The weighted complete intersection theorem

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Abstract

The seminal complete intersection theorem of Ahlswede and Khachatrian gives the maximum cardinality of a k-uniform t-intersecting family on n points, and describes all optimal families for $t \geq 2$. We extend this theorem to the weighted setting, in which we consider unconstrained families. The goal in this setting is to maximize the μ_p measure of the family, where the measure μ_p is given by $\mu_p(A) = p^{|A|}(1-p)^{n-|A|}$. Our theorem gives the maximum μ_p measure of a t-intersecting family on n points, and describes all optimal families for $t \geq 2$.

1 Introduction

The Erdős–Ko–Rado theorem [8], a basic result in extremal combinatorics, states that when $k \leq n/2$, a k-uniform intersecting family on n points contains at most $\binom{n-1}{k-1}$ sets; and furthermore, when $k < n/2$ the only families achieving these bounds are *stars*, consisting of all sets containing some fixed point.

The analog of the Erdős–Ko–Rado theorem for t-intersecting families, in which every two sets must have at least t points in common, was proved by Ahlswede and Khachatrian [3, 5], who gave two different proofs (see also the monograph [1]). When $t \geq 2$, the optimal families are always of the form $\mathcal{F}_{t,r} = \{S : |S \cap [t+2r]| \geq t+r\}$, as had been conjectured by Frankl [9]. They also determined the maximum families under the condition that the intersection of all sets in the family is empty [2], as well as the maximum non-uniform t-intersecting families [6] ("Katona's theorem"). They also proved an analogous theorem for the Hamming scheme [4].

Dinur and Safra [7] considered analogous questions in the weighted setting. They were interested in the maximum μ_p measure of a t-intersecting family on n points, where the μ_p measure is given by $\mu_p(A) = p^{|A|}(1-p)^{n-|A|}$. When $p \leq 1/2$, they related this question to the setting of the original Ahlswede–Khachatrian theorem with parameters K, N satisfying $K/N \approx p$. A similar argument appears in work of Ahlswede–Khachatrian [6, 4] in different guise. The μ_p setting has since been widely studied, and has been used by Friedgut [10] and by Keller and Lifshitz [12] to prove stability versions of the Ahlswede–Khachatrian theorem.

While not stated explicitly in either work, the methods of Dinur–Safra [7] and Ahlswede–Khachatrian [4] give a proof of an Ahlswede–Khachatrian theorem in the μ_p setting for all $p < 1/2$, without any constraint on the number of points. More explicitly, let $w(n,t,p)$ be the maximum μ_p-measure of a t-intersecting family on n points, and let $w(t,p) = \sup_n w(n,t,p)$. The techniques of Dinur–Safra and Ahlswede–Khachatrian show that when $\frac{r}{t+2r-1} \leq p \leq \frac{r+1}{t+2r+1}$, $w(t,p) = \mu_p(\mathcal{F}_{t,r})$. This theorem is incomplete, for three different reasons: it describes $w(t,p)$ rather than $w(n,t,p)$, it only works for $p < 1/2$, and it doesn't describe the optimal families.

Katona [11] solved the case $p = 1/2$, which became known as "Katona's theorem". Ahlswede and Khachatrian gave a different proof [6], and their technique applies also to the case $p > 1/2$. We complete the picture by finding $w(n,t,p)$ for all n, t, p and determining all families achieving this bound when $t \geq 2$. We do this by rephrasing the two original proofs [3, 5] of the Ahlswede–Khachatrian theorem in the μ_p setting. Curiously, whereas the classical Ahlswede–Khachatrian theorem can be proven using either of the techniques described in [3, 5], our proof needs to use both.

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2 Preliminaries

We will use $[n]$ for $\{1,\ldots,n\}$, and $\binom{[n]}{k}$ for all subsets of $[n]$ of size k. We also use the somewhat unorthodox notation $\binom{[n]}{\geq k}$ for all subsets of $[n]$ of size at least k. The set of all subsets of a set A will be denoted 2^A.

A *family on n points* is a collection of subsets of $[n]$. A family $\mathcal{F}$ is *t-intersecting* if any $A,B \in \mathcal{F}$ satisfy $|A \cap B| \geq t$. A family is *intersecting* if it is 1-intersecting.

For any $p \in (0,1)$ and any n, the product measure μ_p is a measure on the set of subsets of $[n]$ given by $\mu_p(A) = p^{|A|}(1-p)^{n-|A|}$.

A family $\mathcal{F}$ on n points is *monotone* if whenever $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$. Given a family $\mathcal{F}$, its *up-set* $\langle \mathcal{F} \rangle$ is the smallest monotone family containing $\mathcal{F}$, consisting of all supersets of sets in $\mathcal{F}$.

For $n \geq t \geq 1$ and $p \in (0,1)$, the parameter $w(n,t,p)$ is the maximum of $\mu_p(\mathcal{F})$ over all t-intersecting families on n points, and the parameter $w(t,p)$ is given by $w(t,p) = \sup_n w(n,t,p)$. It is easy to see that we can also define $w(t,p)$ as a limit instead of a supremum.

For $t \geq 1$ and $r \geq 0$, the *(t,r)-Frankl family* on n points is the t-intersecting family

$$\mathcal{F}_{t,r} = \{A \subseteq [n] : |A \cap [t+2r]| \geq t+r\}.$$

A family $\mathcal{F}$ on n points is *equivalent* to a (t,r)-Frankl family if there exists a set $S \subseteq [n]$ of size $t+2r$ such that $\mathcal{F} = \{A \subseteq [n] : |A \cap S| \geq t+r\}$.

The following result is a straightforward calculation.

Lemma 2.1. *Let $t \geq 1$ and $r \geq 0$ be parameters, and let $p_{t,r} = \frac{r+1}{t+2r+1}$. If $p < p_{t,r}$ then $\mu_p(\mathcal{F}_{t,r}) > \mu_p(\mathcal{F}_{t,r+1})$. If $p = p_{t,r}$ then $\mu_p(\mathcal{F}_{t,r}) = \mu_p(\mathcal{F}_{t,r+1})$. If $p > p_{t,r}$ then $\mu_p(\mathcal{F}_{t,r}) < \mu_p(\mathcal{F}_{t,r+1})$.*

3 Main results

Our main theorem is an analog of the Ahlswede–Khachatrian theorem in the μ_p setting.

Theorem 3.1. *Let $n \geq t \geq 1$ and $p \in (0,1)$. If $\mathcal{F}$ is t-intersecting then*

$$\mu_p(\mathcal{F}) \leq \max_{r\colon t+2r \leq n} \mu_p(\mathcal{F}_{t,r}).$$

Moreover, unless $t=1$ and $p \geq 1/2$, equality holds only if $\mathcal{F}$ is equivalent to a Frankl family $\mathcal{F}_{t,r}$.

When $t=1$ and $p>1/2$, the same holds if $n+t$ is even, and otherwise $\mathcal{F} = \mathcal{G} \cup \binom{[n]}{\geq \frac{n+t+1}{2}}$ where $\mathcal{G} \subseteq \binom{[n]}{\frac{n+t-1}{2}}$ contains exactly $\binom{n-1}{\frac{n+t-1}{2}}$ sets.

When $t=1$ and $p=1/2$ there are many optimal families. For example, the families $\mathcal{F}_{1,r}$ all have $\mu_{1/2}$-measure $1/2$, as does the family $\{S : 1 \in S\} \setminus \{\{1\}\} \cup \{\{2,\ldots,n\}\}$.

Similarly, when $t=1$, $p>1/2$ and $n+1$ is odd there are many optimal families, for example $\binom{[n]}{\geq n/2+1} \cup \binom{[n]}{n/2} \cap \mathcal{F}_{1,0}$, and $\binom{[n]}{\geq n/2+1} \cup \binom{[n]}{n/2} \setminus \mathcal{F}_{1,0}$.

Our proof implies the following more detailed corollary.

