\ldots,x_n \in \{0,1\}^{n-L}$. Since $x_{L+1}+\cdots+x_n = k - x_1 - \cdots - x_L$, it follows that for inputs x in $\binom{[n]}{k}$, we have

$$f(x_1,\ldots,x_n) = \sum_{\substack{a_1,\ldots,a_L \in \{0,1\}^L \\ a_1+\cdots+a_L \leq d}} \prod_{i=1}^{L} x_i^{a_i} \cdot R_{a_1,\ldots,a_L}(k - x_1 - \cdots - x_L).$$

The right-hand side is a degree d polynomial in $x_1,\ldots,x_L$ which agrees with g on $\{0,1\}^L$. □

Theorem 1.1 immediately follows from combining Lemma 3.8 and Lemma 3.9.

We conclude this section by proving a converse of Lemma 3.10.

Lemma 3.10. *Suppose that g is a Boolean degree d function on $\{0,1\}^L$ depending on all coordinates. Then for all n,k satisfying $L \leq k \leq n-L$ there exists a Boolean degree d function f on $\binom{[n]}{k}$ which is an L-junta but not an $(L-1)$-junta.*

Proof. We define $f(x_1,\ldots,x_n) = g(x_1,\ldots,x_L)$. Clearly, f is an L-junta. Since g has degree at most d, there is a polynomial P which agrees with g on all points of $\{0,1\}^L$. The same polynomial also agrees with f on all points of $\binom{[n]}{k}$, and so f also has degree at most d. It remains to show that f is not an $(L-1)$-junta.

Suppose, for the sake of contradiction, that f were an $(L-1)$-junta. Then there exists a set S of size at most $L-1$ and a Boolean function $h\colon \{0,1\}^S \to \{0,1\}$ such that $f(x) = h(x|_S)$. Since $|S| < L$, there exists some coordinate $i \in \{1,\ldots,L\} \setminus S$. Since g depends on all coordinates, there are two inputs y,z to g differing only in the ith coordinate, say $y_i = 0$ and $z_i = 1$, such that $g(y) \neq g(z)$. Since $n \geq 2L$, there exists a coordinate $j \in [n] \setminus (\{1,\ldots,L\} \cup S)$. Since $L \leq k \leq n-L$, we can extend y to an input $\tilde{y}$ to f such that $x_j = 1$. The input $\tilde{z} = \tilde{y}^{(i\,j)}$ extends z. Since $i,j \notin S$, the inputs $\tilde{y},\tilde{z}$ agree on all coordinates in S, and so $f(\tilde{y}) = h(\tilde{y}|_S) = h(\tilde{z}|_S) = f(\tilde{z})$. On the other hand, $f(\tilde{y}) = g(y) \neq g(z) = f(\tilde{z})$. This contradiction shows that f cannot be an $(L-1)$-junta. □

4 Discussion

Optimality. Lemma 3.10 shows that the size of the junta in Theorem 1.1 is optimal. However, it is not clear whether the bounds on k are optimal. The theorem fails when $k \leq d$ or $k \geq n-d$, since in these cases every function has degree d. This prompts us to define the following two related quantities:

1. $\zeta(d)$ is the minimal value such that every Boolean degree d function on $\binom{[n]}{k}$ is an $O_d(1)$-junta whenever $\zeta(d) \leq k \leq n - \zeta(d)$.

2. $\xi(d)$ is the minimal value such that every Boolean degree d function on $\binom{[n]}{k}$ is a $\gamma(d)$-junta whenever $\xi(d) \leq k \leq n - \xi(d)$.

Clearly $d < \zeta(d) \le \xi(d)$. We can improve this to $\zeta(d) \ge \eta(d) \ge 2\lceil \frac{d+1}{2} \rceil$, where $\eta(d)$ is the maximum integer such that there exists a non-constant univariate degree d polynomial P_d satisfying $P_d(0), \ldots, P_d(\eta(d)-1) \in \{0,1\}$. Given such a polynomial P_d, we can construct a degree d function f_d on $\binom{[n]}{k}$ which is not a junta:

$$f_d(x_1, \ldots, x_n) = P_d(x_1 + \cdots + x_{\lfloor n/2 \rfloor}).$$

When $k < \eta(d)$, the possible values of $x_1 + \cdots + x_{\lfloor n/2 \rfloor}$ are such that f_d is Boolean. One can check that f_d is not an L-junta unless $L \ge \lfloor n/2 \rfloor$. The following polynomial shows that $\eta(d) \ge 2\lceil \frac{d+1}{2} \rceil$:

$$P_d(\sigma) = \sum_{e=0}^{d} (-1)^e \binom{\sigma}{e},$$

where $\binom{\sigma}{0} = 1$. While this construction can be improved for specific d (for example, $\eta(7) = 9$ and $\eta(12) = 16$), the upper bound $\eta(d) \le 2d$ shows that this kind of construction cannot give an exponential lower bound on $\zeta(d)$.

