More complete intersection theorems

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April 4, 2017

Abstract

The seminal complete intersection theorem of Ahlswede and Khachatrian gives the maximum cardinality of a k-uniform t-intersecting family on n points, and describes all optimal families. In recent work, we extended this theorem to the weighted setting, giving the maximum μ_p measure of a t-intersecting family on n points. In this work, we prove two new complete intersection theorems. The first gives the supremum μ_p measure of a t-intersecting family on infinitely many points, and the second gives the maximum cardinality of a subset of $\mathbb{Z}_m^n$ in which any two elements x, y have t positions $i_1, \ldots, i_t$ such that $x_{i_j} - y_{i_j} \in \{-(s-1), \ldots, s-1\}$. In both cases, we determine the extremal families, where possible.

1 Introduction

The complete intersection theorem of Ahlswede and Khachatrian [1, 3] is a generalization of the classical Erdős–Ko–Rado theorem [9] to the case of t-intersecting families. The theorem states the maximum cardinality of a t-intersecting k-uniform family on n points, for all values of n, k, t. Moreover, it describes all extremal families (in all but a few exceptional cases). The extremal families are of the form $\mathcal{F}_{t,r} = \{S : |S \cap [t+2r]| \geq t+r\}$, where r depends on $\frac{k-t+1}{n}$; the set $[t+2r]$ can be replaced by any set of size $t+2r$.

The complete intersection theorem concerns the setting of k-uniform families. Dinur and Safra [7] considered the weighted setting, in which the aim is to find the maximum μ_p measure of a family on n points without uniformity restrictions, where $\mu_p(A) = p^{|A|}(1-p)^{n-|A|}$. They showed that the original complete intersection theorem implies that when $p < 1/2$, the maximum μ_p measure of a t-intersecting family on an unbounded number of points is $\max_r \mu_p(\mathcal{F}_{t,r})$. Ahlswede and Khachatrian [2] had considered the case $p = 1/m$ earlier, and their argument (which differs from that of Dinur and Safra) extends for all $p < 1/2$ as well. Recently [10] we have extended these results to all values of p, determining in addition all extremal families; they are all of the form $\mathcal{F}_{t,r}$.

It is natural to ask what happens when we allow our families to depend on *infinitely* many points rather than on an *unbounded* number of points. In Section 4 we show that when $p < 1/2$, the maximum μ_p measure of a t-intersecting family on infinitely many points is still $\max_r \mu_p(\mathcal{F}_{t,r})$, and furthermore all extremal families are of the form $\mathcal{F}_{t,r}$. We also determine the answer when $p \geq 1/2$.

Ahlswede and Khachatrian [2] considered the analog of their complete intersection theorem to the *Hamming scheme*, in which the objects of study are subsets of $\mathbb{Z}_m^n$ under the uniform measure. Such a subset is t-*agreeing* if any two vectors agree on at least t coordinates. They showed that the original complete intersection theorem implies that the maximum measure of a t-agreeing subset of $\mathbb{Z}_m^n$ for unbounded n is $\max_r \mu_{1/m}(\mathcal{F}_{t,r})$. In Section 5 we extend their work to families in which any two vectors have t coordinates which differ by at most $s-1$, showing that the maximum measure in this case is $\max_r \mu_{s/m}(\mathcal{F}_{t,r})$. We also determine all extremal families.

The proofs of both results rely on new versions of Katona's circle argument, described in Section 3.

2 Preliminaries

We use $[n]$ for $\{1, \ldots, n\}$, $\binom{[n]}{k}$ for all subsets of $[n]$ of size k, and $\binom{[n]}{\geq k}$ for all subsets of $[n]$ of size at least k. We denote by 2^A the set of all subsets of A. The binomial distribution with n trials and success

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probability p is denoted $\mathrm{Bin}(n,p)$.

We will need the following basic definitions.

Definition 2.1. A *family on n points* is a collection of subsets of $[n]$. A family $\mathcal{F}$ is *t-intersecting* if any two sets in $\mathcal{F}$ have at least t points in common. Two families $\mathcal{F},\mathcal{G}$ are *cross-t-intersecting* if any set in $\mathcal{F}$ has at least t points in common with every set in $\mathcal{G}$.

A family $\mathcal{F}$ on n points is *monotone* if whenever $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$. Given a family $\mathcal{F}$, its *up-set* $\langle \mathcal{F} \rangle$ is the smallest monotone family containing $\mathcal{F}$, which is $\langle \mathcal{F} \rangle = \{B \supseteq A : A \in \mathcal{F}\}$.

When $t=1$, we will drop the parameter t: intersecting family, cross-intersecting families.

Definition 2.2. For any $p \in (0,1)$ and any n, the product measure μ_p is a measure on the set of subsets of $[n]$ given by

$$\mu_p(A) = p^{|A|}(1-p)^{n-|A|}.$$

For $n \geq t \geq 1$ and $p \in (0,1)$, the parameter $w(n,t,p)$ is the maximum of $\mu_p(\mathcal{F})$ over all t-intersecting families on n points.

For $t \geq 1$ and $p \in (0,1)$, the parameter $w_{\sup}(t,p)$ is given by

$$w_{\sup}(t,p) = \sup_n w(n,t,p).$$

It is not hard to see that we can also define $w_{\sup}(t,p)$ as a limit instead of a supremum, since $w(n,t,p)$ is non-decreasing in n. Indeed, every t-intersecting family on n points can be extended to a t-intersecting family on $n+1$ points having the same μ_p measure.

The optimal families in the weighted complete intersection theorem, named after Frankl [11], are described in the following definition.

Definition 2.3. For $t \geq 1$ and $r \geq 0$, the (t,r)-*Frankl family* on n points is the t-intersecting family

$$\mathcal{F}_{t,r} = \{A \subseteq [n] : |A \cap [t+2r]| \geq t+r\}.$$

A family $\mathcal{F}$ on n points is *equivalent* to a (t,r)-Frankl family if there exists a set $S \subseteq [n]$ of size $t+2r$ such that

$$\mathcal{F} = \{A \subseteq [n] : |A \cap S| \geq t+r\}.$$

The following theorem, proved in [10], is the μ_p version of Ahlswede and Khachatrian's complete intersection theorem.

Theorem 2.1. *Let $n \geq t \geq 1$ and $p \in (0,1)$. If $\mathcal{F}$ is t-intersecting then*

$$\mu_p(\mathcal{F}) \leq \max_{r \colon t+2r \leq n} \mu_p(\mathcal{F}_{t,r}).$$

Moreover, unless $t=1$ and $p \geq 1/2$, equality holds only if $\mathcal{F}$ is equivalent to a Frankl family $\mathcal{F}_{t,r}$.

When $t=1$ and $p > 1/2$, the same holds if $n+t$ is even, and otherwise $\mathcal{F} = \mathcal{G} \cup \binom{[n]}{\geq \frac{n+t+1}{2}}$ where $\mathcal{G} \subseteq \binom{[n]}{\frac{n+t-1}{2}}$ contains exactly $\binom{n-1}{\frac{n+t-1}{2}}$ sets.

Corollary 2.2. *We have*

$$w_{\sup}(1,p) = \begin{cases} p & p \leq \frac{1}{2}, \\ 1 & p > \frac{1}{2} \end{cases}$$

For $t \geq 2$, we have

$$w_{\sup}(t,p) = \begin{cases} \mu_p(\mathcal{F}_{t,r}) & \frac{r}{t+2r-1} \leq p \leq \frac{r+1}{t+2r+1}, \\ \frac{1}{2} & p = \frac{1}{2}, \\ 1 & p > \frac{1}{2}. \end{cases}$$

3 Katona's circle argument

Katona [15] gave a particularly simple proof of the Erdős–Ko–Rado theorem, using what has become known as the *circle method*. The same proof goes through in the μ_p setting, with much the same proof, as we show in Section 3.1. We are able to use the same argument to obtain a description of all optimal families.

The heart of Katona's argument is the following seemingly trivial observation: if S is a measurable set on the unit circumference circle in which any two points are at distance at most $p < 1/2$, then the length of S is at most p, the optimal sets being intervals. In Section 3.2 we give two versions of this argument for the case of two sets: one in a discrete setting, and the other in a continuous setting. These results will be used in the rest of the paper.

3.1 Intersecting families

The Erdős–Ko–Rado theorem, in our setting, states that if $\mathcal{F}$ is an intersecting family then $\mu_p(\mathcal{F}) \leq p$ for all $p \leq 1/2$. Katona [15] gave a particularly simple proof of the theorem in its original setting, and we adapt his proof to our setting.

Lemma 3.1. *Let $n \geq 1$ and $p \in (0, 1/2)$. Then $\mu_p(\mathcal{F}) \leq p$ for all intersecting families $\mathcal{F}$.*

Moreover, if $\mathcal{F}$ is an intersecting family on n points such that $\mu_p(\mathcal{F}) = p$ then $\mathcal{F}$ is equivalent to a $(1, 0)$-Frankl family. In other words, if $\mu_p(\mathcal{F}) = p$ then for some $i \in [n]$ we have

$$\mathcal{F} = \{A \subseteq [n] : A \ni i\}.$$

Proof. The idea is to come up with a probabilistic model for the distribution μ_p, and use it to show that $\mu_p(\mathcal{F}) \leq p$. Since $\mu_p(\mathcal{F}_{1,0}) = p$, this shows that $w(n, 1, p) = p$. We will then use the same probabilistic model to identify all families satisfying $\mu_p(\mathcal{F}) = p$.

Let $\mathbb{T}$ be the circle of unit circumference. Choose n points $x_1, \ldots, x_n$ at random on $\mathbb{T}$. Choose another point t on $\mathbb{T}$ at random, and consider the arc $(t, t+p)$ (where $t+p$ is taken modulo 1). The set of indices of points S_t which lie inside the arc has distribution μ_p, and so $\mu_p(\mathcal{F}) = \Pr[S_t \in \mathcal{F}]$.

We will prove that for *each* setting of $x_1, \ldots, x_n$ we have $\Pr[S_t \in \mathcal{F}] \leq p$, the probability taken over the choice of t. Let $I = \{t : S_t \in \mathcal{F}\}$, and note that I is a union of intervals. The crucial observation is that if $t_1, t_2 \in I$ then the corresponding arcs $(t_1, t_1 + p), (t_2, t_2 + p)$ must intersect, and so $d(t_1, t_2) < p$, where $d(\cdot, \cdot)$ is shortest distance on the circumference of the circle. Consider now any $t_1 \in I$. All $t_2 \in I$ must lie in the interval $(t_1 - p, t_1 + p)$, and moreover for each $s \in (0, p)$, at most one of $t_1 + s, t_1 - p + s$ can be in I (since $d(t_1 + s, t_1 + s - p) = p$). It follows that the measure of I is at most p. In other words, $\Pr[S_t \in \mathcal{F}] \leq p$, implying that $\mu_p(\mathcal{F}) \leq p$.