Corollary 3.2. *Let $n \geq t \geq 1$. Define r^* as the maximal integer satisfying $t+2r^* \leq n$.*

If $t=1$ then

$$w(n,1,p) = \begin{cases} p & p \leq \frac{1}{2}, \\ \mu_p(\mathcal{F}_{1,r^*}) & p \geq \frac{1}{2}. \end{cases}$$

Furthermore, if $\mathcal{F}$ is an intersecting family of μ_p-measure $w(n,1,p)$ for $p \in (0,1)$ then:

- *If $p < \frac{1}{2}$ then $\mathcal{F}$ is equivalent to $\mathcal{F}_{1,0}$.*
- *If $p > \frac{1}{2}$ and n is odd then $\mathcal{F}$ is equivalent to $\mathcal{F}_{1,\frac{n-1}{2}}$.*
- *If $p > \frac{1}{2}$ and n is even then $\mathcal{F} = \mathcal{G} \cup \binom{[n]}{\geq n/2+1}$, where $\mathcal{G}$ contains half the sets in $\binom{[n]}{n/2}$: exactly one of each pair $A, [n] \setminus A$.*

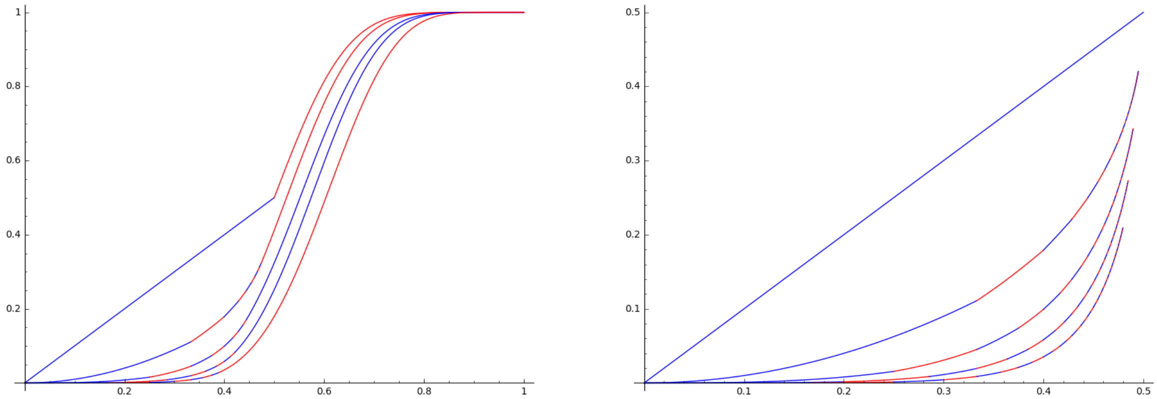

Figure 1: The function $w(20, t, p)$ for $1 \leq t \leq 5$ (left) and the function $w(t, p)$ for $1 \leq t \leq 5$ (right). In both cases, larger functions correspond to smaller t. The colors switch at each of the breakpoints $\frac{r}{t+2r-1}$ for $r \leq r^*$ (left) or for each r (right).

If $t \geq 2$ then

$$w(n,t,p) = \begin{cases} \mu_p(\mathcal{F}_{t,r}) & \frac{r}{t+2r-1} \leq p \leq \frac{r+1}{t+2r+1} \text{ for some } r < r^*, \\ \mu_p(\mathcal{F}_{t,r^*}) & \frac{r^*}{t+2r^*-1} \leq p. \end{cases}$$

Furthermore, if $\mathcal{F}$ is a t-intersecting family of μ_p-measure $w(n,t,p)$ for $p \in (0,1)$ then:

- *If $\frac{r}{t+2r-1} < p < \frac{r+1}{t+2r+1}$ for some $r < r^*$ then $\mathcal{F}$ is equivalent to $\mathcal{F}_{t,r}$.*
- *If $\frac{r^*}{t+2r^*-1} < p$ then $\mathcal{F}$ is equivalent to $\mathcal{F}_{t,r^*}$.*
- *If $p = \frac{r+1}{t+2r+1}$ for some $r < r^*$ then $\mathcal{F}$ is equivalent to $\mathcal{F}_{t,r}$ or to $\mathcal{F}_{t,r-1}$.*

As a corollary, we can compute $w(t, p)$. We leave the straightforward calculations to the reader.

Corollary 3.3. *We have*

$$w(1,p) = \begin{cases} p & p \leq \frac{1}{2}, \\ 1 & p > \frac{1}{2} \end{cases}$$

For $t \geq 2$, we have

$$w(t,p) = \begin{cases} \mu_p(\mathcal{F}_{t,r}) & \frac{r}{t+2r-1} \leq p \leq \frac{r+1}{t+2r+1}, \\ \frac{1}{2} & p = \frac{1}{2}, \\ 1 & p > \frac{1}{2}. \end{cases}$$

Figure 1 illustrates Corollary 3.2 and Corollary 3.3. The proof of Theorem 3.1 occupies the rest of the paper.

4 Shifting and symmetrization

4.1 Shifting

We use the classical technique of *shifting* to obtain families which are easier to analyze.

Let $\mathcal{F}$ be a family on n points and let $i, j \in [n]$ be two different indices. The shift operator $\mathbb{S}_{i,j}$ acts on $\mathcal{F}$ as follows. Let $\mathcal{F}_{i,j}$ consist of all sets in $\mathcal{F}$ containing i but not j. Then

$$\begin{aligned} \mathbb{S}_{i,j}(\mathcal{F}) = (\mathcal{F} \setminus \mathcal{F}_{i,j}) &\cup \{A : A \in \mathcal{F}_{i,j} \text{ and } (A \setminus \{i\}) \cup \{j\} \in \mathcal{F}\} \\ &\cup \{(A \setminus \{i\}) \cup \{j\} : A \in \mathcal{F}_{i,j} \text{ and } (A \setminus \{i\}) \cup \{j\} \notin \mathcal{F}\}. \end{aligned}$$

In words, we try to "shift" each set $A \in \mathcal{F}_{i,j}$ by replacing it with $A' = (A \setminus \{i\}) \cup \{j\}$. If $A' \notin \mathcal{F}$ then we replace A with A', and otherwise we don't change A.

The following lemmas state several well-known properties of shifting.

Lemma 4.1. *For any family $\mathcal{F}$ and indices i, j and for all $p \in (0,1)$, $\mu_p(\mathcal{F}) = \mu_p(\mathbb{S}_{i,j}(\mathcal{F}))$.*

Lemma 4.2. *If $\mathcal{F}$ is t-intersecting then so is $\mathbb{S}_{i,j}(\mathcal{F})$ for any i, j.*

By shifting $\mathcal{F}$ repeatedly we can obtain a left-compressed family. A family $\mathcal{F}$ on n points is *left-compressed* if whenever $A \in \mathcal{F}$, $i \in A$, $j \notin A$, and $j < i$, then $(A \setminus \{i\}) \cup \{j\} \in \mathcal{F}$. (Informally, we can *shift* i to j.)

Lemma 4.3. *Let $\mathcal{F}$ be a t-intersecting family on n points. There is a left-compressed t-intersecting family $\mathcal{G}$ on n points with the same μ_p-measure for all $p \in (0,1)$. Furthermore, $\mathcal{G}$ can be obtained from $\mathcal{F}$ by applying a sequence of shift operators.*

Lemma 4.3 shows that in order to determine $w(n,t,p)$ it is enough to focus on left-compressed families. Moreover, since the up-set of a t-intersecting family is also t-intersecting, we will assume in most of what follows that $\mathcal{F}$ is a monotone left-compressed t-intersecting family. We will show that except for the case $p \geq 1/2$ and $t = 1$, such a family can only have maximum μ_p measure if it is a Frankl family with the correct parameters. We will deduce that general t-intersecting families of measure $w(n,t,p)$ are equivalent to a Frankl family using the following lemma, whose proof closely follows the argument of Ahlswede and Khachatrian [3].

Lemma 4.4. *Let $\mathcal{F}$ be a monotone t-intersecting family on n points, and let $i, j \in [n]$. If $\mathbb{S}_{i,j}(\mathcal{F})$ is equivalent to $\mathcal{F}_{t,r}$ then so is $\mathcal{F}$.*

Proof. Let $S \subseteq [n]$ be the set of size $t + 2r$ such that $\mathbb{S}_{i,j}(\mathcal{F}) = \{A \subseteq [n] : |A \cap S| \geq t + r\}$.

Suppose first that $i, j \in S$ or $i, j \notin S$. If $A \in \mathbb{S}_{i,j}(\mathcal{F})$ then $A \in \mathcal{F}$, since otherwise A would have originated from $A' = (A \setminus \{j\}) \cup \{i\}$, but that is impossible since $A' \in \mathbb{S}_{i,j}(\mathcal{F})$. It follows that $\mathbb{S}_{i,j}(\mathcal{F}) \subseteq \mathcal{F}$ and so $\mathbb{S}_{i,j}(\mathcal{F}) = \mathcal{F}$, since shifting preserves cardinality. Therefore the lemma trivially holds.

The case $i \in S$ and $j \notin S$ cannot happen. Indeed, consider some set $A \subseteq S$ containing i but not j of size $t + r$. Then $A \in \mathbb{S}_{i,j}(\mathcal{F})$ and so, by definition of the shift, $A' = (A \setminus \{i\}) \cup \{j\} \in \mathbb{S}_{i,j}(\mathcal{F})$. However, $|A' \cap S| = t + r - 1$, and so $A' \notin \mathbb{S}_{i,j}(\mathcal{F})$, and we reach a contradiction.

