Research Statement

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1 Summary

My research is split between two areas, combinatorics and theoretical computer science. My interest in combinatorics lies in two related fields: extremal combinatorics and analysis of Boolean functions. In extremal combinatorics, my research has focused on Erdős–Ko–Rado-type intersection theorems, culminating in the proof of the 30 year old conjecture of Simonovits and Sós on triangle-intersecting families, joint work with David Ellis and Ehud Friedgut.

Analysis of Boolean functions, lying at the interface of combinatorics, probability theory, analysis and theoretical computer science, studies 0/1-valued functions using their Fourier–Walsh expansion. Recently it has become an essential tool in theoretical computer science, used in such diverse areas as hardness of approximation, learning theory, property testing, and many others. My research in this area has focused on structural results and applications to extremal combinatorics, including basic structural results for Boolean functions on the symmetric group.

My contributions to theoretical computer science span several areas: circuit complexity, proof complexity, approximation algorithms, and social choice theory. Recent highlights include a strong separation of the monotone NC hierarchy, a combinatorial optimal approximation algorithm for the problem of maximizing a submodular function over a matroid, and limitations of current techniques for fast matrix multiplication.

2 Combinatorics

My research in this field has focused on two major directions: extremal combinatorics and structure theorems. The two directions are strongly related: structure theorems are used to prove stability results in extremal combinatorics.

Triangle-intersecting families A classical result in extremal combinatorics is the fundamental theorem of Erdős, Ko and Rado. It states that a k-uniform family of subsets of $\{1, \ldots, n\}$, in which any two sets intersect, contains at most $\binom{n-1}{k-1}$ sets whenever $k < n/2$; and furthermore, this is achieved only for the so-called *dictators* consisting of all sets containing some point $i \in \{1, \ldots, n\}$. This theorem has led to many generalizations and conjectured generalizations. One of these, which has puzzled combinatorialists for decades, is the Simonovits–Sós conjecture: a family of graphs on n vertices (alternatively, a family of subsets of the complete graph K_n), any two of which contain a triangle in common, can contain at most $2^{\binom{n}{2}-3}$ graphs. Using a method of Friedgut [Fri08] essentially equivalent to computing the Lovász theta function, Ellis, Friedgut and myself succeeded in proving the Simonovits–Sós conjecture [EFF12]. Since our proof uses spectral methods, it has the added advantage of a stability result: not only are the only extremal families triangle-juntas (all graphs containing a fixed triangle), but moreover, families containing $(1-\epsilon)2^{\binom{n}{2}-3}$ graphs are *close* to triangle-juntas.

Structure theorems Friedgut, Kalai and Naor have proved the following classical structure theorem [FKN02]. Consider a function $\{0,1\}^n \to \{0,1\}$ which is close in L2 distance to an affine combination of its inputs. The

FKN theorem states that such a function is close to a Boolean function depending on at most one input. A generalization of this theorem due to Kindler and Safra [KS02] is instrumental in proving the stability version of the Simonovits–Sós conjecture, alluded to above. Together with Ellis and Friedgut, we have generalized the FKN theorem and the Kindler–Safra theorem to functions on the symmetric group [EFF14, EFF13, EFF]. Recently, I have generalized the FKN theorem to functions on the set $\{x \in \{0,1\}^n : \sum_i x_i = k\}$, known as a *slice* of the Boolean cube [Fil14a]. These theorems can be used to prove stability versions of the classical Erdős–Ko–Rado theorem as well as its generalization (due to Ellis, Friedgut and Pilpel [EFP11]) to intersecting families of permutations.

3 Theoretical computer science

My contributions to theoretical computer science are in diverse areas. My work in circuit complexity has studied comparator circuits [CFL14] and monotone switching networks [FPRC13]. My work in proof complexity has concentrated on space complexity [FLN$^+$12, FLM$^+$13, FLM$^+$14], with earlier work on simulating strong proof systems by weaker ones [FPS11]. Together with Justin Ward I have designed an optimal approximation algorithm for a fundamental optimization problem, monotone submodular maximization over a matroid [FW12, FW14]. My work in social choice theory, an area at the intersection of economics and game theory, includes models for the spread of influence in social networks [BFO10] and results on positional scoring rules in partial information scenarios [BFO13, FO14].

Below I describe three recent highlights of my work, belonging to three different subareas: approximation algorithms, circuit complexity, and algebraic complexity.

Fast matrix multiplication How fast can we multiply two $n \times n$ matrices? The high school algorithm for matrix multiplication uses $O(n^3)$ arithmetic operations. Unexpectedly, in 1969 Strassen described an algorithm using only $O(n^{\log_2 7})$ arithmetic operations. Ever since, mathematicians and computer scientists alike have been interested in the arithmetic complexity of matrix multiplication, and specifically in the minimal ω such that for all $\epsilon > 0$, two $n \times n$ matrices can be multiplied using $O(n^{\omega+\epsilon})$ arithmetic operations. The fastest algorithms known [DS13, VW12, LG14] all rely on a sophisticated method due to Coppersmith and Winograd [CW90]. Coppersmith and Winograd described an identity T which can be used to bound the matrix multiplication constant ω. Analyzing the tensor square of T, they obtained even better results. Since then, higher powers of T have been analyzed, leading to ever better bounds, culminating in Le Gall's recent bound $\omega < 2.3728639$. Numerical evidence suggests that taking higher powers improves the bound only slightly. Together with Ambainis and Le Gall, we have recently shown that these methods cannot lead to a bound better than $\omega < 2.3725$ [AFLG14]. We also identify a broader class of methods based on T, and show that even these cannot yield a bound better than $\omega < 2.31$. This suggests that research should be focused on finding new identites rather than analyzing higher powers.

