Ahlswede–Khachatrian theorem

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Abstract

The Erdős–Ko–Rado theorem determines the largest μ_p-measure of an intersecting family of sets. We consider the analogue of this theorem to t-intersecting families (families in which any two sets have at least t elements in common), following Ahlswede and Khachatrian [1, 2]. We present a proof of the μ_p version of their theorem, which is adapted from the earlier proofs. Due to the simpler nature of the μ_p setting, our proof is simpler and cleaner.

1 Introduction

Friedgut [5] considered t-intersecting families, showing that if $\mathcal{F}$ is a t-intersecting family of sets and $p \leq 1/(t+1)$ then $\mu_p(\mathcal{F}) \leq p^t$. The upper bound on p arises naturally in his proof. This limitation is not arbitrary. Indeed, when $p > 1/(t+1)$, the bound p^t is incorrect. The correct bound was found by Ahlswede and Khachatrian [1, 2] in the k-uniform setting. We state it in the language of *slices*: for a family of sets $\mathcal{F}$, $\operatorname{Sl}(\mathcal{F}, k) = \{A \in \mathcal{F} : |A| = k\}$.

Definition 1.1. The (t, r) Frankl family $\mathcal{F}_{t,r}$ is the t-intersecting family defined by
$$\mathcal{F}_{t,r} = \{S \subseteq [t + 2r] : |S| \geq t + r\}.$$
○

Theorem 1.1 (Ahlswede–Khachatrian)**.** *Let $1 \leq t \leq k \leq n$ and $r \geq 0$, and let $\mathcal{F}$ be a t-intersecting family. When*
$$(k - t + 1)\left(2 + \frac{t-1}{r+1}\right) < n < (k - t + 1)\left(2 + \frac{t-1}{r}\right),$$
we have $|\operatorname{Sl}(\mathcal{F}, k)| \leq |\operatorname{Sl}(\mathcal{F}_{t,r}, k)|$, with equality only if the slices are equivalent.

When
$$n = (k - t + 1)\left(2 + \frac{t-1}{r+1}\right),$$
we have $|\operatorname{Sl}(\mathcal{F}, k)| \leq |\operatorname{Sl}(\mathcal{F}_{t,r}, k)| = |\operatorname{Sl}(\mathcal{F}_{t,r+1}, k)|$, with equality only if $\operatorname{Sl}(\mathcal{F}, k)$ is equivalent to either $\operatorname{Sl}(\mathcal{F}_{t,r}, k)$ or $\operatorname{Sl}(\mathcal{F}_{t,r+1}, k)$.

The Dinur–Safra argument [4] implies the following counterpart in the μ_p setting.

Corollary 1.2. *If $\mathcal{F}$ is t-intersecting then for $r \geq 0$, when*

$$\frac{r}{t+2r-1} < p < \frac{r+1}{t+2r+1},$$

we have $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r})$ with equality only if $\mathcal{F}$ is equivalent to $\mathcal{F}_{t,r}$.

If $p = (r+1)/(t+2r+1)$ then $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r}) = \mu_p(\mathcal{F}_{t,r+1})$.

Corollary 1.2 covers all $p < 1/2$ (and for $t = 1$, all $p \leq 1/2$). For $p > 1/2$, there is no meaningful bound in sight: the μ_p-measure of the t-intersecting family consisting of all sets of size at least $(n+t)/2$ approaches 1. For $p = 1/2$, the measure of this family approaches 1/2.

The Dinur–Safra argument isn't strong enough to handle equality when there are two different optimal families. In the rest of this chapter, we adapt the proof of the Ahlswede–Khachtrian theorem to the μ_p setting, thereby settling the cases $p = (r+1)/(t+2r+1)$. We will prove the following version of the Ahlswede–Khachatrian theorem, which uses the notion of *extension*: for a family of sets $\mathcal{F}$ on m points, $U^n(\mathcal{F}) = \{A \subseteq [n] : A \cap [m] \in \mathcal{F}\}$.

Theorem 1.3. *Let $\mathcal{F}$ be a t-intersecting family on n points for $t \geq 2$. If $r/(t+2r-1) < p < (r+1)/(t+2r+1)$ for some $r \geq 0$ then $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r})$, with equality if and only if $\mathcal{F}$ is equivalent to $U^n(\mathcal{F}_{t,r})$.*

If $p = (r+1)/(t+2r+1)$ for some $r \geq 0$ then $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r}) = \mu_p(\mathcal{F}_{t,r+1})$, with equality if and only if $\mathcal{F}$ is equivalent to either $U^n(\mathcal{F}_{t,r})$ or $U^n(\mathcal{F}_{t,r+1})$.

