High-dimensional Hoffman bound

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1 Introduction

Spectral techniques have many applications in combinatorics. One central technique is the Hoffman–Lovász bound. Since the bound is so simple, let us start by stating and proving the version due to Lovász:

Theorem 1 (Lovász bound)**.** *Let $G = (V, E)$ be a graph, and let A be a $V \times V$ symmetric matrix such that $A(x, y) = 1$ if $x = y$ or $(x, y) \notin E$. Then*

$$\alpha(G) \leq \lambda_{\max}(A).$$

Proof. Let f be the indicator function of an independent set, and let $|f|$ denote its size. Then

$$|f|^2 = \langle f, Af \rangle \leq \lambda_{\max}(A)\langle f, f \rangle = \lambda_{\max}(A)|f|. \qquad \square$$

The best bound obtainable in this way is denoted $\theta(G)$, and can be computed efficiently using semidefinite programming. The original application of Lovász was to compute the Shannon capacity of C_5, but it has many other applications, such as:

1. Delsarte's linear programming bound in coding theory. This application also inaugurated the theory of association schemes. Navon and Samorodnitsky found a proof which circumvents the Lovász bound.

2. Various t-intersecting Erdős–Ko–Rado theorems: vector spaces (Frankl–Wilson), permutations (Ellis–Friedgut–Pilpel), graphs (Ellis–Filmus–Friedgut). These are the only known proofs of these theorems.

3. Planted clique (Feige–Krauthgamer). Arguably cleaner than original algorithm of Alon–Krivelevich–Sudakov.

4. Separation between $\mathsf{P/poly}$ and its monotone version (Éva Tardos).

What all of these applications have in common is that the constraints involved are on pairs of elements. For example, in the LP bound, the constraint is minimum distance, in Erdős–Ko–Rado the constraint is intersection of pairs, and in planted clique the constraint is on pairs of vertices. In other circumstances, we are interested in constraints on larger tuples, for example:

1. Erdős matching conjecture: How large can a k-uniform family be if no s sets are pairwise disjoint?
2. s-wise Erdős–Ko–Rado: How large can a k-uniform family be if any s sets have a common element? (solved by Frankl–Tokushige)
3. Mantel's theorem: How many edges can a graph contain if it contains no triangles?
4. Frankl's triangle problem: How large can a k-uniform family be if it contains no solution to $x + y + z = 0$ (sets identified with vectors over $\mathbb{F}_2$)?

All of these can be modeled as independent sets in certain *hypergraphs* (in the case of Mantel's theorem, the encoding is not obvious). This calls for a version of the spectral bound which is appropriate for complexes. Such versions have been suggested by Golubev and by Bachoc, Gundert and Passuello, but our result subsumes the former and seems more useful than the latter. For example, we are able to reprove Mantel's theorem and the Frankl-Tokushige bound, and we prove nearly tight bounds for Frankl's triangle problem.

2 Hoffman bound

We start with the standard Hoffman bound, which is a slightly weaker form of the Lovász bound (though we do not explore this connection). Let V be a ground set, and let μ_2 be a symmetric distribution on ordered pairs of vertices. Denote the marginal by μ_1. We say that a subset of V is *independent* if it spans no edge in the support of μ_2. The characteristic vector ϕ of an independent set thus satisfies

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu_2}[\phi(x)\phi(y)] = 0.$$

Let T be the operator

$$(Tf)(x) = \sum_y \frac{\mu_2(x,y)}{\mu_1(x)} f(y).$$

(Note that $y \mapsto \mu_2(x,y)/\mu_1(x)$ is a probability distribution.) If we define an inner product

$$\langle f, g\rangle = \mathop{\mathbb{E}}_{\mu_1}[fg] = \sum_x \mu_1(x) f(x) g(x),$$

then we have

$$\langle f, Tg\rangle = \sum_{x,y} \mu_2(x,y) f(x) g(y).$$

Since μ_2 is symmetric, the operator T is self-adjoint, and so has real eigenvectors. In fact, it is a Markov operator: $T1 = 1$.