For future use, we also need to identify the cases in which I has measure exactly p. We will show that I must be an interval of length p. The argument above shows that I is a union of non-empty intervals of two types, $J_1, \ldots, J_a \subseteq (t_1 - p, t_1]$ and $K_1, \ldots, K_b \subseteq [t_1, t_1 + p)$, such that the intervals $J_i + p, K_j$ together partition $[t_1, t_1 + p)$. If $a = 0$ then $I = [t_1, t_1 + p)$, and if $b = 0$ then $I = (t_1 - p, t_1]$. If there is a unique interval K_1 which is of the form $[t_1, t_1 + s)$ (or $[t_1, t_1 + s]$) then $I = [t_1 + s - p, t_1 + s)$ (or $(t_1 + s - p, t_1 + s]$). Otherwise, there must be some interval K_j whose left end-point y is larger than t_1. There is a corresponding interval J_i whose right end-point is $y - p$. However, since $p < 1/2$, a point on J_i slightly to the left of $y - p$ has distance larger than p from a point on K_j slightly to the right of y, contradicting our assumptions. We conclude that I must be an interval of length p.

We proceed to identify the families $\mathcal{F}$ which achieve the upper bound, that is, satisfy $\mu_p(\mathcal{F}) = p$. Clearly any family $\mathcal{F}$ equivalent to $\mathcal{F}_{1,0}$ satisfies $\mu_p(\mathcal{F}) = p$. We will show that these are the only such families. First, notice that if $\mu_p(\mathcal{F}) = p$ then $\mathcal{F}$ must be monotone (otherwise its upset is an intersecting family of measure larger than p). Moreover, the set I defined above is an interval of length p almost surely, with respect to the choice of $x_1, \ldots, x_n$. In particular, there is a choice of $x_1, \ldots, x_n$, all distinct and none at distance exactly p, such that I is an interval of length p, say $I = [y, y + p)$. For small $\epsilon > 0$, the arcs $(y + \epsilon, y + p + \epsilon), (y + p - \epsilon, y + 2p - \epsilon)$ intersect at a small neighborhood of $y + p$. Since the corresponding sets $S_{y+\epsilon}, S_{y+p-\epsilon}$ intersect, there must be some point $x_i = y + p$. We will show that $\{x_i\} \in \mathcal{F}$, and so monotonicity implies that $\mathcal{F} = \{A \subseteq [n] : A \ni i\}$.

Let A consist of all points in $(y, y + p)$. Thus $A \cup \{x_i\} \in \mathcal{F}$ (since $y + \epsilon \in I$ for small ϵ) while $A \notin \mathcal{F}$ (since $y - \epsilon \notin I$ for small ϵ). Consider now the set of configurations $x'_1, \ldots, x'_n$ in which $x'_j \in (x'_i - p, x'_i)$

for all $j \in A$ and $x'_j \notin (x'_i - p, x'_i + p)$ for all $j \notin A \cup \{i\}$. This set of configurations has positive measure, and so there must exist one whose corresponding set I' is an interval of measure p. By construction, I' contains all points t just to the right of $x'_i - p$ (since the corresponding S_t is $A \cup \{i\}$) but not $x'_i - p$ (since the corresponding $S_{x'_i - p}$ is A), and so $x'_i - p$ is the left end-point of the interval. The right end-point is thus x'_i itself. By construction, $S_{x'_i - \epsilon} = \{i\}$ for small enough $\epsilon > 0$, showing that $\{i\} \in \mathcal{F}$. This completes the proof. □

Other proofs of the upper bound which employ similar arguments appear in Dinur and Friedgut [6] and in Friedgut [12]. Unfortunately, it seems that Katona's idea doesn't extent to t-intersecting families for $t > 1$. For a discussion of this, see Howard, Károlyi and Székely [14].

3.2 Cross-intersecting settings

As explained in the introduction to this section, the heart of Katona's proof is a result on sets in which any two points are close. The proof of Lemma 3.1 closely follows the original argument. In this section we give alternative arguments for the cross-intersecting counterparts in both discrete and continuous settings.

3.2.1 Discrete setting

We start with the easier, discrete setting. First, a few definitions.

Definition 3.1. A set $A \subseteq \mathbb{Z}_m$ is *s-agreeing* if every $a, b \in A$ satisfy $a - b \in \{-(s-1), \ldots, s-1\}$. Two sets $A, B \subseteq \mathbb{Z}_m$ are *cross-s-agreeing* if every $a \in A$ and $b \in B$ satisfy $a - b \in \{-(s-1), \ldots, s-1\}$.

Definition 3.2. A set $A \subseteq \mathbb{Z}_m$ is an *interval* if it is of the form $\{x - \ell, \ldots, x + \ell\}$ (for $2\ell + 1 < m$) or $\{x - \ell, \ldots, x + \ell - 1\}$ (for $2\ell < m$). In the first case, x is the *center* of A, and in the second, $x - 1/2$ is the center of A.

The following simple lemma will simplify the argument below.

Lemma 3.2. *Let $s \le m/2$. If $A \subseteq \mathbb{Z}_m$ is cross-s-agreeing with the non-empty interval $\{x, \ldots, y\}$ of length at most $2s - 1$ then $A \subseteq \{y - (s-1), \ldots, x + (s-1)\}$.*

Proof. The proof is by induction on the length of the interval. If $y = x$ then trivially $A \subseteq \{x - (s-1), \ldots, x + (s-1)\}$. Suppose now that we have already shown that if A is cross-s-agreeing with $\{x, \ldots, y\}$ then $A \subseteq \{y - (s-1), \ldots, x + (s-1)\}$. If A is cross-s-agreeing with $\{x, \ldots, y+1\}$ then

$$A \subseteq \{y-(s-1), \ldots, x+(s-1)\} \cap \{y+1-(s-1), \ldots, x+1+(s-1)\} = \{y+1-(s-1), \ldots, x+(s-1)\}. \ \square$$

We can now state and prove the result in the discrete setting.

Lemma 3.3. *Let $m, s \ge 1$ be integers satisfying $s < m/2$. If $A, B \subseteq \mathbb{Z}_m$ are non-empty s-agreeing sets then $|A| + |B| \le 2s$. Moreover, if $|A| + |B| = 2s$ then A, B are intervals centered at the same point or half-point. In particular, if A is s-agreeing then $|A| \le s$, with equality only when A is an interval.*

When $s = m/2$ it still holds that $|A| + |B| \le 2s$, with equality when $B = \{x : x + s \notin A\}$, and this holds even without the assumption that A, B be non-empty. In particular, if A is s-agreeing then $|A| \le s$, with equality only when A contains one point out of each pair $x, x + s$.

Proof. We start by proving that $|A| + |B| \le 2s$ when $s < m/2$. The idea is to use a shifting argument to transform A, B into intervals without decreasing $|A| + |B|$. We construct a sequence of non-empty s-agreeing sets, starting with $(A_0, B_0) = (A, B)$. Given (A_i, B_i), note first that $A_i \ne \mathbb{Z}_m$, since otherwise B_i would have to be empty. Therefore there exists a point $x \in A_i$ such that $x + 1 \notin A_i$. We take $A_{i+1} = A_i \cup \{x + 1\}$. If A_{i+1}, B_i are not cross-s-agreeing then there must be a point $y \in B_i$ such that $x - y \in \{-(s-1), \ldots, s-1\}$ but $x + 1 - y \notin \{-(s-1), \ldots, s-1\}$. This can only happen if $x - y = s - 1$, and so there is at most one such point $y = x - (s-1)$. We therefore take $B_{i+1} = B_i \setminus \{x - (s-1)\}$. If $B_{i+1} = \emptyset$, then we stop the sequence at (A_i, B_i), and otherwise we continue. Note that $|A_{i+1}| = |A_i| + 1$ and $|B_{i+1}| \ge |B_i| - 1$, and so $|A_{i+1}| + |B_{i+1}| \ge |A_i| + |B_i|$.

Since the size of A_i keeps increasing, there must be a last pair in the sequence, say (A_t, B_t). Our stopping condition guarantees that $|B_t| = 1$, say $B_t = \{y\}$. This forces $A_t \subseteq \{y-(s-1), \ldots, y+(s-1)\}$, and so $|A_t| + |B_t| \le (2s-1)+1 = 2s$. Since $|A_0| + |B_0| \le |A_t| + |B_t| \le 2s$, this completes the proof that $|A| + |B| \le 2s$.

Next, we show that if $|A| + |B| = 2s$ then A, B are intervals centered at the same point. We do this by reverse induction on the sequence $(A_0, B_0), \ldots, (A_t, B_t)$. For the base case, the argument in the preceding paragraph shows that $|A_t| + |B_t| = 2s$ if $A_t = \{y-(s-1), \ldots, y+(s-1)\}$ and $B_t = \{y\}$, and so both sets are intervals centered at the point y.

Suppose now that (A_{i+1}, B_{i+1}) are intervals centered (without loss of generality) at 0 or $-1/2$, depending on the parity of $|A_{i+1}|$. Suppose first that $|A_{i+1}|$ is odd. Then for some $0 \le \ell \le s-1$ we have $A_{i+1} = \{-\ell, \ldots, \ell\}$ and $B_{i+1} = \{-(s-1-\ell), \ldots, s-1-\ell\}$. By construction, $-\ell \in A_i$. Suppose that also $\ell \in A_i$, so that $A_i = \{-\ell, \ldots, x\} \cup \{x+2, \ldots, \ell\}$. Lemma 3.2 shows that

$$B_i \subseteq \{x-(s-1), \ldots, s-1-\ell\} \cap \{-(s-1-\ell), \ldots, x+2+(s-1)\} \subseteq \{-(s-1-\ell), \ldots, s-1-\ell\},$$

and so $B_i \subseteq B_{i+1}$, implying that $|A_i| + |B_i| < 2s$. We conclude that $\ell \notin A_i$, and so $A_i = \{-\ell, \ldots, \ell-1\}$, corresponding to $x = \ell-1$. By construction, $B_i = B_{i+1} \cup \{x-(s-1)\} = \{-(s-\ell), \ldots, s-1-\ell\}$. Both sets are centered at $-1/2$.

The second case, when $|A_{i+1}|$ is even, is similar. In this case for some $1 \le \ell \le s-1$ we have $A_{i+1} = \{-\ell, \ldots, \ell-1\}$ and $B_{i+1} = \{-(s-\ell), \ldots, s-\ell-1\}$. By construction, $-\ell \in A_i$. Suppose that also $\ell-1 \in A_i$, so that $A_i = \{-\ell, \ldots, x\} \cup \{x+2, \ldots, \ell-1\}$. Lemma 3.2 shows that

$$B_i \subseteq \{x-(s-1), \ldots, s-1-\ell\} \cap \{-(s-\ell), \ldots, x+2+(s-1)\} \subseteq \{-(s-\ell), \ldots, s-1-\ell\},$$

and so $B_i \subseteq B_{i+1}$. As before, this leads to a contradiction, and we conclude that $A_i = \{-\ell, \ldots, \ell-2\}$ is centered at -1. Moreover, $B_i = A_i \cup \{\ell-2-(s-1)\} = \{-(s-\ell)-1, \ldots, s-1-\ell\}$ is also centered at -1. This completes the proof.

It remains to consider the case $s = m/2$. The s-agreeing condition states that if $a \in A$ and $b \in B$ then $a - b \ne s$. Thus $B \subseteq \{x : x+s \notin A\}$, which implies $|A| + |B| \le m = 2s$. This bound is tight only when $B = \{x : x+s \notin A\}$. □

As an easy corollary, we can derive the classical Erdős–Ko–Rado theorem.