Monotone submodular maximization over a matroid Monotone submodular maximization over a matroid is a vast generalization of two classical optimization problems, MaxCover and MaxSAT. Given a monotone submodular function f and a matroid $\mathfrak{m}$, the goal is to find a base $B \in \mathfrak{m}$ which maximizes $f(B)$. The greedy algorithm, which is optimal in terms of approximation ratio for MaxCover, is no longer optimal for the more general problems. The problem has attracted a lot of attention recently, culminating in the continuous greedy algorithm [CCPV11] which attains the optimal approximation ratio $1 - 1/e$. This algorithm first solves a continuous relaxaion of the original problem using the Frank–Wolfe algorithm, and then uses sophisticated rounding techniques to obtain an integral solution. Together with Justin Ward, we have presented a much simpler algorithm based on the elementary but neglected technique of non-oblivious local search [FW14]. Our algorithm has a particularly intuitive form when the objective function is a coverage function [FW12], a setting which includes both MaxCover and MaxSAT.

Monotone switching networks Monotone switching networks are models used to study monotone circuit depth: roughly speaking, the size of a monotone switching network is related exponentially to the depth of the corresponding monotone circuit. In their classical paper, Raz and McKenzie [RM99] proved that the monotone NC hierarchy is strict, that is, some functions computable by polynomial size monotone circuits of depth $O(\log^{k+1} n)$ can only be computed by exponential size monotone circuits if we restrict the depth to $O(\log^k n)$. Recently, Chan and Potechin [CP12] used a completely different and beautiful technique to reprove this result using monotone switching networks. At the same time simplifying and significantly strengthening the latter technique, together with Pitassi, Robere and Cook we exhibit a function f_{k+1} computable by polynomial size monotone circuits of depth $O(\log^{k+1} n)$ which cannot be computed by subexponential size monotone circuits of depth $O(\log^k n)$ even approximately: any such circuit agrees with f_k on at most a $1/2+O(n^{-1/3})$ fraction of the inputs. This is the first average case lower bound for functions in monotone P.

4 Future research

My research interests are wide, and I have many ideas for future research. The following three are some of the directions that I currently find the most interesting.

Fast matrix multiplication My recent work with Ambainis and Le Gall on matrix multiplication shows that analyzing further powers of the Coppersmith–Winograd identity is a dead end. Past progress in the area has always involved a combination of new identities and new techniques for analyzing them. This suggests two avenues of progress. The first is an analysis based on a new framework, described in our recent work. Even though such an analysis cannot prove that $\omega = 2$ (we show in the paper that it cannot prove a bound better than $\omega < 2.31$), it could be useful in analyzing future identities. The second is coming up with novel identities, which have the potential of significantly improving the bound on ω. Since Coppersmith and Winograd's work (published in a conference already in 1987) no new identities have surfaced, and I believe that it is prudent to concentrate effort in this direction.

Analysis of Boolean functions in non-product domains Most work in the field of analysis of Booleean functions has concentrated on functions on product domains such as the Boolean cube $\{0,1\}^n$. However, in many combinatorial applications and in some applications to theoretical computer science, functions with more complicated domains need to be analyzed. Such domains include the symmetric group, a slice of the Boolean cube (consisting of all vectors in the Boolean cube with a given Hamming weight), and the set of all dimension-k subspaces of a dimension-n vector space over a finite field, which is the q-analog of the slice. Recently some of the classical theorems in the field have been extended to such domains: Ellis, Friedgut and myself have extended the Friedgut–Kalai–Naor theorem and the Kindler–Safra theorem to functions on the symmetric group [EFF14, EFF13, EFF], and several results have been extended to the slice: the Kahn–Kalai–Linial theorem [OW09], Friedgut's junta theorem [Wim14, Fil14b], and the Friedgut–Kalai–Naor theorem [Fil14a]. I find this direction of research fascinating, and intend to take an active role in developing it further: extending other results such as the invariance principle and its many corollaries to the slice, and addressing other domains such as the q-analog of the slice.

Beyond the Lovász theta function Lovász, in his seminal paper on his theta function, was the first to apply spectral methods to questions in extremal combinatorics of the Erdős–Ko–Rado type. His methods were taken up by Wilson [Wil84], Frankl and Wilson [FW86], Friedgut [Fri08], Ellis, Friedgut and Pilpel [EFP11], and Ellis, Friedgut and myself [EFF12] in many different settings. However, this method has two shortcomings. First, it can only handle questions in which the intersection of only two sets at a time is being considered. Second, the method doesn't always yield optimal bounds, and for this reason it can't be used to prove Ahlswede and Khachatrian's vast generalization of the Erdős–Ko–Rado theorem [AK97]. Addressing these shortcomings, I intent to look into the nascent spectral theory of tensors, which can be used to handle intersections of larger tuples of sets; and to study semidefinite hierarchies such as the Parrilo hierarchy in order to obtain tight bounds in cases where the Lovász bound is lax.

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