2 Proof overview

Our proof of the Ahlswede–Khachatrian theorem in the μ_p setting combines the approaches in the two papers [1, 2] in which Ahlswede and Khachatrian proved their theorem in the classical setting (the two papers present two different proofs). The classical Erdős–Ko–Rado theorem can be proved using Katona's circle argument, and here we will concentrate on t-intersecting families for $t \geq 2$.

Given $t \geq 2$ and $p \in (0, 1/2)$, our goal is to determine the t-intersecting families of maximum μ_p-measure. In general, the maximum μ_p-measure of a t-intersecting family depends on the size of its support: for example, the maximum μ_p-measure of a 2-intersecting family on 2 points is p^2 for all $p < 1/2$, but for any $p > 1/3$ there is a 2-intersecting family of larger measure $4p^3 - 3p^4$ on 4 points, namely the Frankl family $\mathcal{F}_{2,1}$. We will not be interested in the maximum μ_p-measure of a t-intersecting family on n points. Rather, we will be interested in the supremum of the μ_p-measures of t-intersecting families on any number of points; we will show that for all $p < 1/2$, the supremum is attained at one of the Frankl families.

The proof uses the technique of *shifting*. A t-intersecting family $\mathcal{F}$ on n points is *left-compressed* if for all $A \in \mathcal{F}$, $j \in A$ and $i \in [n] \setminus A$ satisfying $i < j$, we have $A \setminus \{j\} \cup \{i\} \in \mathcal{F}$. Using shifting, we can show that given any t-intersecting family, there is a left-compressed t-intersecting family with the same μ_p-measure for all p. Therefore as far as upper bounds are concerned, it is enough to consider left-compressed families.

Let $\mathcal{F}$ be a left-compressed t-intersecting family, let $r \geq 0$ be an integer, and suppose that $r/(t+2r-1) < p < (r+1)/(t+2r+1)$. We can also assume that $\mathcal{F}$ is monotone (if $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$). The proof consists of two steps. In the first step, we show that if $\mathcal{F}$ depends (as a Boolean function) on some $i > t + 2r$ then we can construct from $\mathcal{F}$ a t-intersecting family of larger μ_p-measure. This implies that the maximum μ_p-measure of a t-intersecting family is attained at some family on $t + 2r$ points. In the second step, we show that if $\mathcal{F}$ is not symmetric with respect to its first $t + 2r$ coordinates then we can construct from $\mathcal{F}$ a t-intersecting family of larger μ_p-measure. This implies that the maximum μ_p-measure of a t-intersecting family is attained (uniquely) at a family of the form $\{A \subseteq [t + 2r] : |A| \geq k\}$, and so at the Frankl family $\mathcal{F}_{t,r}$.

A similar but more delicate argument handles the case $p = (r+1)/(t+2r+1)$, and this completes the proof for left-compressed t-intersecting families. The upper bound on the μ_p-measure holds for arbitrary t-intersecting families. An argument similar in spirit to the one used by Chung et al. [3] to prove the equivalence of intersection problems and agreement problems shows that t-intersecting families of maximum μ_p-measure are equivalent to the corresponding Frankl family or families.

For the duration of the proof, we will use $\mu_p^X(\mathcal{F})$ to denote the μ_p-measure of a family $\mathcal{F}$ as a subset of 2^X.

3 Shifting

In this section we develop formally the classical technique of shifting. We start by defining the shifting operator.

Definition 3.1. Let $\mathcal{F}$ be a family of sets on n points, and let $i, j \in [n]$, $i \neq j$. For $A \in \mathcal{F}$, let $\mathbb{S}_{i \leftarrow j}(A) = A \setminus \{j\} \cup \{i\}$ if $j \in A$, $i \notin A$ and $A \setminus \{j\} \cup \{i\} \notin \mathcal{F}$, and let $\mathbb{S}_{i \leftarrow j}(A) = A$ otherwise. The *shifted family* $\mathbb{S}_{i \leftarrow j}(\mathcal{F})$ consists of the sets $\mathbb{S}_{i \leftarrow j}(A)$ for all $A \in \mathcal{F}$. ◯

As an example, let $\mathcal{F} = \{\{2\}, \{13\}, \{23\}\}$. Then $\mathbb{S}_{1 \leftarrow 2}(\mathcal{F}) = \{\{1\}, \{13\}, \{23\}\}$. Since $|\mathbb{S}_{i \leftarrow j}(A)| = |A|$, shifting doesn't change the μ_p-measure of a family. Shifting also maintains the property of being t-intersecting.