Corollary 3.4. *Let $n \ge k \ge 1$ be parameters such that $k \le n/2$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting then $|\mathcal{F}| \le \binom{n-1}{k-1}$. Furthermore, if $k < n/2$ and $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F}$ consists of all sets containing some $i \in [n]$.*

Proof. The proof is very similar to the proof of Lemma 3.1. Let π be a random permutation of $[n]$, and choose $t \in [n]$ at random. Let $S_t = \{\pi(t+1), \ldots, \pi(t+k)\}$, where indices are taken modulo n. Since S_t is a random set from $\binom{[n]}{k}$, we see that $|\mathcal{F}|/\binom{n}{k}$ is the probability that $S_t \in \mathcal{F}$.

For any setting of π, let $I = \{t \in [n] : S_t \in \mathcal{F}\}$. Since $\mathcal{F}$ is intersecting, if $a, b \in I$ then $\{\pi(a+1), \ldots, \pi(a+k)\}, \{\pi(b+1), \ldots, \pi(b+k)\}$ must intersect, and this implies that I is k-agreeing (in the sense of Definition 3.1). Lemma 3.3 shows that $|I| \le k$, and so $|\mathcal{F}|/\binom{n}{k} \le k/n$, or $|\mathcal{F}| \le \binom{n-1}{k-1}$.

Suppose now that $k < n/2$ and $|\mathcal{F}| = \binom{n-1}{k-1}$. Lemma 3.3 shows that for every permutation π, the set I must be an interval of length k. In particular, this is the case for the identity permutation. Suppose without loss of generality that in this case, $I = \{1, \ldots, k\}$. Thus $\{2, \ldots, k+1\}, \ldots, \{k+1, \ldots, 2k\} \in \mathcal{F}$ but $\{1, \ldots, k\} \notin \mathcal{F}$ (since $0 \notin I$). Let $S \in \binom{[n]}{k}$ be any set containing $k+1$ but not 1, and write $S = \{k+1\} \cup A \cup B$, where $A \subseteq \{2, \ldots, k\}$. Let π be any permutation which starts $1, \{2, \ldots, k\} \setminus A, A, k+1, B$. Since $\{\pi(1), \ldots, \pi(k)\} = \{1, \ldots, k\} \notin \mathcal{F}$ whereas $\{\pi(2), \ldots, \pi(k+1)\} = \{2, \ldots, k+1\} \in \mathcal{F}$, the set I contains 1 but not 0, and so must be $\{1, \ldots, k\}$. This implies that $S \in \mathcal{F}$.

To finish the proof, let $T \in \binom{[n]}{k}$ be any set not containing $k+1$. Let U be $k-1$ elements disjoint from T and not containing 1 or $k+1$; such elements exist since $n-(k+2) \ge (2k+1)-(k+2) = k-1$. Let π be any permutation starting $T, k+1, U$, where we put 1 first if $1 \in T$. The earlier paragraph shows that $I \supseteq \{1, \ldots, k\}$, and in particular $T \notin \mathcal{F}$. Thus all sets in $\mathcal{F}$ contain $k+1$, and since $|\mathcal{F}| = \binom{n-1}{k-1}$, it must contain all such sets. This completes the proof. □

3.2.2 Continuous setting

We proceed with a continuous analog of Lemma 3.3. First, the pertinent definitions.

Definition 3.3. The *unit circle* $\mathbb{T}$ consists of the interval $[0,1]$ with its two end-points pasted. The distance between two points $x, y \in \mathbb{T}$ is their distance on the unit circle. We denote the Lebesgue measure on $\mathbb{T}$ by μ.

Let $p \leq 1/2$. Two measurable sets $A, B \subseteq \mathbb{T}$ are *p-agreeing* if any $a \in A$ and $b \in B$ are at distance less than p.

We now state and prove the analog of Lemma 3.3. We only state and prove the upper bound part, leaving the identification of optimal sets to the reader.

Lemma 3.5. *Let $p < 1/2$. If two non-empty measurable sets $A, B \subseteq \mathbb{T}$ are p-agreeing then $\mu(A)+\mu(B) \leq 2p$.*

Proof. The proof uses an approximation argument. Let $\epsilon > 0$ be a parameter satisfying $p + \epsilon < 1/2$. Since A is measurable, there is a sequence A_i of intervals of total length smaller than $\mu(A) + \epsilon$ such that $A \subseteq \bigcup_i A_i$. Note that every point in $\bigcup_i A_i$ is at distance at most $\epsilon/2$ from a point in A, since otherwise $\mu(\bigcup_i A_i \setminus A) \geq \epsilon$. Similarly, there is a sequence B_j of intervals of total length at most $\mu(B) + \epsilon$ such that $B \subseteq \bigcup_j B_j$, and any point in $\bigcup_j B_j$ is at distance at most $\epsilon/2$ from a point in B. Thus $\bigcup_i A_i$ and $\bigcup_j B_j$ are cross-$(p + \epsilon)$-agreeing.

Choose I so that $\sum_{i>I} \mu(A_i) < \epsilon$, and set $A^* = \bigcup_{i=1}^{I} A_i$. Similarly, choose J so that $\sum_{j>J} \mu(B_j) < \epsilon$, and set $B^* = \bigcup_{j=1}^{J} B_j$. Thus A^*, B^* are cross-$(p + \epsilon)$-agreeing, and each is a union of finitely many intervals.

Let M be a large integer, and define $A^*_M = \bigcup_{x=0}^{M-1}\{[x/M, (x+1)/M] : [x/M, (x+1)/M] \subseteq A^*\}$. Note that $\mu(A^*_M) \geq \mu(A^*) - 2I/M$. Define B^*_M similarly. We can view A^*_M, B^*_M as subsets of $\mathbb{Z}_M$. These subsets are cross-$\lfloor (p+\epsilon)M \rfloor$-agreeing: if $[a/M, (a + 1)/M] \in A^*_M$ and $[b/M, (b + 1)/M] \in B^*_M$ then $a/M, (b+1)/M$ are at distance less than $p+\epsilon$. Lemma 3.3 thus shows that $\mu(A^*_M)+\mu(B^*_M) \leq 2(p+\epsilon)$. On the other hand, $\mu(A^*_M) + \mu(B^*_M) \geq \mu(A^*) + \mu(B^*) - 2(I + J)/M$. Taking the limit $M \to \infty$, we deduce that $\mu(A^*) + \mu(B^*) \leq 2(p + \epsilon)$. Since $\mu(A^*) + \mu(B^*) \geq \mu(A) + \mu(B) - 2\epsilon$, we conclude that $\mu(A) + \mu(B) \leq 2p + 4\epsilon$. Taking the limit $\epsilon \to 0$, we deduce the lemma. □

As a corollary, we obtain the following useful result.

Corollary 3.6. *Let $\mathcal{F}, \mathcal{G}$ be cross-intersecting families on n points. For any $p \leq 1/2$, $\mu_p(\mathcal{F})+\mu_p(\mathcal{G}) \leq 1$.*

Proof. We first settle the case $p = 1/2$. If $\mathcal{F}, \mathcal{G}$ are cross-intersecting then $\mathcal{G}$ is disjoint from $\{\overline{A} : A \in \mathcal{F}\}$. Since the latter set has measure $\mu_p(\mathcal{F})$, we conclude that $\mu_p(\mathcal{F}) + \mu_p(\mathcal{G}) \leq 1$.

Suppose now that $p < 1/2$. We will follow the argument of Lemma 3.1. We choose n points $x_1, \ldots, x_n$ at random on $\mathbb{T}$, and two starting points $t_\mathcal{F}, t_\mathcal{G} \in \mathbb{T}$ at random. Let S_t be the set of points that lie inside the interval $(t, t + p)$, let $I_\mathcal{F}$ be the set of indices $t_\mathcal{F}$ such that $S_{t_\mathcal{F}} \in \mathcal{F}$, and define $I_\mathcal{G}$ analogously. Thus $\mu_p(\mathcal{F}) + \mu_p(\mathcal{G}) = \mathbb{E}[\mu(I_\mathcal{F}) + \mu(I_\mathcal{G})]$. Since $\mathcal{F}, \mathcal{G}$ are cross-intersecting, we see that $I_\mathcal{F}, I_\mathcal{G}$ are cross-p-agreeing. The lemma shows that $\mu(I_\mathcal{F}) + \mu(I_\mathcal{G}) \leq 2p \leq 1$ if neither $I_\mathcal{F}$ nor $I_\mathcal{G}$ are empty, and $\mu(I_\mathcal{F}) + \mu(I_\mathcal{G}) \leq 1$ trivially holds if one of the sets is empty. The corollary follows. □

We comment that the corollary remains holding if the families in question are on infinitely many points, in the sense of Section 4, with the same proof; the only difference is that we choose infinitely many points rather than just n.

4 Infinite families

Theorem 2.1 and its corollary determine the quantities $w(n, t, p)$ and $w_{\sup}(t, p)$ which are, respectively, the maximum μ_p-measure of a t-intersecting family on n points, and the supremum μ_p-measure of a t-intersecting family on any number of points. In this section we consider what happens when we allow families on infinitely many points.

Definition 4.1. The infinite product measure μ_p on $\mathcal{F} \subseteq 2^\mathbb{N}$ is the infinite product measure extending the finite μ_p measures.

Given $p \in (0, 1)$, A *family on infinitely many points* is a subset $\mathcal{F} \subseteq 2^\mathbb{N}$ which is measurable with respect to μ_p. The family is *t-intersecting* if any two sets $A, B \in \mathcal{F}$ have at least t points in common; here t can also be $\aleph_0$.

A family $\mathcal{F}$ is *finitely determined* if it is the *extension* of some family $\mathcal{G}$ on n points: $\mathcal{F} = \{A : A \cap [n] \in \mathcal{G}\}$. Given $p \in (0,1)$, the family $\mathcal{F}$ is *essentially finitely determined* if it differs from a finitely determined family by a μ_p-null set.

Definition 4.2. For $t \geq 1$ and $p \in (0,1)$, the parameter $w_\infty(t,p)$ is the supremum of $\mu_p(\mathcal{F})$ over all t-intersecting families on infinitely many points.

The following theorem summarizes the results proved in this section.

Theorem 4.1. *Let $t \geq 1$ and $p \in (0,1)$.*

If $p < 1/2$ then $w_\infty(t,p) = w_{\sup}(t,p)$. Furthermore, if $\mathcal{F}$ is a t-intersecting family on infinitely many points satisfying $\mu_p(\mathcal{F}) = w_\infty(t,p)$ then $\mathcal{F}$ is essentially finitely determined.

If $p = 1/2$ then $w_\infty(t,p) = 1/2$. Furthermore, if $t > 1$ then no t-intersecting family on infinitely many points has measure $1/2$.

If $p > 1/2$ then $w_\infty(t,p) = 1$. In fact, there is an $\aleph_0$-intersecting family $\mathcal{F}$ on infinitely many points which satisfies $\mu_p(\mathcal{F}) = 1$ for all $p > 1/2$.

Note that there are many intersecting families with $\mu_{1/2}$-measure $1/2$, for example $\mathcal{F}_{1,r}$ for every r. There are also families which are not essentially finitely determined. One example is the family of sets S such that for some $n \geq 0$,

(a) S contains exactly one of $\{2m+1, 2m+2\}$ for all $m < n$.

(b) S contains both $\{2n+1, 2n+2\}$.