It remains to consider the case $i \notin S$ and $j \in S$. Suppose first that $r = 0$. Then $S \in \mathbb{S}_{i,j}(\mathcal{F})$, and so either $S \in \mathcal{F}$ or $S' = (S \setminus \{j\}) \cup \{i\} \in \mathcal{F}$. In both cases, since $\mathcal{F}$ is monotone, it contains all supersets of S or of S'. Since shifting preserves cardinality, $\mathcal{F}$ must consist exactly of all supersets of S or of S', and thus is equivalent to a $(t,0)$-Frankl family.

Suppose next that $r > 0$. Let V be the collection of all subsets of $S \setminus \{j\}$ of size exactly $t + r - 1$. For each $A \in V$ we have $A \cup \{j\} \in \mathbb{S}_{i,j}(\mathcal{F})$, and so either $A \cup \{j\} \in \mathcal{F}$ or $A \cup \{i\} \in \mathcal{F}$.

If $\mathcal{F}$ contains $A \cup \{j\}$ for all $A \in V$ then $\mathcal{F}$ contains all subsets of S of size $t + r$ (since other subsets are not affected by the shift). Monotonicity forces $\mathcal{F}$ to contain all of $\mathbb{S}_{i,j}(\mathcal{F})$, and thus $\mathcal{F} = \mathbb{S}_{i,j}(\mathcal{F})$ as before.

If $\mathcal{F}$ contains $A \cup \{i\}$ for all $A \in V$, then in a similar way we deduce that $\mathcal{F}$ is equivalent to the (t,r)-Frankl family based on $(S \setminus \{j\}) \cup \{i\}$.

It remains to consider the case in which $\mathcal{F}$ contains $A \cup \{i\}$ for some $A \in V$, and $B \cup \{j\}$ for some other $B \in V$. We will show that in this case, $\mathcal{F}$ is not t-intersecting. Consider the graph on V in which two vertices are connected if their intersection has the minimal size $t - 1$. This graph is a generalized Johnson graph, and we show below that it is connected. This implies that there must be two sets A, B satisfying $|A \cap B| = t - 1$ such that $A \cup \{i\}, B \cup \{j\} \in \mathcal{F}$. Since $|A \cap B| = t - 1$, we have reached a contradiction.

To complete the proof, we prove that the graph is connected. For reasons of symmetry, it is enough to give a path connecting $x = \{1, \ldots, t + r - 1\}$ and $y = \{2, \ldots, t + r\}$. Indeed, the vertex $\{2, \ldots, t, t + r + 1, \ldots, t + 2r\}$ is connected to both x and y. □

The preceding lemmas allow us to reduce the proof of Theorem 3.1 to the left-compressed case.

4.2 Generating sets

The goal of the first part of the proof, which follows [3], is to show that any monotone left-compressed t-intersecting family of maximum μ_p-measure has to depend on a small number of points. We will use a representation of monotone families in which this property has a simple manifestation. Our definition is simpler than the original one, due to the different setting.

A family $\mathcal{F}$ on n points is *non-trivial* if $\mathcal{F} \notin \{\emptyset, 2^{[n]}\}$. Let $\mathcal{F}$ be a non-trivial monotone family. A *generating set* is an inclusion-minimal set $S \in \mathcal{F}$. The *generating family* of $\mathcal{F}$ consists of all generating sets of $\mathcal{F}$. The *extent* of $\mathcal{F}$ is the maximal index appearing in a generating set of $\mathcal{F}$. The *boundary generating family* of $\mathcal{F}$ consists of all generating sets of $\mathcal{F}$ containing its extent.

If $\mathcal{G}$ is the generating family of $\mathcal{F}$ then we use the notation $\mathcal{G}^*$ for the boundary generating family of $\mathcal{F}$. For each integer a, we use the notation $\mathcal{G}^*_a$ for the subset of $\mathcal{G}^*$ consisting of sets of size a.

Generating sets are also known as *minterms*. If $\mathcal{G}$ is the generating family of $\mathcal{F}$ then $\mathcal{G}$ is an antichain and $\mathcal{F}$ is the up-set of $\mathcal{G}$ (and this gives an alternative definition of $\mathcal{G}$). If $\mathcal{F}$ has extent m then $\mathcal{F}$ depends only on the first m coordinates: $S \in \mathcal{F}$ iff $S \triangle \{i\} \in \mathcal{F}$ for all $i > m$. For this reason, for the rest of the section we treat a family having extent m as a family on m points.

One reason to focus on the boundary generating family of $\mathcal{F}$ is the following simple observations.

Lemma 4.5. *Let $\mathcal{F}$ be a non-trivial monotone left-compressed family of extent m with generating family $\mathcal{G}$ and boundary generating family $\mathcal{G}^*$. For any subset $G \subseteq \mathcal{G}^*$, $\langle \mathcal{G} \setminus G \rangle = \mathcal{F} \setminus G$.*

Proof. Since $\mathcal{G}$ is an antichain, no $A \in G$ is a superset of any other set in $\mathcal{G}$. For this reason, $\langle \mathcal{G} \setminus G \rangle \subseteq \mathcal{F} \setminus G$.

On the other hand, let $S \in \mathcal{F} \setminus G$. If S is not a superset of any $A \in G$ then clearly $S \in \langle \mathcal{G} \setminus G \rangle$. If $S \supseteq A$ for some $A \in G$ then since $S \neq A$, there is an element $i \in S \setminus A$. The set $S' = S \setminus \{m\}$ is a superset of $(A \setminus \{m\}) \cup \{i\}$, and so $S' \in \mathcal{F}$. Thus S' is a superset of some $B \in \mathcal{G}$. Since $m \notin S'$, necessarily $B \notin G$. As $S \supseteq B$, we conclude that $S \in \langle \mathcal{G} \setminus G \rangle$. □

Lemma 4.6. *Let $\mathcal{F}$ be a non-trivial monotone left-compressed family of extent m with generating family $\mathcal{G}$ and boundary generating family $\mathcal{G}^*$. For any subset $G \subseteq \mathcal{G}^*$,*

$$\langle (\mathcal{G} \setminus G) \cup \{A \setminus \{m\} : A \in G\} \rangle = \mathcal{F} \cup \{A \setminus \{m\} : A \in G\}.$$

Proof. Denote by $\mathcal{F}'$ the left-hand side. Clearly $\mathcal{F}' \supseteq \mathcal{F} \cup \{A \setminus \{m\} : A \in G\}$.

On the other hand, suppose that $S \in \mathcal{F}' \setminus \mathcal{F}$. Then for some $A \in G$, S is a superset of $A \setminus \{m\}$ but not of A. In particular, $m \notin S$. We claim that $S = A \setminus \{m\}$. Otherwise, there exists an element $i \in S \setminus (A \setminus \{m\})$. Since $\mathcal{F}$ is left-compressed, $A' = (A \setminus \{m\}) \cup \{i\} \in \mathcal{F}$. Since $\mathcal{F}$ is monotone and $S \supseteq A'$, we conclude that $S \in \mathcal{F}$, contradicting our assumption. Thus $\mathcal{F}' \setminus \mathcal{F} = \{A \setminus \{m\} : A \in G\}$. □

The following crucial observation drives the entire approach, and explains why we want to classify the sets in the boundary generating family according to their size.

Lemma 4.7. *Let $\mathcal{F}$ be a non-trivial monotone left-compressed t-intersecting family with extent m and boundary generating family $\mathcal{G}^*$. If $A, B \in \mathcal{G}^*$ intersect in exactly t elements then $|A| + |B| = m + t$.*

Proof. We will show that $A \cup B = [m]$. It follows that $|A| + |B| = |A \cup B| + |A \cap B| = m + t$.

Since $A \cup B \subseteq [m]$ and $m \in A \cap B$ by definition, we have to show that every element $i < m$ belongs to either A or B. Suppose that some element i belongs to neither. Since $\mathcal{F}$ is left-compressed, the set $B' = (B \setminus \{m\}) \cup \{i\}$ also belongs to $\mathcal{F}$. However, $|A \cap B'| = |A \cap B| - 1 = t - 1$, contradicting the assumption that $\mathcal{F}$ is t-intersecting. □

Our goal now is to show that if m is too large then we can remove the dependency on m while keeping the family t-intersecting and increasing its μ_p-measure, for appropriate values of p. The idea is to remove m from sets in the boundary generating family. The only obstructions for doing so are sets A, B in the boundary generating family whose intersection contains *exactly* t elements, and here we use Lemma 4.7 to guide us: this can only happen if $|A| + |B| = m + t$. Accordingly, our modification will involve generating sets in $\mathcal{G}^*_a$ and $\mathcal{G}^*_b$ for $a + b = m + t$. There are two cases to consider: $a \neq b$ and $a = b$. The first case is simpler.