Lemma 3.1. *Let $\mathcal{F}$ be a family of sets on n points, and let $i, j \in [n]$, $i \neq j$. If $\mathcal{F}$ is t-intersecting for some $t \geq 1$ then $\mathbb{S}_{i \leftarrow j}(\mathcal{F})$ is also t-intersecting.*

Proof. Let $A' = \mathbb{S}_{i \leftarrow j}(A), B' = \mathbb{S}_{i \leftarrow j}(B) \in \mathbb{S}_{i \leftarrow j}(\mathcal{F})$, where $A, B \in \mathcal{F}$. We consider several cases. If $A' = A$ and $B' = B$ then $|A' \cap B'| = |A \cap B| \geq t$ since $\mathcal{F}$ is t-intersecting. If $A' \neq A$ and $B' \neq B$ then $i \in A', B'$ and $j \in A, B$, and so $|A' \cap B'| = |(A \cap B) \setminus \{j\} \cup \{i\}| = |A \cap B| \geq t$. The remaining case is when $A' \neq A$ and $B' = B$. If $j \notin B$ then $|A' \cap B'| \geq |(A \setminus \{j\}) \cap B| = |A \cap B| \geq t$. If $j \in B$ and $i \in B$ then $|A' \cap B'| = |(A \setminus \{j\} \cup \{i\}) \cap B| = |A \cap B| \geq t$. If $j \in B$ and $i \notin B$ then by the definition of $\mathbb{S}_{i \leftarrow j}(B)$, we must have $B'' = B \setminus \{j\} \cup \{i\} \in \mathcal{F}$.

Hence $|A' \cap B| = |(A' \setminus \{i\} \cup \{j\}) \cap (B \setminus \{j\} \cup \{i\})| = |A \cap B''| \geq t$. Therefore $\mathbb{S}_{i \leftarrow j}(\mathcal{F})$ is t-intersecting. □

By shifting a given family toward smaller elements, we can obtain a left-compressed family.

Definition 3.2. A family $\mathcal{F}$ on n points is *left-compressed* if $\mathbb{S}_{i \leftarrow j}(\mathcal{F}) = \mathcal{F}$ for all $i, j \in [n]$ such that $i < j$. ◯

Lemma 3.2. *Let $\mathcal{F}$ be a t-intersecting family on n points. There is a left-compressed t-intersecting family $\mathcal{G}$ on n points such that $\mu_p(\mathcal{G}) = \mu_p(\mathcal{F})$ for all $p \in [0, 1]$. Furthermore, $\mathcal{G}$ can be obtained from $\mathcal{F}$ by a sequence of applications of the operators $\mathbb{S}_{i \leftarrow j}$ for various i, j.*

Proof. Let $\Phi(\mathcal{F})$ be the sum of all elements in all sets in $\mathcal{F}$. It is easy to see that $\Phi(\mathbb{S}_{i \leftarrow j}(\mathcal{F})) \leq \Phi(\mathcal{F})$ whenever $i < j$, with equality only if $\mathbb{S}_{i \leftarrow j}(\mathcal{F}) = \mathcal{F}$. Let $\mathbb{S}(\mathcal{F})$ result from applying in sequence the operators $\mathbb{S}_{i \leftarrow j}$ for all $i, j \in [n]$ such that $i < j$, and define a sequence $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_{s+1} = \mathbb{S}(\mathcal{F}_s)$. Since $\Phi(\mathcal{F}_{s+1}) \leq \Phi(\mathcal{F}_s)$ and $\Phi(\mathcal{F}_s)$ is a non-negative integer, $\Phi(\mathcal{F}_s)$ reaches its minimum at some $s = T$. Since $\Phi(\mathcal{F}_{T+1}) = \Phi(\mathcal{F}_T)$ and so $\mathcal{F}_{T+1} = \mathcal{F}_T$, we conclude that $\mathbb{S}_{i \leftarrow j}(\mathcal{F}_T) = \mathcal{F}_T$ for all $i, j \in [n]$ such that $i < j$, and so $\mathcal{F}_T$ is left-compressed. Lemma 3.1 shows that $\mathcal{F}_T$ is t-intersecting. Finally, it is easy to check that shifting preserves the μ_p-measure for all $p \in [0, 1]$. □

From now on until Section 6 we will only be interested in left-compressed families.