We prove the different cases in the theorem one by one, starting with the case $p < 1/2$. We start with the following technical proposition, which follows from the definition of μ_p.

Proposition 4.2. *Let $\mathcal{F}$ be a family on infinitely many points. For every $p \in (0,1)$ and for every $\epsilon > 0$ there is a finitely determined family $\mathcal{G}$ such that $\mu_p(\mathcal{F} \triangle \mathcal{G}) < \epsilon$.*

Our proof will need a stability version of Theorem 2.1, due to Keller and Lifshitz [16].

Proposition 4.3. *Fix $p \in (0,1/2)$ and $t \geq 1$. If $\mathcal{F}$ is a t-intersecting family on n points of measure $(1-\epsilon)w(n,t,p)$ then there exists a t-intersecting family $\mathcal{G}$ on n points of measure $w(n,t,p)$ such that $\mu_p(\mathcal{F} \triangle \mathcal{G}) = O(\epsilon^{\log_{1-p} p})$. Here the hidden constant depends on p,t but not on n.*

This will allow us to settle the case $p < 1/2$.

Lemma 4.4. *Let $t \geq 1$ and $p \in (0,1/2)$. Then $w_\infty(t,p) = w_{\sup}(t,p)$. Furthermore, if $\mathcal{F}$ is a t-intersecting family on infinitely many points satisfying $\mu_p(\mathcal{F}) = w_\infty(t,p)$ then $\mathcal{F}$ is essentially finitely determined.*

Proof. Clearly $w_\infty(t,p) \geq w_{\sup}(t,p)$. For the other direction, let $\mathcal{F}$ be a t-intersecting family on infinitely many points. Let $\epsilon \in (0,1/4)$ be a parameter. Proposition 4.2 shows that there is a family $\mathcal{G}$ depending on N points such that $\mu_p(\mathcal{F} \triangle \mathcal{G}) < \epsilon$. We can assume that $\mathcal{G}$ depends on the first N points. For $S \subseteq [N]$, let $\mathcal{F}_S = \{A \in 2^{\mathbb{N} \setminus [N]} : S \cup A \in \mathcal{F}\}$.

Let $\operatorname{dist}(x, \{0,1\}) = \min(|x|, |x-1|)$, and notice that

$$\mathop{\mathbb{E}}_{S \sim \mu_p([N])}[\operatorname{dist}(\mu_p(\mathcal{F}_S), \{0,1\})] \leq \mathop{\mathbb{E}}_{S \sim \mu_p([N])}[\mu_p(\mathcal{F}_S \triangle \mathcal{G}_S)] = \mu_p(\mathcal{F} \triangle \mathcal{G}) < \epsilon.$$

Thus with probability at least $1 - \sqrt{\epsilon}$, $\operatorname{dist}(\mu_p(\mathcal{F}_S), \{0,1\}) < \sqrt{\epsilon}$.

Define now $\mathcal{F}' = \{A \in \mathcal{F} : \mu_p(\mathcal{F}_{A \cap [N]}) > 1 - \sqrt{\epsilon}\}$. In words, $\mathcal{F}'$ is the subset of $\mathcal{F}$ obtained by removing all fibers $\mathcal{F}_S$ whose μ_p-measure is at most $1 - \sqrt{\epsilon}$. In particular, we remove all fibers whose μ_p-measure is less than $\sqrt{\epsilon}$, and all fibers whose measure is between $\sqrt{\epsilon}$ and $1 - \sqrt{\epsilon}$. The former have measure at most $\sqrt{\epsilon}$, and the latter also have a measure at most $\sqrt{\epsilon}$ due to the preceding paragraph. Thus $\mu_p(\mathcal{F} \setminus \mathcal{F}') \leq 2\sqrt{\epsilon}$.

If $S, T \subseteq [N]$ are such that $\mu_p(\mathcal{F}_S), \mu_p(\mathcal{F}_T) > 1 - \sqrt{\epsilon} > 1/2$ then $\mathcal{F}_S, \mathcal{F}_T$ cannot be cross-intersecting, due to Corollary 3.6 (while we stated the corollary for finite families, it holds for infinite families with exactly the same proof). It follows that $|S \cap T| \geq t$, and so the family $\mathcal{G} = \{A : \mu_p(\mathcal{F}_{A \cap [N]}) > 1 - \sqrt{\epsilon}\}$

containing $\mathcal{F}'$ is t-intersecting. Since $\mathcal{F}'$ is contained in a t-intersecting family depending on N points, $\mu_p(\mathcal{F}') \le w_{\sup}(t,p)$, and so $\mu_p(\mathcal{F}) \le w_{\sup}(t,p) + 2\sqrt{\epsilon}$. Taking the limit $\epsilon \to 0$, we deduce that $\mu_p(\mathcal{F}) \le w_{\sup}(t,p)$, and so $w_\infty(t,p) = w_{\sup}(t,p)$.

Suppose now that $\mu_p(\mathcal{F}) = w_{\sup}(t,p)$. For every $\epsilon > 0$, we have constructed above a t-intersecting family $\mathcal{F}' = \mathcal{F}'_\epsilon$ which is contained in a t-intersecting family $\mathcal{G}_\epsilon$ depending on N_ϵ points and satisfies $\mu_p(\mathcal{F}'_\epsilon) \ge \mu_p(\mathcal{F}) - 2\sqrt{\epsilon}$. In particular, $\mu_p(\mathcal{G}_\epsilon) \ge \mu_p(\mathcal{F}) - 2\sqrt{\epsilon}$. For small enough $\epsilon > 0$, this can only happen if $w(N_\epsilon, t, p) = w_{\sup}(t,p)$. Furthermore, Proposition 4.3 shows that $\mu_p(\mathcal{G}_\epsilon \triangle \mathcal{H}_\epsilon) = O(\epsilon^{O(1)})$ for some t-intersecting family $\mathcal{H}_\epsilon$ on N points of μ_p-measure $w_{\sup}(t,p)$. Corollary 2.2 shows that such a family is a Frankl family with appropriate parameters. In particular, $\mathcal{H}_\epsilon$ depends on a constant number of coordinates (depending on t, p).

Notice that $\mu_p(\mathcal{G}_\epsilon \setminus \mathcal{F}'_\epsilon) \le \sqrt{\epsilon}$ (since each fiber we retained had μ_p-measure at least $1 - \sqrt{\epsilon}$), and so $\mu_p(\mathcal{G}_\epsilon \triangle \mathcal{F}) = O(\sqrt{\epsilon})$. Thus $\mu_p(\mathcal{F} \triangle \mathcal{H}_\epsilon) = O(\epsilon^{O(1)})$. In particular, if $\epsilon_1, \epsilon_2 \le \epsilon$ we get $\mu_p(\mathcal{H}_{\epsilon_1} \triangle \mathcal{H}_{\epsilon_2}) = O(\epsilon^{O(1)})$. Since $\mathcal{H}_{\epsilon_1} \triangle \mathcal{H}_{\epsilon_2}$ depends on a constant number of coordinates, for small enough ϵ this forces $\mathcal{H}_{\epsilon_1} = \mathcal{H}_{\epsilon_2}$. There is therefore a Frankl family $\mathcal{H}$ satisfying $\mu_p(\mathcal{F} \triangle \mathcal{H}) = O(\epsilon^{O(1)})$ for all small enough $\epsilon > 0$. Taking the limit $\epsilon \to 0$ concludes the proof. □

We now consider the case $p = 1/2$. We thank Shay Moran for help with the proof of the following lemma.

Lemma 4.5. *For all $t \ge 1$, $w_\infty(t, 1/2) = 1/2$. Furthermore, if $t \ge 2$ then no t-intersecting family on infinitely many points has μ_p-measure $1/2$.*

Proof. If $\mathcal{F}$ is an intersecting family on infinitely many points then $\mathcal{F}$ is disjoint from $\{\overline{A} : A \in \mathcal{F}\}$. Since both families have the same measure, it follows that $\mu_p(\mathcal{F}) \le 1/2$. Thus $w_\infty(t, 1/2) \le 1/2$. On the other hand, clearly $w_\infty(t, 1/2) \ge w_{\sup}(t, 1/2)$, and so $w_\infty(t, 1/2) = 1/2$ due to Corollary 2.2.

Suppose now that $t \ge 2$ and that $\mathcal{F}$ is a t-intersecting family on infinitely many points of $\mu_{1/2}$-measure $1/2$. By possibly taking the up-set of $\mathcal{F}$, we can assume that $\mathcal{F}$ is monotone. Let $\mathcal{F}_- = \{A \in 2^{\mathbb{N} \setminus \{1\}} : A \in \mathcal{F}\}$ and $\mathcal{F}_+ = \{A \in 2^{\mathbb{N} \setminus \{1\}} : A \cup \{1\} \in \mathcal{F}\}$. Since $\mathcal{F}$ is monotone, $\mathcal{F}_- \subseteq \mathcal{F}_+$. Since $t \ge 2$, $\mathcal{F}_+$ is intersecting, and so $\mu_{1/2}(\mathcal{F}_+) \le 1/2$. It follows that $\mu_{1/2}(\mathcal{F}_+) = \mu_{1/2}(\mathcal{F}_-) = 1/2$ as well, and so $\mathcal{F}_1 = \{A \in 2^{\mathbb{N}} : A \cap (\mathbb{N} \setminus \{1\}) \in \mathcal{F}_-\}$ also has $\mu_{1/2}$-measure $1/2$. Note that $\mathcal{F}_1$ does not depend on 1. Note also that $\mathcal{F}_1 \subseteq \mathcal{F}$ and that $\mathcal{F}_1$ is t-intersecting (since $\mathcal{F}_-$ is).

Starting with $\mathcal{F}_1$ but working with the element 2 instead of 1, construct a t-intersecting family $\mathcal{F}_2 \subseteq \mathcal{F}_1$ of $\mu_{1/2}$-measure $1/2$, and note that $\mathcal{F}_2$ depends on neither 1 nor 2. Continuing in this way, we construct a sequence of families $\mathcal{F}_n$ such that $\mathcal{F}_n \subseteq \mathcal{F}_{n-1}$ is a t-intersecting family of $\mu_{1/2}$-measure $1/2$ that does not depend on $1, \ldots, n$. Thus $\mathcal{F}' = \bigcap_{n \in \mathbb{N}} \mathcal{F}_n$ is a t-intersecting family of $\mu_{1/2}$-measure $1/2$ which represents a tail event. However, such a family must have measure 0 or 1 due to Kolmogorov's zero-one law. This contradiction shows that the original family $\mathcal{F}$ could not have existed. □

Finally, we dispense of the case $p > 1/2$.

Lemma 4.6. *There exists a family $\mathcal{F}$ on infinitely many points which is $\aleph_0$-intersecting and satisfies $\mu_p(\mathcal{F}) = 1$ for all $p \in (1/2, 1)$.*

Proof. Let $f(r)$ be any integer function which is $\omega(1)$ and $o(r)$, for example $\lfloor \sqrt{r} \rfloor$. We define $\mathcal{F}$ to consist of all sets A such that $|A \cap [2r + f(r)]| \ge r + f(r)$ for all large enough r. For any $A, B \in \mathcal{F}$ it holds that for large enough r, $|A \cap B \cap [2r + f(r)]| \ge 2(r + f(r)) - (2r + f(r)) = f(r)$, and so the intersection $A \cap B$ is infinite.