Lemma 4.8. *Let $\mathcal{F}$ be a non-trivial monotone left-compressed t-intersecting family with extent m, generating family $\mathcal{G}$, and boundary generating family $\mathcal{G}^*$. Let $a \neq b$ be parameters such that $a + b = m + t$ and $\mathcal{G}^*_a, \mathcal{G}^*_b$ are not both empty. Consider the families $\mathcal{F}_1 = \langle \mathcal{G}_1 \rangle$ and $\mathcal{F}_2 = \langle \mathcal{G}_2 \rangle$, where*

$$\mathcal{G}_1 = (\mathcal{G} \setminus (\mathcal{G}^*_a \cup \mathcal{G}^*_b)) \cup \{S \setminus \{m\} : S \in \mathcal{G}^*_b\}, \qquad \mathcal{G}_2 = (\mathcal{G} \setminus (\mathcal{G}^*_a \cup \mathcal{G}^*_b)) \cup \{S \setminus \{m\} : S \in \mathcal{G}^*_a\}.$$

Both families $\mathcal{F}_1, \mathcal{F}_2$ are t-intersecting. Moreover, if $p < 1/2$ then $\max(\mu_p(\mathcal{F}_1), \mu_p(\mathcal{F}_2)) > \mu_p(\mathcal{F})$; and if $p = 1/2$, $\max(\mu_p(\mathcal{F}_1), \mu_p(\mathcal{F}_2)) \geq \mu_p(\mathcal{F})$, with equality only if $\mu_p(\mathcal{F}_1) = \mu_p(\mathcal{F}_2) = \mu_p(\mathcal{F})$.

Proof. We start by showing that $\mathcal{F}_1$ (and so $\mathcal{F}_2$) is t-intersecting. Clearly, it is enough to show that its generating family $\mathcal{G}_1$ is t-intersecting. Suppose that $S, T \in \mathcal{G}_1$. We consider several cases.

If $S, T \in \mathcal{G}$ then $|S \cap T| \geq t$ since $\mathcal{G}$ is t-intersecting.

If $S \in \mathcal{G}$ and $T \notin \mathcal{G}$ then $T' = T \cup \{m\} \in \mathcal{G}$ and so $|T'| = b$. If $m \notin S$ then $|S \cap T| = |S \cap T'| \geq t$. If $m \in S$ then by construction $|S| \neq a$, and so $|S \cap T| = |S \cap T'| - 1 \geq t$, using Lemma 4.7.

If $S, T \notin \mathcal{G}$ then $S' = S \cup \{m\} \in \mathcal{G}$ and $T' = T \cup \{m\} \in \mathcal{G}$, and so $|S'| = |T'| = b$. As in the preceding case, $|S \cap T| = |S' \cap T'| - 1 \geq t$ due to Lemma 4.7.

Lemma 4.5 and Lemma 4.6 show that $\mathcal{F}_1 = (\mathcal{F} \setminus \mathcal{G}_a^*) \cup \{S \setminus \{m\} : S \in \mathcal{G}_b^*\}$. Since $\mu_p(S \setminus \{m\}) = \frac{1-p}{p}\mu_p(S)$ whenever $m \in S$, $\mu_p(\mathcal{F}_1) = \mu_p(\mathcal{F}) - \mu_p(\mathcal{G}_a^*) + \frac{1-p}{p}\mu_p(\mathcal{G}_b^*)$. Similarly, $\mu_p(\mathcal{F}_2) = \mu_p(\mathcal{F}) - \mu_p(\mathcal{G}_b^*) + \frac{1-p}{p}\mu_p(\mathcal{G}_a^*)$. Taking the average of both estimates, we obtain

$$\frac{\mu_p(\mathcal{F}_1) + \mu_p(\mathcal{F}_2)}{2} = \mu_p(\mathcal{F}) + \frac{1}{2}\left(\frac{1-p}{p} - 1\right)(\mu_p(\mathcal{G}_a^*) + \mu_p(\mathcal{G}_b^*)).$$

When $p < 1/2$, the second term is positive, and so $\max(\mu_p(\mathcal{F}_1), \mu_p(\mathcal{F}_2)) > \mu_p(\mathcal{F})$. When $p = 1/2$ it vanishes, and so $\max(\mu_p(\mathcal{F}_1), \mu_p(\mathcal{F}_2)) \geq \mu_p(\mathcal{F})$. □

When $a = b$ the construction in Lemma 4.8 cannot be executed, and we need a more complicated construction. The new construction will only work for small enough p, mirroring the fact that the optimal families for larger p depend on more points.

Lemma 4.9. *Let $\mathcal{F}$ be a non-trivial monotone left-compressed t-intersecting family with extent $m > 1$, generating family $\mathcal{G}$, and boundary generating family $\mathcal{G}^*$. Suppose that $a = \frac{m+t}{2}$ is an integer and that $\mathcal{G}_a^*$ is non-empty. For each $i \in [m-1]$, let $\mathcal{G}_{a,i}^* = \{S \in \mathcal{G}_a^* : i \in S\}$, and define*

$$\mathcal{G}_i = (\mathcal{G} \setminus \mathcal{G}_a^*) \cup \{S \setminus \{m\} : S \in \mathcal{G}_a^* \setminus \mathcal{G}_{a,i}^*\}.$$

All families $\mathcal{F}_i = \langle \mathcal{G}_i \rangle$ are t-intersecting. Moreover, if $p < \frac{m-t}{2(m-1)}$ then $\mu_p(\mathcal{F}_i) > \mu_p(\mathcal{F})$ for some $i \in [m-1]$.

Proof. We start by showing that the families $\mathcal{F}_i$ are t-intersecting. Clearly, it is enough to show that $\mathcal{G}_i$ is t-intersecting. Let $S, T \in \mathcal{G}_i$. We consider several cases.

If $S, T \in \mathcal{G}$ then $|S \cap T| \geq t$ since $\mathcal{G}$ is t-intersecting.

If $S \in \mathcal{G}$ and $T \notin \mathcal{G}$ then $T' = T \cup \{m\} \in \mathcal{G}_a^*$ and $i \notin S$. If $m \notin S$ then $|S \cap T| = |S \cap T'| \geq t$. If $m \in S$ and $S \notin \mathcal{G}_a^*$ then $|S \cap T| \geq |S \cap T'| - 1 \geq t$, according to Lemma 4.7. If $m \in S$ and $S \in \mathcal{G}_a^*$ then by construction $i \notin S$. Since $\mathcal{F}$ is left-compressed, $S' = (S \setminus \{m\}) \cup \{i\} \in \mathcal{F}$. Therefore $|S \cap T| = |S' \cap T'| \geq t$.

If $S, T \notin \mathcal{G}$ then $S' = S \cup \{m\}$ and $T' = T \cup \{m\}$ both belong to $\mathcal{G}_a^*$, and i belongs to neither. Since $\mathcal{F}$ is left-compressed, $T'' = T \cup \{i\} \in \mathcal{F}$, and so $|S \cap T| = |S' \cap T''| \geq t$.

Lemma 4.5 and Lemma 4.6 show that $\mathcal{F}_i = (\mathcal{F} \setminus \mathcal{G}_{a,i}^*) \cup \{S \setminus \{m\} : S \in \mathcal{G}_a^* \setminus \mathcal{G}_{a,i}^*\}$. Since $\mu_p(S \setminus \{m\}) = \frac{1-p}{p}\mu_p(S)$ whenever $m \in S$,

$$\mu_p(\mathcal{F}_i) = \mu_p(\mathcal{F}) - \mu_p(\mathcal{G}_{a,i}^*) + \frac{1-p}{p}(\mu_p(\mathcal{G}_a^*) - \mu_p(\mathcal{G}_{a,i}^*)) = \mu_p(\mathcal{F}) + \frac{1-p}{p}\mu_p(\mathcal{G}_a^*) - \frac{1}{p}\mu_p(\mathcal{G}_{a,i}^*).$$

Averaging over all $i \in [m-1]$, we obtain

$$\frac{1}{m-1}\sum_{i=1}^{m-1}\mu_p(\mathcal{F}_i) = \mu_p(\mathcal{F}) + \frac{1-p}{p}\mu_p(\mathcal{G}_a^*) - \frac{1}{p(m-1)}\sum_{i=1}^{m-1}\mu_p(\mathcal{G}_{a,i}^*).$$

Since the sets in $\mathcal{G}_a^*$ contain exactly a elements, each set is counted $a-1$ times in $\sum_{i=1}^{m-1}\mu_p(\mathcal{G}_{a,i}^*)$, and so

$$\frac{1}{m-1}\sum_{i=1}^{m-1}\mu_p(\mathcal{F}_i) = \mu_p(\mathcal{F}) + \left(\frac{1-p}{p} - \frac{a-1}{p(m-1)}\right)\mu_p(\mathcal{G}_a^*).$$

When $1 - p > \frac{a-1}{m-1} = \frac{m+t-2}{2(m-1)}$, the bracketed quantity is positive, and so $\max_i \mu_p(\mathcal{F}_i) > \mu_p(\mathcal{F})$. □

4.3 Pushing-pulling

The goal of the second part of the proof, which follows [5], is to show that any monotone left-compressed t-intersecting family of maximum μ_p-measure is symmetric within its extent, or in other words, of the form $\mathcal{F}_{t,r}$.