4 Generating sets

In this section we implement the first step of the proof, following [1]. In this step, we show that if $\mathcal{F}$ is a monotone left-compressed t-intersecting family and $p < (r+1)/(t+2r+1)$, then either $\mathcal{F}$ depends only on the first $t+2r$ points, or we can modify $\mathcal{F}$ to obtain a t-intersecting family of larger measure. The tool we will use is *generating sets*.

Definition 4.1. Let $\mathcal{F}$ be a family of sets on n points. Its *generating set* $G(\mathcal{F})$ is the family of inclusion-minimal sets in $\mathcal{F}$. Its *extent* $m(\mathcal{F})$ is the largest integer appearing in any set in $G(\mathcal{F})$.

Let G be a family of sets on n points. Its *upset* $U^n(G)$ is the family $\mathcal{F} = \{A \subseteq [n] : A \supseteq B \text{ for some } B \in G\}$.

A family of sets $\mathcal{F}$ on n points is *monotone* if for all $B \in \mathcal{F}$, we have $A \in \mathcal{F}$ whenever $B \subseteq A \subseteq [n]$. An upset is always monotone. If $\mathcal{F}$ is monotone then $\mathcal{F} = U^n(G(\mathcal{F}))$. ◯

For example, $G(\mathcal{F}_{t,r}) = \{A \subseteq [t+2r] : |A| = t+r\}$ and $m(\mathcal{F}_{t,r}) = t+2r$. In the language of monotone Boolean functions, if $\mathcal{F}$ is monotone then $G(\mathcal{F})$ is its set of minterms.

Our goal in this section is to show that if $\mathcal{F}$ is a monotone t-intersecting family, $p < (r+1)/(t+2r+1)$ and $m(\mathcal{F}) > t+2r$ then there is another t-intersecting family $\mathcal{G}$ with $\mu_p(\mathcal{G}) > \mu_p(\mathcal{F})$. We will construct $\mathcal{G}$ by modifying the generating set of $\mathcal{F}$, guided by the following easy lemma.

Lemma 4.1. *Let $\mathcal{F}$ be a left-compressed t-intersecting family with $m = m(\mathcal{F})$, and suppose that $A, B \in \mathcal{F}$ both contain m. If $|A \cap B| = t$ then $A \cup B = [m]$ and so $|A| + |B| = m + t$.*

Proof. Let $A, B \in \mathcal{F}$ be as indicated. Clearly $A \cup B \subseteq [m]$. Suppose that for some $i \in [m]$, $i \notin A \cup B$. By assumption, $i < m$. Since $\mathcal{F}$ is left-compressed, $A' = A \setminus \{m\} \cup \{i\} \in \mathcal{F}$. However, $|A' \cap B| = |A \cap B| - 1 = t - 1$, contradicting the assumption that $\mathcal{F}$ is t-intersecting. We conclude that $A \cup B = [m]$ and so $|A| + |B| = |A \cup B| + |A \cap B| = m + t$. □

This lemma suggests separating the sets in $G(\mathcal{F})$ containing m according to their size.

Definition 4.2. Let $\mathcal{F}$ be a family of sets with $m = m(\mathcal{F})$. We define $G^*(\mathcal{F}) = \{A \in G(\mathcal{F}) : m \in A\}$ and $G^*_a(\mathcal{F}) = \{A \in G^*(\mathcal{F}) : |A| = a\}$. In words, $G^*(\mathcal{F})$ consists of those sets in $G(\mathcal{F})$ containing m, and $G^*_a(\mathcal{F})$ consists of those sets in $G(\mathcal{F})$ containing m and of size a.

For a family G on n points and $m \in [n]$, we define $G \setminus m = \{A \setminus \{m\} : A \in G\}$. ◯

Suppose $a + b = m(\mathcal{F}) + t$ and $a \neq b$. Lemma 4.1 implies that $U^n\big(G(\mathcal{F}) \setminus (G^*_a(\mathcal{F}) \cup G^*_b(\mathcal{F})) \cup (G^*_a(\mathcal{F}) \setminus m(\mathcal{F}))\big)$ is t-intersecting. Moreover, it turns out that this transformation can be used to increase the μ_p-measure.

We start by proving two easy auxiliary results.