Let us get back to ϕ, decomposing it orthonormally as

$$\phi = \mathop{\mathbb{E}}_{\mu_1}[\phi]1 + \sum_i c_i v_i,$$

where $1, v_i$ are the eigenvectors of T, say with eigenvalues $1, \ell_i$. A quick calculation shows that

$$0 = \mathop{\mathbb{E}}_{(x,y)\sim\mu_2}[\phi(x)\phi(y)] = \langle \phi, T\phi\rangle = \mathop{\mathbb{E}}_{\mu_1}[\phi]^2 + \sum_i \ell_i c_i^2 \geq \mathop{\mathbb{E}}_{\mu_1}[\phi]^2 + \lambda_{\min}(T)\sum_i c_i^2 =$$
$$\mathop{\mathbb{E}}_{\mu_1}[\phi]^2 + \lambda_{\min}(T)(\|\phi\|^2 - \mathop{\mathbb{E}}_{\mu_1}[\phi]^2) = \mathop{\mathbb{E}}_{\mu_1}[\phi]^2 + \lambda_{\min}(T)(\mathop{\mathbb{E}}_{\mu_1}[\phi] - \mathop{\mathbb{E}}_{\mu_1}[\phi]^2) =$$
$$\mathop{\mathbb{E}}_{\mu_1}[\phi]\left[1 - (1 - \lambda_{\min}(T))(1 - \mathop{\mathbb{E}}_{\mu_1}[\phi])\right].$$

Rearrangement yields

$$1 - \mathop{\mathbb{E}}_{\mu_1}[\phi] \geq \frac{1}{1 - \lambda_{\min}(T)}.$$

In fact, everything would work even with a *signed* measure, and this is important for many applications.

Generalization to higher dimensions Since all the ideas appear already in the generalization to triplets, let us describe this case to simplify notation. Again V is a ground set, and μ_3 is a symmetric distribution on ordered triplets of vertices. A subset of V is independent if it spans no hyperedge in the support of μ_3. We define μ_2 and μ_1 to be the marginal distributions of pairs and singletons, respectively.

Recall that T is the operator defined by

$$(Tf)(x) = \frac{\mu_2(x,y)}{\mu_1(x)} f(y),$$

and that we defined an inner product

$$\langle f, g\rangle = \sum_x \mu_1(x) f(x) g(x).$$

Repeating the calculation above, if ϕ is an independent set then

$$\mathop{\mathbb{E}}_{\mu_1}[\phi]\left[1 - (1 - \lambda_{\min}(T))(1 - \mathop{\mathbb{E}}_{\mu_1}[\phi])\right] \leq \langle \phi, T\phi\rangle = \mathop{\mathbb{E}}_{(x,y)\sim\mu_2}[\phi(x)\phi(y)].$$

Previously, the right-hand side was simply zero. This time, we need to use a different bound. The idea is that if $\phi(x) = 1$ (which we also write as $x \in \phi$), then we can bound the total contribution of $\phi(y)$ by computing a Hoffman bound *relative to* x. In more detail,

$$\mathop{\mathbb{E}}_{(x,y)\sim\mu_2}[\phi(x)\phi(y)] = \sum_x \mu_1(x)\phi(x) \sum_y \frac{\mu_2(x,y)}{\mu_1(x)}\phi(y) \leq$$

$$\sum_x \mu_1(x)\phi(x) \times \max_{x\in\phi} \sum_y \frac{\mu_2(x,y)}{\mu_1(y)}\phi(y) = \mathop{\mathbb{E}}_{\mu_1}[\phi] \max_{x\in\phi} \mathop{\mathbb{E}}_{\mu_1^x}[\phi],$$

where μ_1^x is the measure given by $\mu_1^x(y) = \mu_2(x,y)/\mu_1(x)$. Cancelling $\mathbb{E}_{\mu_1}[\phi]$, this gives

$$1 - (1-\lambda_{\min}(T))(1 - \mathop{\mathbb{E}}_{\mu_1}[\phi]) \leq \max_{x\in\phi} \mathop{\mathbb{E}}_{\mu_1^x}[\phi] \Longrightarrow 1 - \mathop{\mathbb{E}}_{\mu_1}[\phi] \geq \frac{\min_{x\in\phi}(1 - \mathbb{E}_{\mu_1^x}[\phi])}{1-\lambda_{\min}(T)}.$$