The analog of extent in this part is the symmetric extent. Let $\mathcal{F}$ be a left-compressed family on n points. Its *symmetric extent* is the largest integer ℓ such that $\mathbb{S}_{ij}(\mathcal{F}) = \mathcal{F}$ for $i, j \leq \ell$.

If $\ell < n$ then the *boundary* of $\mathcal{F}$ is the collection

$$\mathcal{X} = \{A \in \mathcal{F} : \ell + 1 \notin A \text{ and } (A \setminus \{i\}) \cup \{\ell + 1\} \text{ for some } i \in A \cap [\ell]\}.$$

In other words, $\mathcal{X}$ consists of those sets in $\mathcal{F}$ preventing it from having larger symmetric extent.

The definition of symmetric extension guarantees that $\mathcal{X}$ can be decomposed as $\mathcal{X} = \sum_{a=0}^{\ell} \binom{[\ell]}{a} \times \mathcal{X}_a$, where $\mathcal{X}_a$ is a collection of subsets of $[n] \setminus [\ell + 1]$, a notation we use below.

The symmetric extent of a family is always bounded by its extent, apart from one trivial case.

Lemma 4.10. *Let $\mathcal{F}$ be a non-trivial monotone family on n points having extent m and symmetric extent ℓ. Then $\ell \leq m$.*

Proof. The family $\mathcal{F}$ has the general form $\mathcal{F} = \bigcup_{i=0}^{\ell} \binom{[\ell]}{i} \times \mathcal{F}_i$, where $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ are collections of subsets of $[n] \setminus [\ell]$. We claim that if $m < \ell$ then all $\mathcal{F}_i$ are equal. Indeed, let $i < \ell$. For each $A \in \mathcal{F}_i$, we have $[i] \cup A \in \mathcal{F}$. Since the extent of $\mathcal{F}$ is smaller than ℓ, $[i] \cup \{\ell\} \cup A \in \mathcal{F}$, implying $A \in \mathcal{F}_{i+1}$. Similarly, for each $A \in \mathcal{F}_{i+1}$ we have $[i] \cup \{\ell\} \cup A \in \mathcal{F}$, and so $[i] \cup \{\ell\} \in \mathcal{F}$, implying $A \in \mathcal{F}_i$.

We have shown that $\mathcal{F} = 2^{[\ell]} \times \mathcal{F}_0$. Since the extent of m is at most ℓ, necessarily $\mathcal{F} = 2^{[n]}$. □

The following crucial observation is the counterpart of Lemma 4.7.

Lemma 4.11. *Let $\mathcal{F}$ be a left-compressed t-intersecting family of symmetric extent ℓ and boundary $\mathcal{X}$. If $|A \cap B| = t$ for some $A, B \in \mathcal{X}$ then $|A \cap [\ell]| + |B \cap [\ell]| = \ell + t$.*

Proof. We start by showing that $A \cap B \subseteq [\ell]$. Indeed, suppose that $i \in A \cap B$ for some $i > \ell$. Since neither of A, B contains $\ell + 1$, in fact $i > \ell + 1$. Since $\mathcal{F}$ is left-compressed, $A' = (A \setminus \{i\}) \cup \{\ell + 1\} \in \mathcal{F}$. However, $|A' \cap B| = |A \cap B| - 1 = t - 1$, contradicting the assumption that $\mathcal{F}$ is t-intersecting.

Next, we show that $A \cup B \supseteq [\ell]$. Indeed, suppose that $i \notin A \cup B$ for some $i \in \ell$. By definition of $\mathcal{X}$, the set A must contain some element $j \in [\ell]$. By definition of symmetric extent (if $j < i$) or by the fact that $\mathcal{F}$ is left-compressed (if $j > i$), $A' = (A \setminus \{j\}) \cup \{i\} \in \mathcal{F}$. However, $|A' \cap B| = |A \cap B| - 1 = t - 1$, contradicting the assumption that $\mathcal{F}$ is t-intersecting.

Finally, let $A' = A \cap [\ell]$ and $B' = B \cap [\ell]$. Since $A' \cap B' = A \cap B$ and $A' \cup B' = [\ell]$, we deduce that $|A'| + |B'| = |A' \cup B'| + |A' \cap B'| = \ell + t$.. □

Our goal now is to try to eliminate $\mathcal{X}$, thus increasing the symmetric extent. We do this by trying to add $\binom{[\ell]}{a-1} \times \{\ell + 1\} \times \mathcal{X}_a$ to $\mathcal{F}$. The obstructions are described by Lemma 4.11, which explains why we decompose $\mathcal{X}$ according to the size of the intersection with $[\ell]$. Accordingly, our modification will act on the sets in $\mathcal{X}_a, \mathcal{X}_b$ for $a + b = \ell + t$. As in the preceding section, we have to consider two cases, $a \neq b$ and $a = b$, and the first case is simpler.

Lemma 4.12. *Let $\mathcal{F}$ be a left-compressed t-intersecting family on n points of symmetric extent $\ell < n$. Let $a \neq b$ be parameters such that $a + b = \ell + t$ and $\mathcal{X}_a, \mathcal{X}_b$ are not both empty. Consider the two families*

$$\mathcal{F}_1 = \left(\mathcal{F} \setminus \binom{[\ell]}{b} \times \mathcal{X}_b\right) \cup \binom{[\ell]}{a-1} \times \{\ell + 1\} \times \mathcal{X}_a, \quad \mathcal{F}_2 = \left(\mathcal{F} \setminus \binom{[\ell]}{a} \times \mathcal{X}_a\right) \cup \binom{[\ell]}{b-1} \times \{\ell + 1\} \times \mathcal{X}_b.$$

Both families $\mathcal{F}_1, \mathcal{F}_2$ are t-intersecting. Moreover, if $t > 1$ then for all $p \in (0, 1)$, $\max(\mu_p(\mathcal{F}_1), \mu_p(\mathcal{F}_2)) > \mu_p(\mathcal{F})$.

Proof. We start by showing that $\mathcal{F}_1$ (and so $\mathcal{F}_2$) is t-intersecting. Suppose that $S, T \in \mathcal{F}_1$. We consider several cases.

If $S, T \in \mathcal{F}$ then $|S \cap T| \geq t$ since $\mathcal{F}$ is t-intersecting.

If $S \in \mathcal{F}$ and $T \notin \mathcal{F}$ then $T \in \binom{[\ell]}{a-1} \times \{\ell + 1\} \times \mathcal{X}_a$. Choose $i \in [\ell] \setminus T$ arbitrarily, and notice that $T' = (T \setminus \{\ell + 1\}) \cup \{i\} \in \binom{[\ell]}{a} \times \mathcal{X}_a$, and so $T' \in \mathcal{F}$. If $i \notin S$ or $\ell + 1 \in S$ then $|S \cap T| \geq |S \cap T'| \geq t$.

Suppose therefore that $i \in S$ and $\ell+1 \notin S$. If $S' = (S \setminus \{i\}) \cup \{\ell+1\} \in \mathcal{F}$ then $|S \cap T| = |S' \cap T'| \geq t$. Otherwise, by definition of $\mathcal{X}$, $S \in \mathcal{X}$. By definition of $\mathcal{F}_1$, $|S \cap [\ell]| \neq b$, and so Lemma 4.11 shows that $|S \cap T| \geq |S \cap T'| - 1 \geq t$.

If $S, T \notin \mathcal{F}$ then $S, T \in \binom{[\ell]}{a-1} \times \{\ell+1\} \times \mathcal{X}_a$. Choose $i \in [\ell] \setminus S$ and $j \in [\ell] \setminus T$ arbitrarily, and define $S' = (S \setminus \{\ell+1\}) \cup \{i\}$ and $T' = (T \setminus \{\ell+1\}) \cup \{j\}$. As before, $S', T' \in \mathcal{X}$, and so Lemma 4.11 shows that $|S' \cap T'| \geq t+1$. Since $S \cap T \supseteq ((S' \cap T') \setminus \{i,j\}) \cup \{\ell+1\}$, we see that $|S \cap T| \geq |S' \cap T'| - 1 \geq t$.

