Harper's isoperimetric inequality

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Abstract

We reproduce Harper's proof of his isoperimetric inequality from his book *Global methods for combinatorial isoperimetric problems*.

Let Q_d be the dth dimensional hypercube with vertex set $\{0,1\}^d$ and edges connecting $x0y$ to $x1y$ for all $xy \in \{0,1\}^{d-1}$. For a subset $S \subseteq Q_d$, the *edge boundary* ∂S is the number of edges (a,b) such that $a \in S$ and $b \notin S$. We order the vertices of Q_d lexicographically. For example, for $d = 3$ the order is

$$000, 001, 010, 011, 100, 101, 110, 111.$$

Let $L_d[n] \subseteq Q_d$ denote the first n vertices according to the lexicograph order. Here is Harper's isoperimetric inequality.

Theorem 1. *For every set $S \subseteq Q_d$,*

$$\partial S \geq \partial L_d[|S|].$$

In words, a set of size n minimizing the edge boundary is $L_d[n]$.

In his book *Global methods for combinatorial isoperimetric problems*, Harper offers two proofs of Theorem 1. The first proof, presented in Chapter 1, closely follows the original one and uses intricate induction. The second proof, presented in Chapter 3, also uses induction but is much simpler, and this is the proof that we present here. It uses the technique of *compression*, a generalization of *shifting*.

Proof of Theorem 1. The proof is by induction on d. When $d = 1$ the theorem is trivial. Suppose now that $d \geq 2$, and that the theorem holds for all $d - 1$.

The first step of the proof is to compress the set S. To that end, we need to define an order on subsets of $\{0,1\}^d$. We think of such subsets as vectors in $\{0,1\}^{2^d}$, with the coordinates ordered according to the lexicographic order of Q_d. We order the subsets using the *reverse* lexicographic order (that is, the lexicographic order with the order of coordinates reversed). Thus, if $x, y \in Q_d$ satisfy $x < y$ and $y \in T \subseteq \{0,1\}^d$ then replacing y with x decreases T in the order.

Let $T \subseteq \{0,1\}^n$. For every coordinate $i \in [d]$, we can decompose T into two subsets $T_{i=0}, T_{i=1} \subseteq \{0,1\}^{d-1}$ according to the value of the ith coordinate. Let $C_i(T)$ be the set obtained by replacing $T_{i=0}$ with $L_{d-1}[|T_{i=0}|]$ and $T_{i=1}$ with $L_{d-1}[|T_{i=1}|]$, and note that $|C_i(T)| = |T|$ and that $C_i(T) \leq T$ with respect to the order on subsets of $\{0,1\}^d$. The important property is $\partial C_i(T) \leq \partial T$:

$$\begin{aligned}
\partial C_i(T) &= \partial L_{d-1}[|T_{i=0}|] + \partial L_{d-1}[|T_{i=1}|] + |L_{d-1}[|T_{i=0}|] \triangle L_{d-1}[|T_{i=1}|]| \\
&= \partial L_{d-1}[|T_{i=0}|] + \partial L_{d-1}[|T_{i=1}|] + \big|\,|T_{i=0}| - |T_{i=1}|\,\big| \\
&\leq \partial T_{i=0} + \partial T_{i=1} + |T_{i=0} \triangle T_{i=1}| \\
&= \partial T.
\end{aligned}$$

On the first line, the first two summands account for the edge boundary in all directions but i, and the third for the edge boundary in direction i. The second line equals the first since $L_{d-1}[|T_{i=0}|] \subseteq L_{d-1}[|T_{i=1}|]$ or vice versa. The inequality holds by induction and since $|A \triangle B| \geq ||A| - |B||$.

Starting with S, apply the operations $C_1, \ldots, C_d$ in a cyclic fashion repeatedly. The set keeps decreasing in the order of subsets of $\{0,1\}^{2^d}$, and so eventually we reach a fixed point T of all operations. As we have shown above, $|T| = |S|$ and $|\partial T| \leq |\partial S|$. We say that T is *compressed*.

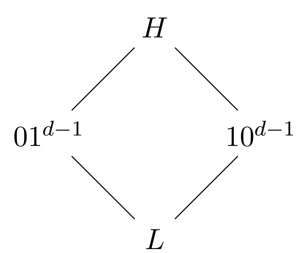
If all compressed sets containing $y \in \{0,1\}^d$ also contain $x \in \{0,1\}^d$ then we write $x \prec y$. The order $\prec$ is known as the *compressibility order*. We proceed show that if $x < y$ then also $x \prec y$ unless $x = 01^{d-1}$ and $y = 10^{d-1}$; in the latter case we say that x, y are *bad*.

Indeed, suppose that $x < y$ and that x, y are good. If $x_i = y_i$ for some coordinate i then $T = C_i(T)$ implies that $x \prec y$. It remains to consider the case that $x_i \neq y_i$ for all i. Clearly $x_1 = 0, y_1 = 1$, and since x, y are not bad, $x_i = 0, y_i = 1$ for some $i > 1$. We can write $x = 0a0b$ and $y = 1\bar{a}1\bar{b}$. Let $z = 0a1b$:

$$
\begin{aligned}
y &= 1\bar{a}1\bar{b} \\
z &= 0a1b \\
x &= 0a0b
\end{aligned}
$$

Since $x < z$ and $x_1 = z_1$, $x \prec z$. Since $z < y$ and $z_i = y_i$, $z \prec y$. We conclude that $x \prec y$, as required.

We can picture the compressiblity order using a Hasse diagram, in which L consists of all vectors smaller than 01^{d-1}, and H consists of all vectors larger than 10^{d-1}:



The sets L, H are ordered linearly according to the lexicographic order. The only incomparable elements are 01^{d-1} and 10^{d-1}. This shows that unless $|S| = 2^{d-1}$, T must be a prefix of the lexicographic order on the vertices of Q_d, and so $T = L_d[|S|]$, completing the proof in this case.

It remains to handle the case $|S| = 2^{d-1}$. In that case, either $T = L_d[2^{d-1}]$ or $T = L_d[2^{d-1}] \setminus 01^{d-1} \cup 10^{d-1}$. We complete the proof by calculating the corresponding edge boundaries:

$$
\begin{aligned}
&\partial L_d[2^{d-1}] = 2^{d-1}, \\
&\partial(L_d[2^{d-1}] \setminus 01^{d-1} \cup 10^{d-1}) = 2^{d-1} + 2(d-2).
\end{aligned}
$$

The theorem follows since $\partial L_d[2^{d-1}] \leq \partial(L_d[2^{d-1}] \setminus 01^{d-1} \cup 10^{d-1})$. $\square$