Lemma 4.2. *Let $\mathcal{F}$ be a monotone left-compressed family on n points with $m = m(\mathcal{F})$ and let $A \in G^*(\mathcal{F})$. Then*

$$\mathcal{F} \setminus U^n(G(\mathcal{F}) \setminus \{A\}) = \{A\} \times 2^{[n]\setminus[m]}.$$

In words, if $A \in G^*(\mathcal{F})$ then the sets generated by A are exactly $\{A\} \times 2^{[n]\setminus[m]}$.

Proof. Suppose $B \in \mathcal{F} \setminus U^n(G(\mathcal{F}) \setminus \{A\})$. Clearly $B \supseteq A$. We would like to show that $B \cap [m] = A$. If not, then let $x \in (B \cap [m]) \setminus A$. Since $\mathcal{F}$ is left-compressed, $C = \mathbb{S}_{x \leftarrow m}(A) \in \mathcal{F}$. Clearly $C \in U^n(G(\mathcal{F}) \setminus \{A\})$, and since $B \supseteq C$, also $B \in U^n(G(\mathcal{F}) \setminus \{A\})$, contrary to the assumption. Hence $B \cap [m] = A$.

For the other direction, let $B = A \cup C$, where $C \subseteq [n] \setminus [m]$. If $B \in U^n(G(\mathcal{F}) \setminus \{A\})$ then $B \supseteq D$ for some $D \in G(\mathcal{F}) \setminus \{A\}$. Since $\max D \le m$, necessarily $D \subseteq B \cap [m] = A$, contradicting the fact that A is inclusion-minimal. This completes the proof of the lemma. □

Lemma 4.3. *Let $\mathcal{F}$ be a family of sets on n points with $m = m(\mathcal{F})$ and let $A \in G^*(\mathcal{F})$. If $B \in \mathcal{F}$ and $B \cap [m-1] = A \setminus \{m\}$ then $m \in B$.*

Proof. Suppose that $m \notin B$. Since $B \in \mathcal{F}$, $B \supseteq C$ for some $C \in G(\mathcal{F})$. Since $\max C \leq m$ and $m \notin B$, $C \subseteq B \cap [m] = A \setminus \{m\}$, contradicting the fact that A is inclusion-minimal. $\square$

Next, we describe the transformation itself.

Lemma 4.4. *Let $\mathcal{F}$ be a monotone left-compressed t-intersecting family on n points with $m = m(\mathcal{F})$, and let $a + b = m + t$ for some non-negative integers $a \neq b$. Define*

$$H_a = G(\mathcal{F}) \setminus (G_a^*(\mathcal{F}) \cup G_b^*(\mathcal{F})) \cup (G_a^*(\mathcal{F}) \setminus m), \qquad \mathcal{G}_a = U^n(H_a),$$
$$H_b = G(\mathcal{F}) \setminus (G_a^*(\mathcal{F}) \cup G_b^*(\mathcal{F})) \cup (G_b^*(\mathcal{F}) \setminus m), \qquad \mathcal{G}_b = U^n(H_b).$$

The families $\mathcal{G}_a, \mathcal{G}_b$ are t-intersecting. Furthermore, if $G_a^*(\mathcal{F}) \neq \emptyset$ or $G_b^*(\mathcal{F}) \neq \emptyset$ then for all $p < 1/2$, $\max(\mu_p(\mathcal{G}_a), \mu_p(\mathcal{G}_b)) > \mu_p(\mathcal{F})$.

Proof. In order to show that $\mathcal{G}_a$ is t-intersecting, it is enough to show that H_a is t-intersecting. Let $A, B \in H_a$. If $A, B \notin G_a^*(\mathcal{F}) \setminus m$ then $A, B \in G(\mathcal{F})$ and so $|A \cap B| \geq t$, so suppose that $A \in G_a^*(\mathcal{F}) \setminus m$. Notice that $A \cup \{m\} \in G_a^*(\mathcal{F})$. If $B \notin G^*(\mathcal{F})$ then $m \notin B$ and $B \in G(\mathcal{F})$, and so $|A \cap B| = |(A \cup \{m\}) \cap B| \geq t$. If $B \in G_c^*(\mathcal{F})$ then $c \neq b$ and so $|A \cup \{m\}| + |B| = a + c \neq a + b = m + t$. Therefore Lemma 4.1 implies that $|(A \cup \{m\}) \cap B| \geq t + 1$ and so $|A \cap B| \geq t$. A similar argument applies if $B \in G_a^*(\mathcal{F}) \setminus \{m\}$ (with a in place of c), and we conclude that $\mathcal{G}_A$ is t-intersecting. The proofs for $\mathcal{G}_b$ are analogous.