Curiously, essentially the same function appears in [3, Section 7] as an example of a degree d function on the biased hypercube which is almost Boolean but somewhat far from being constant.

Extensions. It would be interesting to extend Theorem 1.1 to other domains. In recent work [7], we explored Boolean degree 1 functions on various domains, including various association schemes and finite groups, and the multislice (consult the work for the appropriate definitions). Inspired by these results, we make the following conjectures:

1. If f is a Boolean degree d function on the Grassmann scheme then there are $O(1)$ points and hyperplanes such that $f(S)$ depends only on which of the points is contained in S, and which of the hyperplanes contain S. The case $d = 1$ has been proved in [7] for $q = 2, 3, 4, 5$.

2. If f is a Boolean degree d function on the multislice $M(k_1, \ldots, k_m)$ for $k_1, \ldots, k_m \ge \exp(d)$ then f is a $\gamma_m(d)$-junta, where $\gamma_m(d)$ is the maximum number of coordinates that a Boolean degree d function on the Hamming scheme $H(n, m)$ can depend on. This conjecture has subsequently been proved in [10], by closely following the argument in this paper.

3. If f is a Boolean degree d function on the symmetric group then it is computed by a constant depth n-way decision tree whose internal nodes correspond to queries of the form $\pi(i)$ or $\pi^{-1}(j)$. The case $d = 1$ has been proved by Ellis, Friedgut and Pilpel [4].[1]

We leave it to the reader to show that $\gamma_m(d)$ exists. In fact, simple arguments show that $m^{d-1} \le \gamma_m(d) \le \gamma(\lceil \log_2 m \rceil d)$.

[1]Theorem 27 of [4] would imply this conjecture, but unfortunately the result is false [6].

References

[1] John Chiarelli, Pooya Hatami, and Michael Saks. Tight bound on the number of relevant variables in a bounded degree Boolean function. *CoRR*, abs/1801.08564, 2018.

[2] Persi Diaconis and Laurent Saloff-Coste. Logarithmic Sobolev inequalities for finite Markov chains. *Ann. Appl. Probab.*, 6(3):695–750, Aug 1996.

[3] Irit Dinur, Yuval Filmus, and Prahladh Harsha. Low degree almost Boolean functions are sparse juntas. *CoRR*, abs/1711.09428, 2017.

[4] David Ellis, Ehud Friedgut, and Haran Pilpel. Intersecting families of permutations. *J. Amer. Math. Soc.*, 24:649–682, 2011.

[5] Yuval Filmus. Orthogonal basis for functions over a slice of the Boolean hypercube. *Electronic J. Comb.*, 23(1):P1.23, 2016.

[6] Yuval Filmus. A comment on Intersecting Families of Permutations. *CoRR*, abs/1706.10146, 2017.

[7] Yuval Filmus and Ferdinand Ihringer. Boolean degree 1 functions on some classical association schemes. *Journal of Combinatorial Theory, Series A*, 162:241–279, February 2019.

[8] Yuval Filmus, Guy Kindler, Elchanan Mossel, and Karl Wimmer. Invariance principle on the slice. In *31st Computational Complexity Conference*, 2016.

[9] Yuval Filmus and Elchanan Mossel. Harmonicity and invariance on slices of the Boolean cube. *Probability Theory and Related Fields*, to appear.

[10] Yuval Filmus, Ryan O'Donnell, and Xinyu Wu. A log-Sobolev inequality for the multislice, with applications. In *Proceedings of the 10th Innovations in Theoretical Computer Science conference (ITCS'19)*, 2019.

[11] Nathan Keller and Ohad Klein. Kindler–Safra theorem for the slice. Manuscript, 2017.

[12] Tzong-Yau Lee and Horng-Tzer Yau. Logarithmic Sobolev inequality for some models of random walks. *Ann. Prob.*, 26(4):1855–1873, 1998.

[13] Marvin L. Minsky and Seymour A. Papert. *Perceptrons: Expanded Edition*. MIT Press, Cambridge, MA, USA, 1988.

[14] Noam Nisan and Mario Szegedy. On the degree of Boolean functions as real polynomials. *Comp. Comp.*, 4(4):301–313, 1994.

[15] Ryan O'Donnell. *Analysis of Boolean functions*. Cambridge University Press, 2014.

[16] Jake Wellens. A tighter bound on the number of relevant variables in a bounded degree Boolean function. *CoRR*, abs/1903.08214, 2019.

[17] Karl Wimmer. Low influence functions over slices of the Boolean hypercube depend on few coordinates. In *CCC*, 2014.