We calculate the measures of $\mathcal{F}_1$ and $\mathcal{F}_2$ in terms of the quantities $m_a = \mu_p(\binom{[\ell]}{a} \times \mathcal{X}_a)$ and $m_b = \mu_p(\binom{[\ell]}{b} \times \mathcal{X}_b)$:

$$\mu_p(\mathcal{F}_1) = \mu_p(\mathcal{F}) - m_b + \frac{\binom{\ell}{a-1}}{\binom{\ell}{a}} m_a = \mu_p(\mathcal{F}) - m_b + \frac{a}{\ell - a + 1} m_a,$$

$$\mu_p(\mathcal{F}_2) = \mu_p(\mathcal{F}) - m_a + \frac{\binom{\ell}{b-1}}{\binom{\ell}{b}} m_b = \mu_p(\mathcal{F}) - m_a + \frac{b}{\ell - b + 1} m_b.$$

Multiply the first inequality by $\frac{\ell-a+1}{\ell-t+2}$, the second inequality by $\frac{\ell-b+1}{\ell-t+2}$, and add; note that $\ell - t + 2 = (\ell - a + 1) + (\ell - b + 1) > 0$. The result is

$$\begin{aligned} \frac{\ell-a+1}{\ell-t+2}\mu_p(\mathcal{F}_1) + \frac{\ell-b+1}{\ell-t+2}\mu_p(\mathcal{F}_2) &= \mu_p(\mathcal{F}) + \left[\frac{a}{\ell-t+2} - \frac{\ell-b+1}{\ell-t+2}\right] m_a + \left[\frac{b}{\ell-t+2} - \frac{\ell-a+1}{\ell-t+2}\right] m_b \\ &= \mu_p(\mathcal{F}) + \frac{t-1}{\ell-t+2}(m_a + m_b), \end{aligned}$$

using $a + b = \ell + t$. We conclude that when $t > 1$, $\max(\mu_p(\mathcal{F}_1), \mu_p(\mathcal{F}_2)) > \mu_p(\mathcal{F})$. □

When $a = b$, the construction increases the extent m (defined in the preceding section), and works only for large enough p.

Lemma 4.13. *Let $\mathcal{F}$ be a non-trivial monotone left-compressed t-intersecting family on n points of extent $m < n$ and symmetric extent ℓ, and let $s \in [n]$ be an index satisfying $s > m$ and $s \neq \ell+1$ (such an element exists if $\ell < m$ or if $m \leq n-2$). Suppose that $a = \frac{\ell+t}{2}$ is an integer and that $\mathcal{X}_a$ is non-empty. Let $\mathcal{X}'_a = \{S \in \mathcal{X}_a : s \in S\}$ and define*

$$\mathcal{F}' = \left(\mathcal{F} \setminus \binom{[\ell]}{a} \times \mathcal{X}_a\right) \cup \binom{[\ell+1]}{a} \times \mathcal{X}'_a.$$

The family $\mathcal{F}'$ is t-intersecting. Moreover, if $p > \frac{\ell-t+2}{2(\ell+1)}$ then $\mu_p(\mathcal{F}') > \mu_p(\mathcal{F})$.

Proof. We start by showing that $\mathcal{F}'$ is t-intersecting. Suppose that $S, T \in \mathcal{F}'$. We consider several cases.

If $S, T \in \mathcal{F}$ then $|S \cap T| \geq t$ since $\mathcal{F}$ is t-intersecting.

If $S \in \mathcal{F}$ and $T \notin \mathcal{F}$ then $T \in \binom{[\ell+1]}{a} \times \mathcal{X}'_a$ and $\ell+1, s \in T$. Choose $i \in [\ell] \setminus T$ arbitrarily, and notice that $T' = (T \setminus \{\ell+1\}) \cup \{i\} \in \binom{[\ell]}{a} \times \mathcal{X}_a \in \mathcal{F}$. If $i \notin S$ or $\ell+1 \in S$ then $|S \cap T| \geq |S \cap T'| \geq t$. Suppose therefore that $i \in S$ and $\ell+1 \notin S$. If $S' = (S \setminus \{i\}) \cup \{\ell+1\} \in \mathcal{F}$ then $|S \cap T| = |S' \cap T'| \geq t$. Otherwise, $S \in \mathcal{X}$. If $|S \cap [\ell]| \neq a$ then Lemma 4.11 shows that $|S \cap T| \geq |S \cap T'| - 1 \geq t$. If $|S \cap [\ell]| = a$ then by construction, $s \in S$. Since the extent of $\mathcal{F}$ is $m < s$, also $S' = S \setminus \{s\} \in \mathcal{F}$. Therefore $|S \cap T| \geq |S' \cap T'| \geq t$, since $s \in S \cap T$ but $s \notin S'$.

If $S, T \notin \mathcal{F}$ then $S, T \in \binom{[\ell+1]}{a} \times \mathcal{X}'_a$ and $\ell+1, s \in S, T$. Choose $i \in [\ell] \setminus S$ and $j \in [\ell] \setminus T$, so that $S' = (S \setminus \{\ell+1\}) \cup \{i\}$ and $T' = (T \setminus \{\ell+1\}) \cup \{j\}$ are both in $\mathcal{F}$. By construction, s belongs to S and T and so to S' and T'. Since the extent of $\mathcal{F}$ is $m < s$, $S'' = S \setminus \{s\}$ and $T'' = T \setminus \{s\}$ also belong to $\mathcal{F}$. Observe that $S \cap T \subseteq ((S'' \cap T'') \setminus \{i,j\}) \cup \{\ell+1, s\}$, and so $|S \cap T| \geq |S'' \cap T''| \geq t$.

We calculate the measure of $\mathcal{F}'$ in terms of the quantity $m_a = \mu_p(\binom{[\ell]}{a} \times \mathcal{X}_a)$:

$$\mu_p(\mathcal{F}') = \mu_p(\mathcal{F}) - m_a + p\frac{\binom{\ell+1}{a}}{\binom{\ell}{a}} m_a = \mu_p(\mathcal{F}) - m_a + p\frac{\ell+1}{\ell+1-a} m_a = \mu_p(\mathcal{F}) + \frac{a - (1-p)(\ell+1)}{\ell+1-a} m_a.$$

Thus $\mu_p(\mathcal{F}') > \mu_p(\mathcal{F})$ as long as $1 - p < \frac{a}{\ell+1} = \frac{\ell+t}{2(\ell+1)}$. □

Lemma 4.13 cannot be applied when $m = n$. However, if n has the correct parity, we can combine Lemma 4.13 with Lemma 4.8 to handle this issue.

Lemma 4.14. *Let $\mathcal{F}$ be a non-trivial monotone left-compressed t-intersecting family on n points of extent m and symmetric extent ℓ, where either $\ell < m$ or $m < n$. If $n + t$ is even and $\frac{\ell-t+2}{2(\ell+1)} < p \le \frac{1}{2}$ then there exists a t-intersecting family on n points with larger μ_p-measure.*

Proof. Consider first the case $\ell < m$. If $m < n$ then the statement follows from Lemma 4.12 and Lemma 4.13, so suppose that $m = n$. Let $\mathcal{F}' = \mathcal{F} \cup \mathcal{F} \times \{n+1\}$, and note that this is a non-trivial monotone left-compressed t-intersecting family on $n + 1$ points. We can apply Lemma 4.13 to obtain a non-trivial monotone left-compressed t-intersecting family $\mathcal{G}$ on $n + 1$ points satisfying $\mu_p(\mathcal{G}) > \mu_p(\mathcal{F}') = \mu_p(\mathcal{F})$. Since $n+1+t$ is odd, we can apply Lemma 4.8 repeatedly to obtain a non-trivial monotone t-intersecting family $\mathcal{H}$ on $n + 1$ points and extent n which satisfies $\mu_p(\mathcal{H}) \ge \mu_p(\mathcal{G}) > \mu_p(\mathcal{F})$. Since $\mathcal{H}$ has extent n, there is a t-intersecting family on n points having the same μ_p-measure.

Consider next the case $\ell = m < n$. If $m \le n - 2$ then the statement follows from Lemma 4.12 and Lemma 4.13, so suppose that $m = n - 1$. In this case $m + t$ is odd, and so the statement follows from Lemma 4.8. □

5 Proof of main theorem

5.1 The case $p < 1/2$

In this section we prove Theorem 3.1 in the case $p < 1/2$. In view of the results of Section 4.1, it suffices to consider monotone left-compressed families. We first settle the case $t = 1$, which corresponds to the classical Erdős–Ko–Rado theorem.