Let $p < 1/2$. We proceed to calculate $\mu_p(\mathcal{G}_a)$ and $\mu_p(\mathcal{G}_b)$. Lemma 4.2 shows that

$$\mathcal{F} \setminus \mathcal{G}_a = G_b^*(\mathcal{F}) \times 2^{[n] \setminus [m]},$$

and Lemma 4.3 shows that

$$\mathcal{G}_a \setminus \mathcal{F} = (G_a^*(\mathcal{F}) \setminus m) \times 2^{[n] \setminus [m]}.$$

Therefore

$$\begin{aligned}\mu_p(\mathcal{G}_a) &= \mu_p(\mathcal{F}) - \mu_p^{[m]}(G_b^*(\mathcal{F})) + \mu_p^{[m]}(G_a^*(\mathcal{F}) \setminus m) \\ &= \mu_p(\mathcal{F}) - \mu_p^{[m]}(G_b^*(\mathcal{F})) + \frac{1-p}{p} \mu_p^{[m]}(G_a^*(\mathcal{F})).\end{aligned}$$

Without loss of generality, suppose that $\mu_p^{[m]}(G_a^*(\mathcal{F})) \geq \mu_p^{[m]}(G_b^*(\mathcal{F}))$, which implies that $\mu_p^{[m]}(G_a^*(\mathcal{F})) > 0$ by assumption. Then

$$\mu_p(\mathcal{G}_a) \geq \mu_p(\mathcal{F}) + \left(\frac{1-p}{p} - 1\right) \mu_p^{[m]}(G_a^*(\mathcal{F})) > 0,$$

since $p < 1/2$ implies $(1-p)/p > 1$. $\square$

This lemma allows us to achieve our goal whenever $G_a^*(\mathcal{F}) \neq \emptyset$ for some $a \neq (m(\mathcal{F}) + t)/2$. When $a = (m(\mathcal{F}) + t)/2$, the construction in the lemma doesn't result in a t-intersecting family. In order to fix the construction, we will focus on a subset of $G_a^*(\mathcal{F})$ *not* containing some common element. This property will guarantee that the result is t-intersecting. If p is small enough (depending on $m(\mathcal{F})$), then the construction still increases the μ_p-measure.

Lemma 4.5. *Let $\mathcal{F}$ be a monotone left-compressed t-intersecting family on n points with $m = m(\mathcal{F}) > t+2r$ for some $r \geq 0$, and let $a = (m+t)/2$ be integral. For $i \in [m-1]$, define*

$$H_i = G(\mathcal{F}) \setminus G_a^*(\mathcal{F}) \cup \{A \in G_a^*(\mathcal{F}) \setminus m : i \notin A\}, \qquad \mathcal{G}_i = U^n(H_i).$$

The families $\mathcal{G}_i$ are t-intersecting. Furthermore, if $p < (r+1)/(t+2r+1)$ and $G_a^*(\mathcal{F}) \neq \emptyset$ then $\max_{i\in[m-1]} \mu_p(\mathcal{G}_i) > \mu_p(\mathcal{F})$.

Proof. Let $i \in [m-1]$. We proceed to show that $\mathcal{G}_i$ is t-intersecting. As in the proof of the corresponding part of Lemma 4.4, it is enough to show that H_i is t-intersecting. If $A, B \in H_i$ and not both $A, B \in G_a^*(\mathcal{F}) \setminus m$ then the argument in Lemma 4.4 shows that $|A \cap B| \geq t$, so suppose that $A, B \in G_a^*(\mathcal{F}) \setminus m$. Note that $i \notin A, B$. Lemma 4.1 shows that $|(A \cup \{m\}) \cap (B \cup \{m\})| > t$, and so $|A \cap B| \geq t$, unless $(A \cup \{m\}) \cup (B \cup \{m\}) = [m]$. However, the latter is impossible since $i \notin A \cup B$. This shows that $\mathcal{G}_i$ is t-intersecting.

