Thirteenth proof of a result about tiling a rectangle

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We paraphrase the 13th proof of Stan Wagon's well-known "Fourteen proofs of a result about tiling a rectangle".

Define an *integral rectangle* as a rectangle whose sides are parallel to the axes, and at least one of its sides has integral length. The result is as follows. If a rectangle can be tiled by integral rectangles, then it is itself integral.

Here's the proof. Denote by $\{x\}$ the fractional part of a number, i.e. $\{x\} = x - \lfloor x \rfloor$. For a rectangle R with corners $(x_1, y_1), (x_2, y_2)$, define its dummy area $\alpha(R)$ as $(\{x_2\} - \{x_1\})(\{y_2\} - \{y_1\})$. Notice that a rectangle is integral iff its dummy area is zero.

Suppose that a grid divides a rectangle R into rectangles $G_{i,j}$. It's easy to see that $\alpha(R) = \sum_{i,j} \alpha(G_{i,j})$. Now suppose that a rectangle R is tiled by rectangles R_i. Extend all sides of all rectangles to form a grid $G_{j,k}$. Then

$$\alpha(R) = \sum_{j,k} \alpha(G_{j,k}) = \sum_i \alpha(R_i).$$

If all rectangles R_i are integral then $\alpha(R_i) = 0$ and so $\alpha(R) = 0$, and R is integral.

The proof can be easily extended to show that if a k-dimensional box is tiled by boxes, each of which having at least l integral sides, then so does the large box.