Thirteenth proof of a result about tiling a rectangle

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We paraphrase the 13th proof of Stan Wagon’s well-known “Fourteen proofs of a result about tiling a rectangle”.

Define an \textit{integral rectangle} as a rectangle whose sides are parallel to the axes, and at least one of its sides has integral length. The result is as follows. If a rectangle can be tiled by integral rectangles, then it is itself integral.

Here’s the proof. Denote by \{x\} the fractional part of a number, i.e. \{x\} = x - \lfloor x \rfloor. For a rectangle \( R \) with corners \((x_1, y_1), (x_2, y_2)\), define its dummy area \( \alpha(R) \) as \((\{x_2\} - \{x_1\})(\{y_2\} - \{y_1\})\). Notice that a rectangle is integral iff its dummy area is zero.

Suppose that a grid divides a rectangle \( R \) into rectangles \( G_{i,j} \). It’s easy to see that \( \alpha(R) = \sum_{i,j} \alpha(G_{i,j}) \). Now suppose that a rectangle \( R \) is tiled by rectangles \( R_i \). Extend all sides of all rectangles to form a grid \( G_{j,k} \). Then

\[
\alpha(R) = \sum_{j,k} \alpha(G_{j,k}) = \sum_i \alpha(R_i).
\]

If all rectangles \( R_i \) are integral then \( \alpha(R_i) = 0 \) and so \( \alpha(R) = 0 \), and \( R \) is integral.

The proof can be easily extended to show that if a \( k \)-dimensional box is tiled by boxes, each of which having at least \( l \) integral sides, then so does the large box.