Hoeffding's bound states (in one version) that $\Pr[\mathrm{Bin}(n,p) \le qn] \le e^{-2n(p-q)^2}$. This implies that for $A \sim \mu_p$,

$$\Pr[|A \cap [2r + f(r)]| < r + f(r)] \le \exp\left[-2(2r + f(r))\left(p - \frac{r + f(r)}{2r + f(r)}\right)^2\right].$$

Since $(r + f(r))/(2r + f(r)) \to 1/2$, there exists $\epsilon > 0$ such that for large enough r we have $\Pr[|A \cap [2r + f(r)]| < r + f(r)] \le e^{-4r\epsilon^2}$. This shows that $\sum_r \Pr[|A \cap [2r + f(r)]| < r + f(r)]$ converges. The Borell–Cantelli lemma thus shows that almost surely, only finitely many of these "bad events" happen, and so $\mu_p(\mathcal{F}) = 1$. □

5 Agreeing families

Chung, Frankl, Graham, and Shearer [5] considered the difference between *intersecting* and *agreeing* families. Let us say that a family $\mathcal{F}$ on n points is $\mathcal{G}$*-intersecting* if any $A, B \in \mathcal{F}$ satisfy $A \cap B \in \mathcal{G}$, where $\mathcal{G}$ is some monotone family on n points. Thus t-intersecting families are $\mathcal{G}$-intersecting for $\mathcal{G} = \{S \subseteq [n] : |S| \geq t\}$. A family $\mathcal{F}$ is $\mathcal{G}$*-agreeing* if any $A, B \in \mathcal{F}$ satisfy $\overline{A \triangle B} \in \mathcal{G}$. Since $\overline{A \triangle B} = (A \cap B) \cup (\overline{A} \cap \overline{B})$, every $\mathcal{G}$-intersecting family is a fortiori $\mathcal{G}$-agreeing, but the converse does not hold in general. Nevertheless, Chung et al. showed that the maximum $\mu_{1/2}$-measure of a $\mathcal{G}$-agreeing family is the same as the maximum $\mu_{1/2}$-measure of a $\mathcal{G}$-intersecting family, using a simple shifting argument. This explains why many arguments, such as ones using Shearer's lemma, seem to work not only for $\mathcal{G}$-intersecting families but also for $\mathcal{G}$-agreeing families. The latter are easier to work with since the definition is more symmetric, and this is taken to full advantage in [8], for example.

Ahlswede and Khachatrian [2], motivated by the geometry of Hamming spaces, considered a more general question: what is the largest subset of $\mathbb{Z}_m^n$ of diameter $n - t$ with respect to Hamming distance? Two vectors $x, y \in \mathbb{Z}_m^n$ have Hamming distance $n - t$ if they agree on exactly t coordinates. Thus a subset of diameter $n - t$ is the same as a collection of vectors, every two of which agree on at least t coordinates. They showed that this corresponds (roughly) to μ_p for $p = 1/m$ (though they stated this in a different language).

Motivated by the recent success of Fourier-analytic techniques to analyze questions in extremal combinatorics, Alon, Dinur, Friedgut and Sudakov [4] studied independent sets in graph products, and in particular rederived the results of [2] for the case $t = 1$. Shinkar [19] extended this to general t and $m \geq t + 1$, using methods of Friedgut [13]. Unfortunately, these spectral techniques cannot at the moment yield all results of [2].

Any bound on t-agreeing families of vectors in $\mathbb{Z}_m^n$ readily yields matching results on the $\mu_{1/m}$-measure of t-intersecting families on n points. Indeed, given a family $\mathcal{F} \subseteq \{0,1\}^n$, let $\mathcal{G} \subseteq \mathbb{Z}_m^n$ consist of all vectors obtained by taking each binary vector in $\mathcal{F}$ and replacing each 0 coordinate with one of the $m - 1$ values $\{2, \ldots, m\}$. The resulting family is t-agreeing, and $|\mathcal{G}| = m^n \mu_{1/m}(\mathcal{F})$. Therefore a bound on $|\mathcal{G}|$ yields a bound on $\mu_{1/m}(\mathcal{F})$. In other words, t-agreeing families give a discrete model for $\mu_{1/m}$.

We extend these results in Section 5.1, by giving a discrete model for $\mu_{s/m}$ for any integers s, m satisfying $s/m < 1/2$. Extending the work of [2], we prove a complete intersection theorem in this setting. Section 5.2 describes a similar continuous model for μ_p for any $p < 1/2$, and proves a complete intersection theorem in that setting.

5.1 Discrete setting

We start by defining the new discrete model.

Definition 5.1. A *family on* $\mathbb{Z}_m^n$ is a subset of $\mathbb{Z}_m^n$. For $s \leq m/2$, a family $\mathcal{F}$ on $\mathbb{Z}_m^n$ is t*-agreeing up to* s if every $x, y \in \mathcal{F}$ have t coordinates $i_1, \ldots, i_t$ such that $x_{i_j} - y_{i_j} \in \{-(s-1), \ldots, s-1\}$ for $1 \leq j \leq t$.

The uniform measure on $\mathbb{Z}_m$ is denoted by μ_m, and the uniform measure on $\mathbb{Z}_m^n$ by μ_m^n.

The maximum measure of a t-agreeing up to s family on $\mathbb{Z}_m^n$ is denoted $w(\mathbb{Z}_m^n, t, s)$.

Our goal is to show that $w(\mathbb{Z}_m^n, t, s) = w(n, t, s/m)$ whenever $s/m \leq 1/2$, and to identify the optimal families when possible. Our candidate optimal families are obtained in the following way.

Definition 5.2. Fix $n, m \geq 1$ and $s \leq m/2$. For $y \in \mathbb{Z}_m^n$, the mapping $\sigma_y \colon \mathbb{Z}_m^n \to 2^{[n]}$ is given by $\sigma_y(x) = \{i \in [n] : 1 \leq x_i - y_i \leq s\}$.

If $\mathcal{F}$ is a family on $\mathbb{Z}_m^n$ and $\mathcal{G}$ is a family on n points then $\mathcal{F} \approx \mathcal{G}$ (or, $\mathcal{F}$ is *equivalent* to $\mathcal{G}$) if $\mathcal{F} = \sigma_y^{-1}(\mathcal{G})$ for some vector $y \in \mathbb{Z}_m^n$. In words, $\mathcal{F}$ arises from $\mathcal{G}$ by going through all sets $S \in \mathcal{F}$, replacing each $i \in S$ by all values in $\{y_1 + 1, \ldots, y_1 + s\}$, and each $i \notin S$ by all other values.

We say that $\mathcal{F}$ is equivalent to a (t, r)-Frankl family if $\mathcal{F} \approx \mathcal{G}$ for some family $\mathcal{G}$ equivalent to a (t, r)-Frankl family (see Definition 2.3).

We will prove the following theorem.

Theorem 5.1. *Let* $n, m, t \geq 1$ *and* $s \leq m/2$. *Then* $w(\mathbb{Z}_m^n, t, s) = w(n, t, s/m)$. *Furthermore, if* $s < m/2$ *and* $\mathcal{F}$ *is a family on* $\mathbb{Z}_m^n$ *of measure* $w(n, t, s/m)$ *then* $\mathcal{F}$ *is equivalent to a* t*-intersecting family on* n *points of* $\mu_{s/m}$*-measure* $w(n, t, s/m)$. *The same holds if* $m = 2$, $s = 1$ *and* $t > 1$.

When $s = m/2$ and $s > 1$ there can be exotic families of maximum measure. For example, when $s = 2$ and $m = 4$ the family $\{00, 01, 12, 13, 20, 21, 32, 33\}$ is intersecting up to 2 but doesn't arise from any family on two points. Adding another coordinate ranging over $\{0, 1\}$, we get a 2-intersecting up to 2 family which doesn't arise from any family on three points.

When $m = 2$, $s = 1$ and $t = 1$, a maximum measure family is one which contains exactly one vector of each pair of complementary vectors.

On the way toward proving the theorem, we will need to consider hybrid families in which some of the coordinates come from $\mathbb{Z}_m$, and others from $\{0, 1\}$.

Definition 5.3. A *family on* $\mathbb{Z}_m^n \times \{0,1\}^\ell$ is a subset of $\mathbb{Z}_m^n \times \{0,1\}^\ell$. Such a family $\mathcal{F}$ is *t-agreeing up to s* if every $x, y \in \mathcal{F}$ have t coordinates $i_1, \ldots, i_t$ such that $x_{i_j} - y_{i_j} \in \{-(s-1), \ldots, s-1\}$ (if $i_j \le n$) or $x_{i_j} = y_{i_j} = 1$ (if $i_j > n$) for $1 \le j \le t$.

More generally, two vectors $x, y \in \mathbb{Z}_m^n \times \{0,1\}^\ell$ s-agree on a coordinate $i \le n$ if $x_i - y_i \in \{-(s-1), \ldots, s-1\}$, and they s-agree on a coordinate $i > n$ if $x_i = y_i = 1$. Thus a family on $\mathbb{Z}_m^n \times \{0,1\}^\ell$ is t-agreeing up to s if every $x, y \in \mathcal{F}$ s-agree on at least t coordinates.

We measure families on $\mathbb{Z}_m^n \times \{0,1\}^\ell$ using the product measures $\mu_{s,m}^{n,\ell} = \mu_m^n \times \mu_{s/m}^\ell$.

For $y \in \mathbb{Z}_m^n$, the mapping $\sigma_y \colon \mathbb{Z}_m^n \times \{0,1\}^\ell \to 2^{[n+\ell]}$ is the product of σ_y and the identity mapping.

If $\mathcal{F}$ is a family on $\mathbb{Z}_m^n \times \{0,1\}^\ell$ and $\mathcal{G}$ is a family on $n + \ell$ points then $\mathcal{F} \approx \mathcal{G}$ if $\mathcal{F} = \sigma_y^{-1}(\mathcal{G})$ for some vector $y \in \mathbb{Z}_m^n$. In words, $\mathcal{F}$ arises from $\mathcal{G}$ by going through all sets $S \in \mathcal{F}$, and for each $i \le n$, replacing each $i \in S$ by all values in $\{y_1 + 1, \ldots, y_1 + s\}$, and each $i \notin S$ by all other values.

We start by proving $w(\mathbb{Z}_m^n, t, s) = w(n, t, s/m)$. The identification of optimal families will require further refining the proof, but we present the two proofs separately for clarity. We start with a technical result about the stable set polytope.

Proposition 5.2. *Let $G = (V, E)$ be a graph, and let $\mu \colon V \to \mathbb{R}$ be a set of weights. Consider the program*

$$\begin{array}{lll} \max & \sum_{x \in V} \mu_x v_x & \\ \text{s.t.} & 0 \le v_x \le 1 & \text{for all } x \in V \\ & v_x + v_y \le 1 & \text{for all } (x, y) \in E \end{array}$$

The maximum of the program is attained (not necessarily uniquely) at some half-integral point (a point in which all entries are $\{0, 1/2, 1\}$).

Moreover, if there is a unique half-integral point at which the objective is maximized, then this point is the unique maximum.

