Khintchine-Kahane using Fourier Analysis

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September 2011

Latała and Oleszkiewicz [1] gave a simple proof of the following theorem.

Theorem 1. *Let* $\mathbf{X} = X_1, \ldots, X_n$ *be a vector of non-negative real numbers. Define a random variable* S *by*

$$S = \left| \sum_i s_i X_i \right|, \quad s_i \in_R \{\pm 1\}.$$

We have

$$\frac{1}{\sqrt{2}} \|\mathbf{X}\|_2 \leq \mathbb{E}[S] \leq \|\mathbf{X}\|_2.$$

We rephrase their proof in terms of Fourier analysis.

Proof. The upper bound follows from $\mathbb{E}[S]^2 \leq \mathbb{E}[S^2] = \|\mathbf{X}\|_2^2$. For the lower bound, define a function $f \colon \{\pm 1\}^n \to \mathbb{R}_+$ by

$$f(s_1, \ldots, s_n) = \left| \sum_i s_i X_i \right|.$$

Note $\hat{f}(\mathbf{0}) = \mathbb{E}_{\mathbf{s}} f(\mathbf{s}) = \mathbb{E}[S]$. Let $\mathbf{s}^i$ denote the vector obtained from $\mathbf{s}$ by flipping coordinate i. Define the *total derivative* $\mathsf{D}f$ of f by

$$\mathsf{D}f(\mathbf{s}) = \frac{1}{2} \sum_i \left(f(\mathbf{s}) - f(\mathbf{s}^i) \right).$$

It is well-known that $\widehat{\mathsf{D}f}(\mathbf{t}) = |\mathbf{t}| \hat{f}(\mathbf{t})$, and so

$$\langle f, 2f - \mathsf{D}f \rangle = \sum_{\mathbf{t}} (2 - |\mathbf{t}|) \hat{f}(\mathbf{t})^2.$$

As mentioned, $\hat{f}(\mathbf{0}) = \mathbb{E}[S]$. Also, since f is invariant under flipping all coordinates, it is easy to see that $\hat{f}(\mathbf{t}) = 0$ when $|\mathbf{t}| = 1$. Therefore

$$\langle f, 2f - \mathsf{D}f \rangle \leq 2\mathbb{E}[S]^2.$$

How big can $\mathsf{D}f(\mathbf{s})$ be? Assume, without loss of generality, that $\sum_i s_i X_i \geq 0$. If $s_i = -1$ then $f(\mathbf{s}^i) = f(\mathbf{s}) + 2X_i$, and so $f(\mathbf{s}) - f(\mathbf{s}^i) = -2X_i = 2s_i X_i$. If $s_i = +1$ then $f(\mathbf{s}^i) = |f(\mathbf{s}) - 2X_i| \geq f(\mathbf{s}) - 2X_i$, and so $f(\mathbf{s}) - f(\mathbf{s}^i) \leq 2X_i = 2s_i X_i$. We conclude that $\mathsf{D}f(\mathbf{s}) \leq f(\mathbf{s})$. Since $f \geq 0$, this implies that

$$\langle f, 2f - \mathsf{D}f \rangle \geq \langle f, f \rangle = \mathbb{E}[S^2] = \|\mathbf{X}\|_2^2.$$

We conclude that $\|\mathbf{X}\|_2^2 \leq 2\mathbb{E}[S]^2$. □

References

[1] Rafał Latała and Krzysztof Oleszkiewicz. On the best constant in the Khintchine-Kahane inequality. *Studia Mathematica*, 109(1):101–104, 1994.