Lemma 5.1. *Let $\mathcal{F}$ be a monotone left-compressed intersecting family on n points of maximum μ_p-measure, for some $p \in (0, 1/2)$. Then $\mathcal{F} = \mathcal{F}_{1,0}$.*

Proof. Let m be the extent of $\mathcal{F}$. Since $\frac{m-t}{2(m-1)} = 1/2$, Lemma 4.8 and Lemma 4.9 together show that $m = 1$, and so $\mathcal{F} = \mathcal{F}_{1,0}$. □

The case $t \ge 2$ requires more work.

Lemma 5.2. *Let $\mathcal{F}$ be a monotone left-compressed t-intersecting family on n points of maximum μ_p-measure, for some $p \in (0, 1/2)$ and $t > 1$. Let r be the maximal integer satisfying $p \ge \frac{r}{t+2r-1}$ and $t + 2r \le n$. If $p \ne \frac{r}{t+2r-1}$ then $\mathcal{F} = \mathcal{F}_{t,r}$, and if $p = \frac{r}{t+2r-1}$ then $\mathcal{F} \in \{\mathcal{F}_{t,r}, \mathcal{F}_{t,r-1}\}$.*

Proof. Our definition of r guarantees that one of the following two alternatives holds: either $p < \frac{r+1}{t+2r-1}$, or $n \le t + 2r + 1$.

Let m be the extent of $\mathcal{F}$. We claim that $m \le t + 2r$. If $n \le t + 2r + 1$ then Lemma 4.8 shows that $m + t$ is even, and so $m \le t + 2r$. Suppose therefore that $p < \frac{r+1}{t+2r-1}$ and $m > t + 2r$. Lemma 4.8 shows that in fact $m \ge t + 2r + 2$, and so

$$\frac{m-t}{2(m-1)} = \frac{1}{2} - \frac{t-1}{2(m-1)} \ge \frac{1}{2} - \frac{t-1}{2(t+2r+1)} = \frac{r+1}{t+2r+1}.$$

Therefore Lemma 4.8 and Lemma 4.9 contradict the assumption that $\mathcal{F}$ has maximum μ_p-measure.

We now turn to consider the symmetric extent ℓ of $\mathcal{F}$. We first consider the case in which $p > \frac{r}{t+2r-1}$. We claim that in this case $\ell = m$. If $\ell < m$ then Lemma 4.8 and Lemma 4.12 show that both $m + t$ and $\ell + t$ are even, and so $\ell \le m - 2 \le t + 2r - 2$. This implies that

$$\frac{\ell-t+2}{2(\ell+1)} = \frac{1}{2} - \frac{t-1}{2(\ell+1)} \le \frac{1}{2} - \frac{t-1}{2(t+2r-1)} = \frac{r}{t+2r-1}.$$

Therefore Lemma 4.12 and Lemma 4.14 contradict the assumption that $\mathcal{F}$ has maximum μ_p-measure.

We have shown that if $p > \frac{r}{t+2r-1}$ then $\ell = m \le t + 2r$, and moreover $m + t$ is even. Thus $\ell = m = t + 2s$ for some $s \le r$. Since $\mathcal{F}$ is t-intersecting, $\mathcal{F} \subseteq \mathcal{F}_{t,s}$ for some $s \le r$. The fact that $\mathcal{F}$ has maximum μ_p-measure forces $\mathcal{F} = \mathcal{F}_{t,s}$. In view of Lemma 2.1, necessarily $s = r$.

The case $p = \frac{r}{t+2r-1}$ is slightly more complicated. Suppose first that $\ell = m$. In that case, as before, $\mathcal{F} = \mathcal{F}_{t,s}$ for some $s \leq r$. This time Lemma 2.1 shows that $s \in \{r, r-1\}$.

Suppose next that $\ell < m$. The same argument as before shows that $\ell \geq m-2$. Lemma 4.8 and Lemma 4.12 show that both $\ell + t$ and $m + t$ are even, and so $\ell = m - 2$ in this case. In the remainder of the proof, we show that this leads to a contradiction. To simplify notation, we will assume that $m = n$. As $m + t$ is even, we can write $m = t + 2s$ for some $s \leq r$.

Since $\mathcal{F}$ is monotone and has symmetric extent $m - 2$, it can be decomposed as follows:

$$\mathcal{F} = \binom{[t+2s-2]}{\geq a} \cup \binom{[t+2s-2]}{\geq b} \times \{t+2s+1\} \cup \\ \binom{[t+2s-2]}{\geq c} \times \{t+2s\} \cup \binom{[t+2s-2]}{\geq d} \times \{t+2s-1, t+2s\}.$$

Since the family is t-intersecting, we must have $2d - (t+2s-2) + 2 \geq t$, and so $d \geq t + s - 2$. If $d \geq t + s - 1$ then monotonicity implies that $a, b, c \geq d \geq t + s - 1$, and so $\mathcal{F} \subseteq \mathcal{F}_{t,s-1}$. Since $\mathcal{F}$ has maximum μ_p-measure, necessarily $\mathcal{F} = \mathcal{F}_{t,s-1}$, in which case the extent is $t + 2s - 2$, contrary to assumption.

We conclude that $d = t+s-2$. The fact that $\mathcal{F}$ is t-intersecting implies that $c+d-(t+2s-2)+1 \geq t$, and so $c \geq t + s - 1$. Similarly $b \geq t + s - 1$, and moreover $a + d - (t + 2s - 2) \geq t$, implying $a \geq t + s$. Thus $\mathcal{F} \subseteq \mathcal{F}_{t,s}$. Since $\mathcal{F}$ has maximum μ_p-measure, necessarily $\mathcal{F} = \mathcal{F}_{t,s}$, in which case the symmetric extent is $t + 2s$, contrary to assumption. □

5.2 The case $p = 1/2$

In this section we prove Theorem 3.1 in the case $p = 1/2$. This case is known as Katona's theorem, after Katona's paper [11], and we reprove it here using the techniques of Section 4. The case $t = 1$ is trivial, so we only prove the case $t \geq 2$. Once again, it suffices to consider monotone left-compressed families.

Lemma 5.3. *Let $\mathcal{F}$ be a monotone left-compressed t-intersecting family on n points of maximum $\mu_{1/2}$-measure, for some $t > 1$. If $n \in \{t + 2r, t + 2r + 1\}$ then $\mathcal{F} = \mathcal{F}_{t,r}$.*

Proof. Let m be the extent of $\mathcal{F}$, and ℓ be its symmetric extent.

Suppose first that $n = t + 2r$. Lemma 4.14 shows that $\ell = m = n$, which easily implies $\mathcal{F} = \mathcal{F}_{t,r}$.

Suppose next that $n = t+2r+1$. If $m \leq t+2r$ then the previous case $n = t+2r$ shows that $\mathcal{F} = \mathcal{F}_{t,r}$, so suppose that $m = n$. Since $m + t$ is odd, Lemma 4.8 shows that there is a family $\mathcal{H}$ of extent at most $m - 1 = t + 2r$ such that $\mu_p(\mathcal{H}) \geq \mu_p(\mathcal{F})$. In view of the preceding case, this shows that $\mathcal{H} = \mathcal{F}_{t,r}$, and so $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r})$. It remains to show that $\mathcal{F} = \mathcal{F}_{t,r}$ when $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}_{t,r})$.

The family $\mathcal{H}$ is constructed by repeatedly applying the following operation, where $\mathcal{G}^*$ is the boundary generating family of $\mathcal{F}$, and $a + b = n + t$: remove $\mathcal{G}^*_a$ and $\mathcal{G}^*_b$, and add either $\{S \setminus \{m\} : S \in \mathcal{G}^*_a\}$ or $\{S \setminus \{m\} : S \in \mathcal{G}^*_b\}$. All options must lead eventually to the same family $\mathcal{F}_{t,r}$, and this can only happen if $\mathcal{G}^*_a = \mathcal{G}^*_b = \emptyset$ for all a, b. However, in that case the extent of $\mathcal{F}$ is in fact $n - 1$, contradicting our assumption. □

5.3 The case $p > 1/2$

In this section we prove Theorem 3.1 in the case $p > 1/2$. The proof in this case differs from that of the other cases: it uses a different shifting argument, also due to Ahlswede and Khachatrian [6], who used it for the case $p = 1/2$.

The idea is to use a different kind of shifting. Let $\mathcal{F}$ be a family on n points. For two disjoint sets $A, B \subseteq [n]$, the shift operator $\mathbb{S}_{A,B}$ acts on $\mathcal{F}$ as follows. Let $\mathcal{F}_{A,B}$ consist of all sets in S containing A and disjoint from B. Then

$$\mathbb{S}_{A,B}(\mathcal{F}) = (F \setminus \mathcal{F}_{A,B}) \cup \{S : S \in \mathcal{F}_{A,B} \text{ and } (S \setminus A) \cup B \in \mathcal{F}\} \\ \cup \{(S \setminus A) \cup B : S \in \mathcal{F}_{A,B} \text{ and } (S \setminus A) \cup B \notin \mathcal{F}\}.$$

(This is a generalization of the original shifting operator: $\mathbb{S}_{i,j}$ is the same as $\mathbb{S}_{\{i\},\{j\}}$.)