Proof. The stable set polytope of G is defined as the set of all V-indexed vectors which satisfy the constraints stated in the program. Every linear functional over the polytope is maximized at some vertex. Moreover, if there is a unique vertex at which the maximum is attained then this vertex is the unique maximum. Therefore the proposition follows from the fact that each vertex of the polytope is half-integral. This classical fact is due to Nemhauser and Trotter [17]. We briefly sketch the proof below, following Schrijver [18, Theorem 64.7].

Let v be a vertex of the stable set polytope. Let $I = \{x : 0 < v_x < 1/2\}$ and $J = \{y : 1/2 < v_y < 1\}$. For small enough $\epsilon > 0$, both $v + \epsilon(1_I - 1_J)$ and $v - \epsilon(1_I - 1_J)$ are in the stable set polytope (where 1_I is the characteristic function of I), and so v can only be a vertex if $I = J = \emptyset$. □

The heart of the proof of $w(\mathbb{Z}_m^n, t, s) = w(n, t, s/m)$ is the following lemma.

Lemma 5.3. *Let $\mathcal{F}$ be a family on $\mathbb{Z}_m^n \times \{0,1\}^\ell$ which is t-agreeing up to s, where $s \le m/2$ and $n \ge 1$ (but possibly $\ell = 0$). There is a family $\mathcal{H}$ on $\mathbb{Z}_m^{n-1} \times \{0,1\}^{\ell+1}$ which is t-agreeing up to s and satisfies $\mu_{s,m}^{n-1,\ell+1}(\mathcal{H}) \ge \mu_{s,m}^{n,\ell}(\mathcal{F})$.*

Proof. The first step is to decompose $\mathcal{F}$ according to the values of all coordinates but the n'th one:

$$\mathcal{F} = \bigcup_{x_1 \in \mathbb{Z}_m^{n-1}} \bigcup_{x_2 \in \{0,1\}^\ell} \{x_1\} \times \mathcal{F}_{x_1, x_2} \times \{x_2\}.$$

It will be convenient to refer to the pair x_1, x_2 as a single vector $x = (x_1, x_2) \in \mathbb{Z}_m^{n-1} \times \{0,1\}^\ell$.

We now construct a graph $G = (V, E)$ as follows. The vertices are $V = \{x \in \mathbb{Z}_m^{n-1} \times \{0,1\}^\ell : \mathcal{F}_x \neq \emptyset\}$. We connect two vertices x, y (possibly $x = y$) if $\{x, y\}$ is *not* t-agreeing up to s; in other words, if the vectors x, y s-agree on exactly $t-1$ coordinates. If $(x, y) \in E$ then the sets $\mathcal{F}_x, \mathcal{F}_y \subseteq \mathbb{Z}_m$ must be s-agreeing, in the terminology of Definition 3.1.

There are now two cases to consider: the degenerate case $n + \ell = t$, and the non-degenerate case $n + \ell > t$. In the degenerate case the graph G consists of isolated vertices with self-loops, since the only way that vectors x, y of length $t-1$ can s-agree on $t-1$ coordinates is if $x = y$. Thus each set $\mathcal{F}_x$ must be s-agreeing, and so $|\mathcal{F}_x| \le s$ by Lemma 3.3. We form the new family $\mathcal{H}$ as follows:

$$\mathcal{H} = \bigcup_{x_1 \in \mathbb{Z}_m^{n-1}} \bigcup_{x_2 \in \{0,1\}^\ell} \{x_1\} \times \mathcal{H}_{x_1,x_2} \times \{x_2\}, \quad \mathcal{H}_x = \begin{cases} \{1\} & \text{if } \mathcal{F}_x \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

By construction, $\mu_{s,m}^{1,0}(\mathcal{F}_x) \le \mu_{s,m}^{0,1}(\mathcal{H}_x)$: either both sets are empty, or $\mathcal{F}_x$ has measure at most s/m, while $\mathcal{H}_x$ has measure s/m. Thus $\mu_{s,m}^{n,\ell}(\mathcal{F}) \le \mu_{s,m}^{n-1,\ell+1}(\mathcal{H})$. Moreover, it is not hard to verify that $\mathcal{H}$ is t-agreeing up to s. This completes the proof in the degenerate case.

The proof in the non-degenerate case is more complicated. First, notice that we no longer have self-loops, since every $x \in V$ agrees with itself on $n - 1 + \ell \ge t$ coordinates. If $(x, y) \in E$ then the non-empty sets $\mathcal{F}_x, \mathcal{F}_y$ are s-agreeing, and so $|\mathcal{F}_x| + |\mathcal{F}_y| \le 2s$ by Lemma 3.3. In particular, if $|\mathcal{F}_x| \ge 2s$ then x must be an isolated vertex. This suggests pruning the graph by removing all isolated vertices. We denote the resulting graph by $G' = (V', E')$; note that V' could be empty. All vertices $x \in V'$ now satisfy $0 < |\mathcal{F}_x| < 2s$.

Let $v_x = |\mathcal{F}_x|/(2s)$. Then $0 \le v_x \le 1$, $v_x + v_y \le 1$ for all edges, and

$$\mu_{s,m}^{n,\ell}(\mathcal{F}) = \sum_{x \in V \setminus V'} \mu_{s,m}^{n-1,\ell}(x)\mu_m(\mathcal{F}_x) + \frac{2s}{m} \sum_{x \in V'} \mu_{s,m}^{n-1,\ell}(x) v_x.$$

Proposition 5.2 states that there exists a $\{0, 1/2, 1\}$-valued vector w which satisfies $0 \le w_x \le 1$, $w_x + w_y \le 1$ for all edges, and $\sum_{x \in V'} \mu_{s,m}^{n-1,\ell}(x) v_x \le \sum_{x \in V'} \mu_{s,m}^{n-1,\ell}(x) w_x$. We will construct $\mathcal{H}$ according to this vector:

$$\mathcal{H} = \bigcup_{x_1 \in \mathbb{Z}_m^{n-1}} \bigcup_{x_2 \in \{0,1\}^\ell} \{x_1\} \times \mathcal{H}_{x_1,x_2} \times \{x_2\}, \quad \mathcal{H}_x = \begin{cases} \{0,1\} & \text{if } x \in V \setminus V', \\ \{0,1\} & \text{if } x \in V' \text{ and } w_x = 1, \\ \{1\} & \text{if } x \in V' \text{ and } w_x = 1/2, \\ \emptyset & \text{if } x \in V' \text{ and } w_x = 0, \\ \emptyset & \text{if } x \notin V'. \end{cases}$$

Note that the *support* of $\mathcal{H}$, which is the set of $x \in \mathbb{Z}_m^{n-1} \times \{0,1\}^\ell$ such that $\mathcal{H}_x \neq \emptyset$, is a subset of the support of $\mathcal{F}$.

We start by showing that $\mathcal{H}$ is t-agreeing up to s. Let $(x_1, \alpha, x_2), (y_1, \beta, y_2) \in \mathcal{H}$, where $x_1, y_1 \in \mathbb{Z}_m^{n-1}$, $\alpha, \beta \in \{0, 1\}$, and $x_2, y_2 \in \{0,1\}^\ell$. If there is no edge between (x_1, x_2) and (y_1, y_2) then clearly $(x_1, \alpha, x_2), (y_1, \beta, y_2)$ s-agree on at least t coordinates. If there does exist an edge then $x = (x_1, x_2)$ and $y = (y_1, y_2)$ agree on $t-1$ coordinates, and so it suffices to show that $\alpha = \beta = 1$. By construction, $0 \in \mathcal{H}_x$ if either $x \in V \setminus V'$ or $w_x = 1$. In the former case, since the support of $\mathcal{H}$ is contained in the support of $\mathcal{F}$, x is isolated, and so there cannot be an edge (x, y). In the latter case, $w_y = 0$ (since $w_x + w_y \le 1$ for all edges), and so y is not in the support of $\mathcal{H}$. We conclude that $\mathcal{H}$ is t-agreeing up to s.

We proceed by comparing the measures of $\mathcal{F}$ and $\mathcal{H}$:

$$\begin{aligned} \mu_{s,m}^{n-1,\ell+1}(\mathcal{H}) &= \sum_{x \in \mathbb{Z}_m^{n-1} \times \{0,1\}^\ell} \mu_{s,m}^{n-1,\ell}(x) \mu_{s/m}(\mathcal{H}_x) \\ &= \sum_{x \in V \setminus V'} \mu_{s,m}^{n-1,\ell}(x) + \sum_{\substack{x \in V' \\ w_x = 1}} \mu_{s,m}^{n-1,\ell}(x) + \frac{s}{m} \sum_{\substack{x \in V' \\ w_x = 1/2}} \mu_{s,m}^{n-1,\ell}(x). \end{aligned}$$

The first term is at least as large as $\sum_{x \in V \setminus V'} \mu_{s,m}^{n-1,\ell}(x)\mu_m(\mathcal{F}_x)$, and the other two terms are at least

$(2s/m)\sum_{x\in V'}\mu_{s,m}^{n-1,\ell}(x)w_x$. Therefore

$$\begin{aligned}\mu_{s,m}^{n-1,\ell+1}(\mathcal{H}) &\geq \sum_{x\in V\setminus V'}\mu_{s,m}^{n-1,\ell}(x)\mu_m(\mathcal{F}_x)+\frac{2s}{m}\sum_{x\in V'}\sum_{x\in V'}\mu_{s,m}^{n-1,\ell}(x)w_x\\ &\geq \sum_{x\in V\setminus V'}\mu_{s,m}^{n-1,\ell}(x)\mu_m(\mathcal{F}_x)+\frac{2s}{m}\sum_{x\in V'}\sum_{x\in V'}\mu_{s,m}^{n-1,\ell}(x)v_x\\ &= \mu_{s,m}^{n-1,\ell+1}(\mathcal{F}).\end{aligned}$$

This completes the proof.

We note that when $(s,m)=(1,2)$, necessarily $v_x=1/2$ for all $x\in V'$ (since $0<2v_x<2$ is an integer), and so we can use $w=v$. ☐

As a corollary, we can deduce the part $w(\mathbb{Z}_m^n,t,s)=w(n,t,s/m)$ of Theorem 5.1.

Lemma 5.4. *Let $n\geq t\geq 1$, $m\geq 2$ and $s\leq m/2$. Then $w(\mathbb{Z}_m^n,t,s)=w(n,t,s/m)$.*

Proof. Let $\mathcal{F}$ be a family on $\mathbb{Z}_m^n$ which is t-intersecting up to s. Applying Lemma 5.3 n times, we obtain a t-intersecting family $\mathcal{H}$ on $\{0,1\}^n$ satisfying $\mu_m(\mathcal{F})\leq\mu_{s/m}(\mathcal{H})\leq w(n,t,s/m)$. This shows that $w(\mathbb{Z}_m^n,t,s)\leq w(n,t,s/m)$.

For the other direction, let $\mathcal{H}$ by a t-intersecting family on n points satisfying $\mu_{s,m}(\mathcal{H})=w(n,t,s/m)$. Define $\mathcal{F}=\sigma_0^{-1}(\mathcal{H})$, where 0 is the zero vector. It is straightforward to verify that $\mathcal{F}$ is t-agreeing up to s and that $\mu_m(\mathcal{F})=\mu_{s/m}(\mathcal{H})=w(n,t,s/m)$. It follows that $w(\mathbb{Z}_m^n,t,s)\geq w(n,t,s/m)$. ☐

We proceed with the characterization of families achieving the bound $w(\mathbb{Z}_m^n,t,s)$. We start with the case $s/m<1/2$. The idea is to prove a stronger version of Lemma 5.3 which states that if $\mathcal{F}$ and $\mathcal{H}$ have the same measure and $\mathcal{H}$ is equivalent to a Frankl family then so is $\mathcal{F}$. This suggests analyzing the graph $G=(V,E)$ constructed in the proof of Lemma 5.3 in the case of a family equivalent to a Frankl family.