Let us define, by analogy, $\mu_2^x(y,z) = \mu_3(x,y,z)/\mu_1(x)$. Since ϕ is an independent set with respect to μ_3 and $\phi(x) = 1$, we see that ϕ is an independent set with respect to μ_2^x. Therefore, if we define the operator T^x in the natural way,

$$1 - \mathop{\mathbb{E}}_{\mu_1^x}[\phi] \geq \frac{1}{1-\lambda_{\min}(T^x)} \Longrightarrow 1 - \mathop{\mathbb{E}}_{\mu_1}[\phi] \geq \frac{1}{(1-\lambda_{\min}(T))\max_x(1-\lambda_{\min}(T^x))}.$$

This is the Hoffman bound for triplets. The Hoffman bound for d-tuples includes $d-1$ factors, with an identical proof.

3 Example: Frankl's triangle problem

Recall that the μ_p measure is a tensorial measure given by $\mu_p(S) = p^{|S|}(1-p)^{|\overline{S}|}$. Frankl's triangle problem (in its μ_p version) asks for the maximum μ_p-measure of a subset of $\{0,1\}^n$ without a triplet of elements summing to zero (modulo 2). When $p > 2/3$, the family of all vectors of weight more than $\frac{2}{3}n$ has measure tending to 1, so we concentrate on $p \leq 2/3$.

We start by solving the seemingly trivial case $n = 1$. We are looking for a distribution μ_3 supported on triplets summing to zero, say

$$\mu_3(0,1,1) = \mu_3(1,0,1) = \mu_3(1,1,0) = \alpha, \quad \mu_3(0,0,0) = 1 - 3\alpha.$$

In order to compute α, let us notice that $\mu_1(1) = 2\alpha$, and so $\alpha = p/2$. The distribution μ_3 is thus given by

$$\mu_3(0,1,1) = \mu_3(1,0,1) = \mu_3(1,1,0) = \frac{p}{2}, \quad \mu_3(0,0,0) = 1 - \frac{3}{2}p.$$

The marginal distributions are

$$\mu_2(1,1) = \frac{p}{2}, \quad \mu_2(1,0) = \mu_2(0,1) = \frac{p}{2}, \quad \mu_2(0,0) = 1 - \frac{3}{2}p$$

and

$$\mu_1(1) = p, \quad \mu_1(0) = 1 - p.$$

The relevant operators are

$$T = \begin{pmatrix} \frac{1-3p/2}{1-p} & \frac{p/2}{1-p} \\ \frac{p/2}{p} & \frac{p/2}{p} \end{pmatrix}, \quad T^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding eigenvalues are

$$\lambda(T) = 1, \frac{1-2p}{2(1-p)}; \quad \lambda(T^0) = 1, 1; \quad \lambda(T^1) = 1, -1.$$

(For T, the second eigenvalue can be computed by considering the trace.)

When $n > 1$, we take tensor products of the same distribution μ_3. The eigenvalues of $T^{\otimes n}$ are products of eigenvalues of T, and similarly the eigenvalues of T^x are products of eigenvalues of T^0, T^1.

We immediately see that $\max_x (1 - \lambda_{\min}(T^x)) = 2$. As for $\lambda_{\min}(T^{\otimes n})$, the answer depends on whether $p \geq 1/2$ or not. If $p \geq 1/2$ then $-1 \leq \lambda_{\min}(T) \leq 0$, and so $\lambda_{\min}(T^{\otimes n}) = \lambda_{\min}(T)$. This shows that the measure of a triangle-free set is at most

$$1 - \frac{1}{(1 - \frac{1-2p}{2(1-p)})2} = p,$$

a bound which is met by stars (all vectors v with $v_i = 1$ for some fixed i).

When $p \leq 1/2$ then $\lambda_{\min}(T) \geq 0$, and so all we can say is that $\lambda_{\min}(T^{\otimes n}) \geq 0$, which implies a bound of

$$1 - \frac{1}{1 \cdot 2} = \frac{1}{2}.$$

This bound is asymptotically met by the family of all vectors having odd parity.

4 Open problem

The generalized Hoffman bound (with minor adjustments) is strong enough to prove Mantel's theorem. Can it solve Turán's (4,3) problem, which asks for the maximum number of hyperedges in a 3-uniform graph without a tetrahedron? The answer should be: a 5/9 fraction of them (compared to a half for Mantel's theorem).