Let $K_i = \{A \in G_a^*(\mathcal{F}) : i \notin A\}$. We proceed to calculate $\mu_p(\mathcal{G}_i)$. Lemma 4.2 shows that

$$\mathcal{F} \setminus \mathcal{G}_i = (G_a^*(\mathcal{F}) \setminus K_i) \times 2^{[n]\setminus[m]},$$

and Lemma 4.3 shows that

$$\mathcal{G}_i \setminus \mathcal{F} = (K_i \setminus m) \times 2^{[n]\setminus[m]}.$$

Therefore

$$\begin{aligned} \mu_p(\mathcal{G}_i) &= \mu_p(\mathcal{F}) - \mu_p^{[m]}(G_a^*(\mathcal{F}) \setminus K_i) + \frac{1-p}{p}\mu_p^{[m]}(K_i) \\ &= \mu_p(\mathcal{F}) - \mu_p^{[m]}(G_a^*(\mathcal{F})) + \frac{1}{p}\mu_p^{[m]}(K_i). \end{aligned} \tag{1}$$

In view of this, we would like to maximize $\mu_p^{[m]}(K_i)$. Since all sets in K_i have Hamming weight a, $\mu_p^{[m]}(K_i) = |K_i|p^a(1-p)^{m-a}$, and similarly $\mu_p^{[m]}(G_a^*(\mathcal{F})) = |G_a^*(\mathcal{F})|p^a(1-p)^{m-a}$. We therefore want to maximize $|K_i|$. Since each $A \in G_a^*(\mathcal{F})$ satisfies $|A \cap [m-1]| = a-1$, it is easy to see that

$$\mathop{\mathbb{E}}_{i\in[m-1]} |K_i| = \frac{m-a}{m-1}|G_a^*(\mathcal{F})|.$$

There must be some $i \in [m-1]$ which satisfies $|K_i| \geq (m-a)/(m-1) \cdot |G_a^*(\mathcal{F})|$, and so $\mu_p^{[m]}(K_i) \geq (m-a)/(m-1) \cdot \mu_p^{[m]}(G_a^*(\mathcal{F}))$. Substituting this in (1), we obtain

$$\begin{aligned} \mu_p(\mathcal{G}_i) - \mu_p(\mathcal{F}) &\geq \left(\frac{1}{p} \cdot \frac{m-a}{m-1} - 1\right)\mu_p^{[m]}(G_a^*(\mathcal{F})) \\ &= \frac{m-a-p(m-1)}{p(m-1)}\mu_p^{[m]}(G_a^*(\mathcal{F})). \end{aligned}$$

The proof will be complete if we show that $m - a > p(m-1)$. Since $m > t+2r$ and $m+t$ is even, $m \geq t + 2r + 2$, and so

$$\begin{aligned} 2[m - a - p(m-1)] &= m - t - 2p(m-1) \\ &= (1-2p)m - t + 2p \\ &\geq (1-2p)(t+2r+2) - t + 2p \\ &= 2r + 2 - 2p(t+2r+1) \\ &= 2[r+1-p(t+2r+1)] > 0. \end{aligned}$$ ☐

Combining Lemma 4.4 and Lemma 4.5, we obtain the following result.

Lemma 4.6. *Let $\mathcal{F}$ be a monotone left-compressed t-intersecting family on n points with $m = m(\mathcal{F}) > t + 2r$ for some $r \geq 0$. If $p < (r+1)/(t+2r+1)$ then there exists a t-intersecting family $\mathcal{G}$ on n points satisfying $\mu_p(\mathcal{G}) > \mu_p(\mathcal{F})$.*

Proof. By definition, $G^*(\mathcal{F}) \neq \emptyset$, and so $G^*_a(\mathcal{F}) \neq \emptyset$ for some a. If $a \neq (m+t)/2$ then the result follows from Lemma 4.4, otherwise it follows from Lemma 4.5. ☐

We can conclude an important corollary.

Corollary 4.7. *Let $t \geq 1$, $r \geq 0$ and $p < (r+1)/(t+2r+1)$. There exists a monotone left-compressed t-intersecting family $\mathcal{F}$ on $t+2r$ points such that for every t-intersecting family $\mathcal{G}$, $\mu_p(\mathcal{G}) \leq \mu_p(\mathcal{F})$. Furthermore, equality is only possible if $m(\mathcal{G}) \leq t + 2r$.*

Proof. Lemma 3.2 implies that it is enough to construct a (not necessarily left-compressed) t-intersecting family $\mathcal{F}$ on $t+2r$ points. We let $\mathcal{F}$ be a t-intersecting family of maximal μ_p-measure among those on $t+2r$ points.

