Triangle-Intersecting Families of Graphs

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Outline

Background

Fourier Analysis

Friedgut's Method

Constructing A

Extremal Combinatorics, EKR-style

- What is largest intersecting family of k-subsets of $[n]$? $(k \leq n/2)$
- Erdős, Ko, Rado (1961): Sunflower, relative size k/n
- Many generalizations

Triangle-Intersecting Families

- What is largest family of triangle-intersecting graphs?
- Simonovits, Sós (1976) conjectured: Sunflower, relative size $1/8$
- Chung, Graham, Frankl, Shearer (1986): upper bound $1/4$

Proof Ingredients

- Fourier analysis
- Hoffman's bound (Friedgut's method)
- Some graph theory

Outline

Fourier Analysis on $\mathbb{Z}_2^m$

- $f \colon \mathbb{Z}_2^m \longrightarrow \mathbb{R}$
- Fourier expansion: $f(x) = \sum_{S \subset [m]} \hat{f}(S) \chi_S(x)$
- Fourier character: $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$

Fourier Analysis: Examples

- $\chi_{\emptyset}(\ldots) = 1$
- $\chi_{\{1\}}(0, \ldots) = 1$, $\chi_{\{1\}}(1, \ldots) = -1$
- If $f(x_1, \ldots, x_m) = x_i$ then

$$f = \frac{1}{2}\chi_{\emptyset} + \frac{1}{2}\chi_{\{i\}}$$

- If $g(x_1, \ldots, x_m) = x_i \wedge x_j$ then

$$g = \frac{1}{4}\chi_{\emptyset} - \frac{1}{4}\chi_{\{i\}} - \frac{1}{4}\chi_{\{j\}} + \frac{1}{4}\chi_{\{i,j\}}$$

Fundamental Properties

- Recall $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$
- Fourier characters form orthonormal basis wrt $\langle f, g \rangle = \mathbb{E}_x f(x) g(x)$
  - Fourier transform: $\hat{f}(S) = \langle f, \chi_S \rangle$
  - Parseval: $\langle f, g \rangle = \sum_S \hat{f}(S) \hat{g}(S)$
- $\chi_\varnothing$ is constant 1 so $\hat{f}(\varnothing) = \mathbb{E}_x f(x)$
- f boolean implies $f^2 = f$, so by Parseval

$$\sum_S \hat{f}(S)^2 = \mathbb{E}_x f(x)$$

Why Use Fourier Transform?

- $f\colon \mathbb{Z}_2^{\binom{n}{2}} \to \{0,1\}$: characteristic function of family of graphs on n vertices
- $\hat{f}(\emptyset) = \sum_S \hat{f}(S)^2 = \mathbb{E}_x f(x)$ is relative size
- Sunflowers have simple expansions
- Problem: express being triangle-intersecting in a useful way

Outline

Friedgut's Method

Developed by Friedgut following Hoffman (1969)

- $\mathcal{F}$ is disjoint from co-bipartite "shifts" $\mathcal{F} \triangle \overline{H}$
- Shifts are linear, ev's are Fourier characters
- Combine shifts to a linear operator A with nice eigenvalues
- Apply Hoffman's bound

Step 1

Lemma

$\mathcal{F}$ triangle-intersecting, H bipartite $\Longrightarrow$ $\mathcal{F}$ disjoint from $\mathcal{F} \triangle \overline{H}$.

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$\mathcal{F}$ triangle-intersecting, H bipartite $\Longrightarrow$ $\mathcal{F}$ disjoint from $\mathcal{F} \triangle \overline{H}$.

Proof.

$$G \cap (G \triangle \overline{H}) \subset \overline{G \triangle (G \triangle \overline{H})} = H.$$

□

Step 2

Lemma

For some linear operator S_H,

$$\mathcal{G} = \mathcal{F} \triangle \overline{H} \Rightarrow g = S_H f.$$

Also, $S_H \chi_K = (-1)^{|K \cap \overline{H}|} \chi_K$.

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Proof.

$$(S_H \chi_K)(x) = \chi_K(x \triangle \overline{H}) = \chi_K(x) \chi_K(\overline{H}).$$

□

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$$(S_H \chi_K)(x) = \chi_K(x \triangle \overline{H}) = \chi_K(x)\chi_K(\overline{H}).$$

□

If H bipartite & f triangle-intersecting, $\langle f, S_H f \rangle = 0$.

Step 3

Ellis function $q_i(G)$ is probability that
a random bipartition cuts *exactly* i edges of G.

Lemma

$\exists$ *linear combination of co-bipartite shifts* Q_i
s.t. $Q_i \chi_G = (-1)^{|G|} q_i(G) \chi_G$.

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Proof.