This kind of shift is useful when $p > 1/2$ due to the following obvious property.

Lemma 5.4. *If $|B| > |A|$ then $\mu_p(\mathbb{S}_{A,B}(\mathcal{F})) \geq \mu_p(\mathcal{F})$ for any $p \in (1/2, 1)$, with equality if only if $\mathbb{S}_{A,B}(\mathcal{F}) = \mathcal{F}$.*

When done correctly, $\mathbb{S}_{A,B}$ preserves the property of being t-intersecting, as the following lemma from [6], whose lengthy proof we omit, shows.

Lemma 5.5. *Let $\mathcal{F}$ be a t-intersecting family on n points, and let $A, B \subseteq [n]$ be disjoint sets of cardinalities $|A| = s$ and $|B| = s+1$. If $\mathcal{F}$ is $(r, r+1)$-stable for all $r < s$ then $\mathbb{S}_{A,B}(\mathcal{F})$ is t-intersecting as well.*

A family is $(s, s+1)$-*stable* if $\mathbb{S}_{A,B}(\mathcal{F}) = \mathcal{F}$ for any disjoint sets A, B of cardinalities $|A| = s$ and $|B| = s+1$. As in the case of the simpler shifting operator $\mathbb{S}_{i,j}$, we can convert any family to a stable family while maintaining its being t-intersecting, by repeatedly applying a shifting operation on sets A, B with minimal $|A|$, implying the following lemma.

Lemma 5.6. *Let $p \in (1/2, 1)$. If $\mathcal{F}$ is a t-intersecting family on n points having maximum μ_p-measure then $\mathcal{F}$ is $(s, s+1)$-stable for all s.*

The importance of stable families is the following simple observation from [6].

Lemma 5.7. *If a t-intersecting family $\mathcal{F}$ on n points is $(s, s+1)$-stable for all s then every $A, B \in \mathcal{F}$ satisfy $|A| + |B| \geq n + t - 1$.*

Proof. Let $A, B \in \mathcal{F}$. If $A \cup B = [n]$ then $|A| + |B| = |A \cup B| + |A \cap B| \geq n + t$, so suppose $|A \cup B| < n$.

Define $s = \min(|A \cap B|, n - |A \cup B| - 1) \geq 0$, and choose a subset $C \subseteq A \cap B$ of size s and a subset $D \subseteq [n] \setminus (A \cup B)$ of size $s+1$. Since $\mathcal{F}$ is $(s, s+1)$-stable, $A' = (A \setminus C) \cup D \in \mathcal{F}$. We have $|A' \cap B| = |A \cap B| - |C| = |A \cap B| - s$, showing that $|A \cap B| \geq s + t$. In particular, $s = n - |A \cup B| - 1$, and so

$$|A| + |B| = |A \cup B| + |A \cap B| \geq (n - s - 1) + (s + t) = n + t - 1. \qquad \square$$

The bound $n + t - 1$ is tight: when $n = t + 2r + 1$, the family $\mathcal{F}_{t,r}$ is $(s, s+1)$-stable for all s, and two sets A, B of size $t + r$ satisfy $|A| + |B| = n + r - 1$.

We need one more lemma, on uniform families, which also follows from the classical Ahlswede–Khachatrian theorem; the proof appearing below only relies on the Erdős–Ko–Rado theorem.

Lemma 5.8. *Let $\mathcal{F} \subseteq \binom{[t+2r+1]}{t+r}$ be a t-intersecting family of maximum size, and define a family $\mathcal{F}'_{t,r}$, the uniform analog of $\mathcal{F}_{t,r}$, as follows:*

$$\mathcal{F}'_{t,r} = \{S \in \binom{[t+2r+1]}{t+r} : |S \cap [t+2r]| = t + r\}.$$

If $t \geq 2$ then $\mathcal{F}$ is equivalent to $\mathcal{F}'_{t,r}$ (that is, equals a similar family with $[t+2r]$ possibly replaced by some other subset of $[t+2r+1]$ of size $t+2r$), and if $t = 1$ then $|\mathcal{F}| \leq |\mathcal{F}'_{t,r}|$.

Proof. Define $\mathcal{G} = \{\overline{A} : A \in \mathcal{F}\}$ (where $\overline{A} = [t+2r+1] \setminus A$), so that $\mathcal{G} \subseteq \binom{[t+2r+1]}{r+1}$. Since

$$|\overline{A} \cap \overline{B}| = |\overline{A \cup B}| = t + 2r + 1 - |A \cup B| = t + 2r + 1 - |A| - |B| + |A \cap B| = |A \cap B| - (t-1),$$

we see that the condition that $\mathcal{F}$ is t-intersecting is equivalent to the condition that $\mathcal{G}$ is intersecting.

Since $r + 1 \leq \frac{t+2r+1}{2}$ (with equality only for $t = 1$), the Erdős–Ko–Rado theorem shows that $|\mathcal{G}| \leq \binom{t+2r}{r} = \binom{t+2r}{t+r}$. Moreover, when $t \geq 2$, equality holds only when $\mathcal{G} = \{S \in \binom{[t+2r+1]}{r+1} : i \in S\}$ for some $i \in [t+2r+1]$. In that case, $\mathcal{F} = \{S \in \binom{[t+2r+1]}{t+r} : i \notin S\}$, and so $\mathcal{F}$ is equivalent to $\mathcal{F}'_{t,r}$ (in the family $\mathcal{F}_{t,r}$ itself, $i = t + 2r + 1$). $\square$

We can now prove Theorem 3.1 in the case $p > 1/2$.

Lemma 5.9. *Let $\mathcal{F}$ be a t-intersecting family on n points of maximum μ_p-measure, for some $p \in (1/2, 1)$. Suppose that $n \in \{t+2r, t+2r+1\}$. If $t \geq 2$ or $n = t+2r$ then $\mathcal{F}$ is equivalent to $\mathcal{F}_{t,r}$. If $t = 1$ and $n = t+2r+1$ then $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r})$ and $\mathcal{F} = \mathcal{G} \cup \binom{[t+2r+1]}{\geq t+r+1}$, where $\mathcal{G} \subseteq \binom{[t+2r+1]}{t+r}$ contains exactly $\binom{t+2r}{t+r}$ sets.*

Proof. Lemma 5.6 shows that $\mathcal{F}$ is $(s, s+1)$-stable for all s, and so Lemma 5.7 shows that any $A, B \in \mathcal{F}$ satisfy $|A| + |B| \geq n + t - 1$. In particular, any set A has cardinality at least $\frac{n+t-1}{2}$. We now consider two cases, according to the parity of $n + t$.

Suppose first that $n = t + 2r$. Then $\frac{n+t-1}{2} = t + r - \frac{1}{2}$, and so all sets in $\mathcal{F}$ have cardinality at least $t + r$. In other words, $\mathcal{F} \subseteq \mathcal{F}_{t,r}$. Since $\mathcal{F}$ has maximum μ_p-measure, $\mathcal{F} = \mathcal{F}_{t,r}$.

Suppose next that $n = t + 2r + 1$. Then $\frac{n+t-1}{2} = t + r$, and so all sets in $\mathcal{F}$ have cardinality at least $t + r$. If $|A| \geq t + r$ and $|B| \geq t + r + 1$ then $|A \cap B| \geq |A| + |B| - n = t$, and so the fact that $\mathcal{F}$ has maximum μ_p-measure shows that $\mathcal{F} = \mathcal{G} \cup \binom{[n]}{\geq t+r+1}$, where $\mathcal{G} \subseteq \binom{[n]}{t+r}$ is t-intersecting.

We can now complete the proof using Lemma 5.8. If $t \geq 2$ then $\mathcal{G}$ is equivalent to $\mathcal{F}'_{t,r}$, say

$$\mathcal{G} = \{S \in \binom{[t+2r+1]}{t+r} : |S \cap X| = t + r\},$$

where $|X| = t + 2r$. Since any set of size at least $t + r + 1$ intersects X in at least $t + r$ points, $\mathcal{F} = \{S \subseteq [n] : |S \cap X| \geq t + r\}$, and so $\mathcal{F}$ is equivalent to $\mathcal{F}_{t,r}$.

When $t = 1$, Lemma 5.8 shows that $|\mathcal{G}| \leq |\mathcal{F}'_{t,r}|$, and so the same reasoning shows that $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r})$. □

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