Lemma 5.5. *Let $n,t\geq 1$ and ℓ,s,m be given, and suppose that $s<m/2$. Let $\mathcal{F}$ be a family on $\mathbb{Z}_m^n\times\{0,1\}^\ell$ which is equivalent to some (t,r)-Frankl family, for some $r\geq 0$. The graph $G'=(V',E')$ constructed in the proof of Lemma 5.3 is either empty (has no vertices) or is connected and non-bipartite.*

The same holds when $s=1$ and $m=2$.

Proof. Fix m,t,r. We first consider the case $s<m/2$.

We will first prove the lemma for $n=1$ and $\ell=t+2r-1$. Then we will prove it for $n=1$ and arbitrary ℓ. Finally, we will tackle the general case.

Recall that the graph G' was constructed by first constructing a larger graph $G=(V,E)$ and then removing all isolated vertices (vertices having no edges). We will assume throughout that $\mathcal{F}$ depends on the nth coordinate (that is, the underlying Frankl family has n in its support), since otherwise G has no edges and so G' is empty.

We will prove one more property of G', which we call *property Z*. This property states that if $x\in V'$ and $x_i=0$ for $i>n-1$ (that is, i is a coordinate taking values in $\{0,1\}$) then x is connected to some $y\in V'$ with $y_i=1$.

We start with the case $(n,\ell)=(1,t+2r-1)$. If $r=0$ then G consists of a single vertex $[t-1]$ with a self-loop (we identify zero-one vectors with sets), and so it is connected and non-bipartite. If $r>0$ then $V=\binom{[t+2r-1]}{\geq t+r-1}$. If $A\in\binom{[t+2r-1]}{\geq t+r-1}$ and $B\in\binom{[t+2r-1]}{\geq t+r}$ then $|A\cap B|\geq(t+r-1)+(t+r)-(t+2r-1)=t$, and so $V'\subseteq\binom{[t+2r-1]}{t+r-1}$; symmetry dictates that $V'=\binom{[t+2r-1]}{t+r-1}$. Moreover, the sequence

$$\{1,\ldots,t-1,t,\ldots,t+r-1\},\{1,\ldots,t-1,t+r,\ldots,t+2r-1\},\{2,\ldots,t-1,t,\ldots,t+r\}$$

corresponds to a path connecting $\{1,\ldots,t+r-1\}$ and $\{2,\ldots,t+r\}$. This shows that G' is connected. Since the path has even length, we conclude that there is a path of even length connecting any two vertices. In particular, there is a path of even length connecting $\{1,\ldots,t-1,t,\ldots,t+r-1\}$ and $\{1,\ldots,t-1,t+r,\ldots,t+2r-1\}$, and together with the corresponding edge, we obtain an odd cycle. This shows that G' is non-bipartite.

To prove property Z, we consider two cases. When $r = 0$, property Z holds vacuously, since the only vertex contains no zero coordinates. When $r > 0$, we can assume without loss of generality that $x = \{2, \ldots, t + r\}$ and $i = 1$. In that case, x is connected to $y = \{1, \ldots, t - 1, t + r, \ldots, t + 2r - 1\}$.

We now prove by induction the case $n = 1$ and $\ell \geq t+2r-1$. Let G_ℓ, G'_ℓ be the graphs corresponding to a particular value of ℓ, where we assume without loss of generality that $\mathcal{F}$ depends on the first $t + 2r$ points (we can make this assumption since $\mathcal{F}$ has to depend on the first point). Suppose that we have shown that G'_ℓ is connected and non-bipartite. Notice that $V_{\ell+1} = V \times \{0, 1\}$ and $E_{\ell+1} = \{((x, i), (y, j)) : (x, y) \in E_\ell, (i, j) \neq (1, 1)\}$. This shows that $V'_{\ell+1} = V'_\ell \times \{0, 1\}$. Since G' is connected and non-bipartite, there is an even-length path connecting any two $x, y \in V'$, which lifts to even-length paths connecting $(x, 0), (y, 0)$ and $(x, 1), (y, 1)$ (flipping the extra coordinate at each step); and an odd-length path connecting $x, y \in V'$ which lifts to paths connecting $(x, 0), (y, 1)$ and $(x, 1), (y, 0)$. This shows that $G'_{\ell+1}$ is connected. An odd-length path from x to itself lifts to an odd-length path from $(x, 0)$ to itself (flipping the extra coordinate at each step but the first), showing that G' is non-bipartite.

To prove property Z, let $x \in V'_{\ell+1}$ have $x_i = 0$. If $i < \ell+1$ then property Z for G_ℓ implies the existence of a neighbor y with $y_i = 1$. If $i = \ell + 1$ then x is connected to some y (since x is not isolated), and since $V'_{\ell+1}$ is independent of the last coordinate, we can assume that $y_i = 1$ (the value of this coordinate does not change the number of coordinates on which x and y agree).

Suppose now that the lemma holds for families on $\mathbb{Z}_m^n \times \{0, 1\}^\ell$. We will show that it holds for families on $\mathbb{Z}_m^{n+1} \times \{0, 1\}^{\ell-1}$ as well. We denote the relevant graphs G_n, G'_n and G_{n+1}, G'_{n+1}, respectively. Without loss of generality we can assume that

$$V_{n+1} = \bigcup_{x_1 \in \mathbb{Z}_m^{n-1}} \bigcup_{x_2 \in \{0,1\}^{\ell-1}} \{x_1\} \times \sigma_0^{-1}(\{x' : (x_1, x', x_2) \in V_n\}) \times \{x_2\}.$$

In words, V_{n+1} is obtained by applying σ_0^{-1} to the n'th coordinate.

We claim that V'_{n+1} is obtained from V'_n in the same way, showing that property Z holds for G_{n+1}. Indeed, suppose that $(x_1, x', x_2) \in V'_n$ is not isolated, say it is connected to $(y_1, y', y_2) \in V'_n$, and let $x'' \in \sigma_0^{-1}(x')$. If $x' = y' = 1$ then (x_1, x'', x_2) is connected to (y_1, x'', y_2). If $x' = 1$ and $y' = 0$ then (x_1, x'', x_2) is connected to $(y_1, x'' + s, y_2)$ (note $s + 1 \leq x'' + s \leq 2s$). If $x' = 0$ then property Z allows us to assume that $y' = 1$, and so (x_1, x'', x_2) is connected to (y_1, y'', y_2) for some $1 \leq y'' \leq s$; indeed, Lemma 3.2 shows that the only elements s-agreeing with all of $1, \ldots, s$ are $1, \ldots, s$.

Toward showing that V'_{n+1} is connected, we prove first that $(x_1, a, x_2) \in V'_{n+1}$ is connected to $(x_1, b, x_2) \in V'_{n+1}$, by an even-length path, whenever $\sigma_0(a) = \sigma_0(b)$. Suppose first that $\sigma_0(a) = \sigma_0(b) = 1$. Let $(y_1, y', y_2) \in V'_n$ be a neighbor of $(x_1, 1, x_2)$ in G'_n. If $\sigma_0(y') = 1$ then $(x_1, a, x_2), (y_1, a, y_2), (x_1, b, x_2)$ is a path in G'_{n+1}. If $\sigma_0(y) = 0$ then consider the following path in G'_{n+1}:

$$(x_1, 1, x_2), (y_1, s + 2, y_2), (x_1, 2, x_2), (y_1, s + 3, y_2), \ldots, (x_1, s - 1, x_2), (y_1, 2s, y_2), (x_1, s, x_2).$$

This contains a sub-path connecting (x_1, a, x_2) and (x_1, b, x_2).

Suppose next that $\sigma_0(a) = \sigma_0(b) = 0$. Property Z shows that $(x_1, 0, x_2) \in V'_n$ has a neighbor $(y_1, 1, y_2) \in V'_n$ in G'_n. Consider the following path in G'_{n+1}:

$$(x_1, s + 1, x + 2), (y_1, 1, y_2), (x_1, s + 2, x_2), (y_1, 2, y_2), \ldots, (y_1, s - 1, y_2), (x_1, 2s, x_2), (y_1, s, y_2).$$

This path shows that (x_1, a, x_2) and (x_1, b, x_2) are connected whenever $s + 1 \leq a, b \leq 2s$. Since (y_1, s, y_2) neighbors (x_1, c, x_2) for all $2s \leq c \leq m$, we deduce that (x_1, a, x_2) and (x_1, b, x_2) are connected for all $s + 1 \leq a, b \leq m$.

Consider now any two vertices $(x_1, x, x_2), (y_1, y, y_2) \in V'_{n+1}$ such that $(x_1, \sigma_0(x), x_2) \in V'_n$ neighbors $(y_1, \sigma_0(y), y_2) \in V'_n$ in G'_n. We will show that (x_1, x, x_2) and (y_1, y, y_2) are connected in V'_{n+1} by an odd-length path by showing that (x_1, X, x_2) and (y_1, Y, y_2) are connected for some $X, Y \in \mathbb{Z}_m$ satisfying $\sigma_0(x) = \sigma_0(X)$ and $\sigma_0(y) = \sigma_0(Y)$. Such X, Y are given by the following table:

$\sigma_0(x), \sigma_0(y)$	$0, 0$	$0, 1$	$1, 0$	$1, 1$
X, Y	$s + 1, 2s + 1$	$s + 1, 1$	$1, s + 1$	$1, 1$

This shows that V'_{n+1} is connected. Moreover, any odd-length cycle in V'_n lifts to an odd-length cycle in V'_{n+1}, showing that G'_{n+1} is non-bipartite.

When $s = 1$ and $m = 2$, notice that the general case differs from the case $n = 1$ only by a translation of the coordinates. Since the proof of the case $n = 1$ did not use the bound $s < m/2$, we conclude that the lemma holds even when $s = 1$ and $m = 2$. $\square$

Now we can prove the strengthening of Lemma 5.3.

Lemma 5.6. *Let $n, t \geq 1$ and ℓ, s, m be given, and suppose that $s < m/2$. Let $\mathcal{F}$ be a family on $\mathbb{Z}_m^n \times \{0,1\}^\ell$ which is t-agreeing up to s, and let $\mathcal{H}$ be the family on $\mathbb{Z}_m^{n-1} \times \{0,1\}^{\ell+1}$ constructed in Lemma 5.3. If $\mu_{s,m}^{n-1,\ell+1}(\mathcal{H}) = \mu_{s,m}^{n,\ell}(\mathcal{F})$ and $\mathcal{H}$ is equivalent to a (t,r)-Frankl family then $\mathcal{F}$ is also equivalent to a (t,r)-Frankl family.*

The same holds for $(s,m) = (1,2)$, assuming that in the proof of Lemma 5.3 we use $w_x = 1/2$ (see comment at the end of the proof).