- For each bipartition B, construct $Q_{i,B}$.
- Q_i is convex combination of $Q_{i,B}$.

$\square$

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A is some linear combination of Q_i.

Step 4

Lemma (Hoffman's Bound)

Suppose $A\chi_S = \lambda_S \chi_S$, $\lambda_\emptyset = 1$, $\lambda_S \geq \frac{-\mu}{1-\mu}$.
If $\langle f, Af \rangle = 0$ *then* $\mathbb{E}_x f(x) \leq \mu$.

Step 4

Lemma (Hoffman's Bound)

Suppose $A\chi_S = \lambda_S \chi_S$, $\lambda_\varnothing = 1$, $\lambda_S \geq \frac{-\mu}{1-\mu}$.
If $\langle f, Af \rangle = 0$ *then* $\mathbb{E}_x f(x) \leq \mu$.

Proof.

$$0 = \langle f, Af \rangle = \sum_S \lambda_S \hat{f}(S)^2.$$

Algebra. □

Putting It Together

Theorem

If $\mathcal{F}$ is triangle-intersecting then $|\mathcal{F}| \leq 1/8$.

Proof.

Let f be characteristic function of $\mathcal{F}$.
Construct linear combination of shifts A satisfying $\lambda_\varnothing = 1$ and for all G, $\lambda_G \geq -\frac{1}{7}$.
We have $\langle f, Af \rangle = 0$.
Hoffman's bound implies $\mathbb{E}_x f(x) \leq 1/8$. □

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Some Experimentation

	q_0	q_1	q_2	q_3	q_4
$\varnothing$	1				
$-$	$\frac{1}{2}$	$\frac{1}{2}$			
$\wedge$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$		
$\triangle$	$\frac{1}{4}$	0	$\frac{3}{4}$		
$\wedge\ \wedge$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$\boxslash$	$\frac{1}{8}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$

Implications

- Looking for $A = \sum_{i=0}^{4} c_i Q_i$, $c_0 = 1$
- Need $(-1)^{|G|} \sum_{i=0}^{4} c_i q_i(G) \geq -\frac{1}{7}$
- Constraints must be tight for $-, \wedge, \triangle$
- Table determines $c_1, c_2, 4c_3 + c_4$:

$$A = Q_0 - \frac{5}{7} Q_1 - \frac{1}{7} Q_2 + \frac{3}{28} Q_3.$$

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- Constraints must be tight for $-, \wedge, \triangle$
- Table determines $c_1, c_2, 4c_3 + c_4$:

$$A = Q_0 - \frac{5}{7} Q_1 - \frac{1}{7} Q_2 + \frac{3}{28} Q_3.$$

- Have to show that for all G, $\lambda_G \geq -\frac{1}{7}$, i.e. $(-1)^{|G|}(q_0(G) - \frac{5}{7} q_1(G) - \frac{1}{7} q_2 + \frac{3}{28} q_3(G)) \geq -\frac{1}{7}$.

Cut Statistics

Let $\mathfrak{Q}_G(t) = \sum_{i=0}^{\infty} q_i(G)t^i$.
Block: bridge or biconnected component.

Lemma

If G decomposes into blocks $G_1, \ldots, G_\ell$ then

$$\mathfrak{Q}_G = \prod_{j=1}^{\ell} \mathfrak{Q}_{G_\ell}.$$

Some Graph Theory

Lemma

- $q_0(G) = 2^{cc(G)-v(G)}$.
- $q_1(G) = br(G)q_0(G)$.
- $q_k(G) \leq 1/2$ *if G has odd-degree vertex.*
- $q_k(G) \leq 1/2$ *for odd* k.
- $q_2(G) \leq 3/4$.

Proof that A works

Theorem

$A\chi_G = \lambda_G \chi_G$ where $\lambda_G \geq -\frac{1}{7}$.

Proof.

- Two cases, $|G|$ odd and $|G|$ even.
- Enumerate over number of bridges m.
- If m or $|G|$ is big, result holds.
- Check all small graphs.

□

Summary of Results

$\mathcal{F}$ triangle-intersecting family, relative size $|\mathcal{F}|$.

- Upper bound: $|\mathcal{F}| \leq 1/8$.
- Uniqueness: $|\mathcal{F}| = 1/8 \Rightarrow$ sunflower.
- Stability: $|\mathcal{F}| \approx 1/8 \Rightarrow \approx$ sunflower.
- Generalizations ($p \leq 1/2$, Schur triplets).

Also works for odd-cycle-intersecting families!

Open Questions

- What about cycle-intersecting?
- What happens for other graphs?
  Sunflower not best for path of length 3!
  (Christofides)
- What happens when $p > 1/2$?
- Lots of other EKR-like questions!