Proof. As in the proof of Lemma 5.3, we consider separately the degenerate case $n + \ell = t$ and the non-degenerate case $n + \ell > t$.

When $n + \ell = t$, $\mathcal{H}$ must be equivalent to the family $\{[t]\}$. Without loss of generality, we can assume that $\mathcal{H} = [s]^{n-1} \times \{1\}^{\ell+1}$ (this corresponds to the choice $y = 0$ in the definition of equivalence). Thus $\mathcal{F}$ has the form

$$\mathcal{F} = \bigcup_{x_1 \in [s]^{n-1}} \bigcup_{x_2 = \{1\}^\ell} \{x_1\} \times \mathcal{F}_{x_1,x_2} \times \{x_2\},$$

where each $\mathcal{F}_{x_1,x_2}$ is s-agreeing. We remind the reader that the support of $\mathcal{F}$ is the set of pairs (x_1, x_2) such that $\mathcal{F}_{x_1,x_2} \neq \emptyset$; the support $\mathcal{H}$ is defined in the same way.

Since $\mu_{s,m}^{n-1,\ell+1}(\mathcal{H}) = \mu_{s,m}^{n,\ell}(\mathcal{F})$, moreover $|\mathcal{F}_{x_1,x_2}| = s$, and so Lemma 3.3 shows that $\mathcal{F}_{x_1,x_2}$ is an interval (when $s = 1$ this is trivial). If $(x_1,x_2),(y_1,y_2)$ are in the support of $\mathcal{F}$ then $\mathcal{F}_{x_1,x_2}, \mathcal{F}_{y_1,y_2}$ are cross-s-agreeing, and so Lemma 3.3 shows that they are equal (again, when $s = 1$ this is trivial). This shows that $\mathcal{F}$ is equivalent to $\{[t]\}$ as well.

Consider now the non-degenerate case $n+\ell > t$. Recall that in the proof of Lemma 5.3 we constructed two graphs, $G = (V, E)$ and $G' = (V', E')$, and a vector $w \colon V' \to \mathbb{R}$, and defined $\mathcal{H}$ in terms of this data. When computing the measure of $\mathcal{H}$, we used the estimate

$$\sum_{x \in V \setminus V'} \mu_{s,m}^{m-1,\ell}(x) \geq \sum_{x \in V \setminus V'} \mu_{s,m}^{m-1,\ell}(x) \mu_m(\mathcal{F}_x).$$

This estimate can only be tight if $\mathcal{F}_x = \mathbb{Z}_m$ for $x \in V \setminus V'$. We also used the estimate

$$\sum_{\substack{x \in V' \\ w_x = 1}} \mu_{s,m}^{n-1,\ell}(x) + \frac{s}{m} \sum_{\substack{x \in V' \\ w_x = 1/2}} \mu_{s,m}^{n-1,\ell}(x) \geq \frac{2s}{m} \sum_{x \in V'} \mu_{s,m}^{n-1,\ell}(x) w_x.$$

If $2s < m$, this can only be tight if it is never the case that $w_x = 1$. Recall that w_x was a $\{0, 1/2, 1\}$-valued vector maximizing $\sum_{x \in V'} \mu_{s,m}^{n-1,\ell}(x) w_x$ under the constraints $0 \leq w_x \leq 1$ and $w_x + w_y \leq 1$ whenever $(x, y) \in E'$. Since $w_x \leq 1/2$ for all $x \in V'$, we see that in fact $w_x = 1/2$ for all V' (since the constant $1/2$ vector is always feasible). This shows that $\mathcal{H}$ and $\mathcal{F}$ have the same support.

Moreover (still assuming $2s < m$), if there were a different $\{0, 1/2, 1\}$-valued vector maximizing $\sum_{x \in V'} \mu_{s,m}^{n-1,\ell}(x) w_x$ then the construction of Lemma 5.3 would have shown that $\mathcal{F}$ does not have maximum measure. We conclude that w_x is the unique $\{0, 1/2, 1\}$-valued maximizer, and so the unique maximizer according to Proposition 5.2.

When $(s, m) = (1, 2)$, $w_x = 1/2$ for all V' by assumption, and so $\mathcal{H}$ and $\mathcal{F}$ have the same support; the property proved in the preceding paragraph won't be needed in this case.

Lemma 5.5 shows that G' is either empty or connected. Recall that $v_x = |\mathcal{F}_x|/(2s)$. When computing the measure of $\mathcal{H}$, we used the estimate $\sum_{x \in V'} \mu_{s,m}^{n-1,\ell}(x) w_x \geq \sum_{x \in V'} \mu_{s,m}^{n-1,\ell}(x) v_x$. When $s < m/2$, w is the unique maximizer, and so $v_x = 1/2$ for all $x \in V$, that is, $|\mathcal{F}_x| = s$ for all $x \in V'$; when $(s, m) = (1, 2)$, the same trivially holds. For any two vertices $x, y \in V'$ connected by an edge, the sets $\mathcal{F}_x, \mathcal{F}_y$ must be s-agreeing, and so Lemma 3.3 shows that $\mathcal{F}_x = \mathcal{F}_y$ (when $s = 1$ this is trivial). Since G is connected, we see that all $\mathcal{F}_x$ are equal. It follows that for some $a \in \mathbb{Z}_m$, $\mathcal{F}_x = \sigma_a^{-1}(\mathcal{H}_x)$ for all $x \in V$. Therefore $\mathcal{F}$ is equivalent to the same (t, r)-Frankl family as $\mathcal{H}$. $\square$

Finally, we can determine the maximum measure families.

Lemma 5.7. *Fix $n \geq t \geq 1$, $m \geq 2$, and either $s < m/2$ or both $(s,m) = (1,2)$ and $t > 1$. If $\mathcal{F}$ is a family on $\mathbb{Z}_m^n$ which is t-intersecting up to s and has measure $w(\mathbb{Z}_m^n, t, s)$ then $\mathcal{F}$ is equivalent to a family on n points whose $\mu_{s/m}$-measure is $w(n,t,s/m)$.*

Proof. Let $\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \ldots, \mathcal{F}_n$ be the sequence of families constructed by applying Lemma 5.3, so that $\mathcal{F}_\ell$ is a family on $\mathbb{Z}_m^{n-\ell} \times \{0,1\}^\ell$. The lemma states that $\mu_{s,m}^{n-\ell,\ell}(\mathcal{F}_\ell) \leq \mu_{s,m}^{n-\ell-1,\ell+1}(\mathcal{F}_{\ell+1})$. Since $\mu_{s/m}(\mathcal{F}_n) \leq w(n,t,s/m) = w(\mathbb{Z}_m^n, t, s)$, we conclude that $\mu_{s,m}^{n-\ell,\ell}(\mathcal{F}_\ell) = \mu_{s,m}^{n-\ell-1,\ell+1}(\mathcal{F}_{\ell+1})$ for all $0 \leq \ell < n$. Also, Theorem 2.1 shows that $\mathcal{F}_n$ is equivalent to a (t,r)-Frankl family for some $r \geq 0$. Applying Lemma 5.6, we conclude that the same holds for $\mathcal{F}_0$. □

This completes the proof of Theorem 5.1.

5.2 Continuous setting

The continuous analog of the setting of Section 5.1 is given by the following definitions.

Definition 5.4. A *continuous family on n points* is a measurable subset of $\mathbb{T}^n$ (recall that $\mathbb{T}$ is the unit circumference circle). For $p \leq 1/2$, a continuous family $\mathcal{F}$ on n points is *t-agreeing up to p* if any two vectors $x, y \in \mathcal{F}$ have t coordinates $i_1, \ldots, i_t$ such that the distance between x_{i_j} and y_{i_j} is less than p for all $1 \leq j \leq t$.

We denote the measure of a continuous family $\mathcal{F}$ by $\mu(\mathcal{F})$.

The maximum measure of a continuous family on n points which is t-agreeing up to p is denoted $w(\mathbb{T}^n, t, p)$.

We will prove the following theorem.

Theorem 5.8. *Let $n, m, t \geq 1$ and $p < 1/2$. Then $w(\mathbb{T}^n, t, p) = w(n,t,p)$.*

The proof uses a reduction to Theorem 5.1 which is very similar to the one used to deduce Lemma 3.5 from Lemma 3.3.

Proof. We start with the easy direction: $w(\mathbb{T}^n, t, p) \geq w(n,t,p)$. Let $\mathcal{F}$ be a t-intersecting family on n points with $\mu_p(\mathcal{F}) = w(n,t,p)$. Define a mapping $\tau(\mathbb{T}) \to \{0,1\}$ by $\tau([0,p)) = 1$ and $\tau([p,1)) = 0$. It is not hard to check that $\tau^{-1}(\mathcal{F})$ is a continuous family on n points which is t-agreeing up to p and has measure $w(n,t,p)$. Thus $w(\mathbb{T}^n, t, p) \geq w(n,t,p)$.

For the other direction, let $\mathcal{F}$ be a continuous family on n points which is t-agreeing up to p. Let $\epsilon > 0$ be a parameter satisfying $p + \epsilon^{1/n} < 1/2$. Since $\mathcal{F}$ is measurable, there is a sequence F_i of cylindrical sets of total measure at most $\mu(\mathcal{F}) + \epsilon$ which covers $\mathcal{F}$. Since an L_∞ ball of radius δ around any point has volume $(2\delta)^n$, it follows that any point in $\bigcup_i F_i$ is at L_∞-distance $\epsilon^{1/n}/2$ from $\mathcal{F}$. This implies that $\bigcup_i F_i$ is t-agreeing up to $p + \epsilon^{1/n}$.

Choose I so that $\sum_{i>I} \mu(F_i) < \epsilon$, and let $\mathcal{F}^* = \bigcup_{i \leq I} F_i$. Let M be a large integer, and partition $\mathbb{T}^n$ into M^n cubes of side length $1/M$. Let $\mathcal{F}_M^*$ consist of the union of all cubes contained entirely inside $\mathcal{F}^*$. Thus $\mu(\mathcal{F}_M^*) \geq \mu(\mathcal{F}^*) - O(I/M) \geq \mu(\mathcal{F}) - \epsilon - O(I/M)$. We can view $\mathcal{F}_M^*$ as a family on $\mathbb{Z}_M^n$, and this family is t-agreeing up to $\lfloor (p+\epsilon^{1/n})M \rfloor$. Theorem 5.1 thus shows that $\mu(\mathcal{F}_M^*) \leq w(n,t,\lfloor (p+\epsilon^{1/n})M \rfloor / M)$. Therefore

$$\mu(\mathcal{F}) \leq w(n,t,\lfloor (p+\epsilon^{1/n})M \rfloor / M) + \epsilon + O\left(\frac{I}{M}\right).$$

Since $w(n,t,q)$ is continuous for $q \leq 1/2$, taking the limit $M \to \infty$ we deduce that $\mu(\mathcal{F}) \leq w(n,t,p + \epsilon^{1/n}) + \epsilon$. Taking the limit $\epsilon \to 0$, we conclude that $\mu(\mathcal{F}) \leq w(n,t,p)$, and so $w(\mathbb{T}^n, t, p) \leq w(n,t,p)$. □

It is tempting to try and prove Theorem 5.8 directly, along the lines of Theorem 5.1. Besides requiring a stronger version of Lemma 3.5, one would also need a version of Proposition 5.2 for infinite polytopes, which we doubt holds.

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