Now let $\mathcal{G}$ be a t-intersecting family on n points. In order to show that $\mu_p(\mathcal{G}) \leq \mu_p(\mathcal{F})$, we can assume that $\mathcal{G}$ has maximal μ_p-measure among t-intersecting families on n points. Lemma 4.6 implies that $m(\mathcal{G}) \leq t+2r$, and so $\mu_p(\mathcal{G}) \leq \mu_p(\mathcal{F})$ by definition. The lemma also implies that equality is only possible if $m(\mathcal{G}) \leq t + 2r$. ☐

At this point, [1] considers the complemented family $\bar{\mathcal{F}} = \{[n] \setminus A : A \in \mathcal{F}\}$. When $\mathcal{F}$ is a k-uniform t-intersecting family, $\bar{\mathcal{F}}$ is an $(n-k)$-uniform $(n-2k+t)$-intersecting family, and we can apply Corollary 4.7 to $\bar{\mathcal{F}}$. However, in our setting $\bar{\mathcal{F}}$ need not even be intersecting. Instead, we turn to the argument in [2].

5 Pushing-pulling

In this section we implement the second step of the proof, following [2]. We will show that if $\mathcal{F}$ is a left-compressed t-intersecting family of maximal μ_p-measure, where $p > r/(t+2r-1)$, then the first $t+2r$ coordinates of $\mathcal{F}$ are symmetric. We start by formalizing the notion of symmetry.

Definition 5.1. A family of sets $\mathcal{F}$ is *ℓ-invariant* if for all $i \neq j$ in the range $1 \leq i, j \leq \ell$, $\mathbb{S}_{i \leftarrow j}(\mathcal{F}) = \mathcal{F}$.

The *symmetric extent* $\ell(\mathcal{F})$ of a family of sets $\mathcal{F}$ on n points is the maximal $\ell \leq n$ such that $\mathcal{F}$ is ℓ-invariant. ◯

Our goal in this section is to show that if $p > r/(t+2r-1)$ and $\ell(\mathcal{F}) < t+2r$ for some t-intersecting family $\mathcal{F}$ then we can come up with a t-intersecting family of larger μ_p-measure.

Since we are focusing on left-compressed families, the only way in which ℓ-invariance can fail is if $\mathbb{S}_{\ell \leftarrow i}(\mathcal{F}) \neq \mathcal{F}$ for some $i < \ell$. The following definition singles out the sets which determine the symmetric extent of a family.

Definition 5.2. Let $\mathcal{F}$ be a family of sets on n points with $\ell = \ell(\mathcal{F})$. If $n > \ell$ then its *boundary sets* are given by

$$X(\mathcal{F}) = \{A \in \mathcal{F} : \mathbb{S}_{\ell+1 \leftarrow i}(A) \notin \mathcal{F} \text{ for some } i \leq \ell\}.$$

if $n = \ell$ then we define $X(\mathcal{F}) = \emptyset$. ◯

Our starting point is the following analog of Lemma 4.1.

Lemma 5.1. *Let $\mathcal{F}$ be a left-compressed t-intersecting family with $\ell = \ell(\mathcal{F})$, and let $A, B \in X(\mathcal{F})$. If $|A \cap B| = t$ then $A \cap B \subseteq [\ell]$ and $A \cup B \supseteq [\ell]$, and so $|A \cap [\ell]| + |B \cap [\ell]| = \ell + t$.*

Proof. Let $A, B \in X(\mathcal{F})$ be as given, and note that $\ell + 1 \notin A, B$. We start by showing that $A \cap B \subseteq [\ell]$. Suppose that $x \in A \cap B$ satisfies $x > \ell$. Since $\ell + 1 \notin A, B$, in fact $x > \ell + 1$. Since $\mathcal{F}$ is left-compressed, $\mathbb{S}_{\ell+1 \leftarrow x}(A) \in \mathcal{F}$. However, $|\mathbb{S}_{\ell+1 \leftarrow x}(A) \cap B| = |A \cap B| - 1 = t - 1$, contrary to assumption. We conclude that $A \cap B \subseteq [\ell]$.

Next, we show that $A \cup B \supseteq [\ell]$. Suppose that $x \notin A \cup B$ for some $x \in [\ell]$. Since $t \geq 1$ and $A \cap B \subseteq [\ell]$, there is some $y \in A \cap B \cap [\ell]$. Since $\mathcal{F}$ is ℓ-invariant, $\mathbb{S}_{x \leftarrow y}(A) \in \mathcal{F}$. However, $|\mathbb{S}_{x \leftarrow y}(A) \cap B| = |A \cap B| - 1 = t - 1$, contrary to assumption. We conclude that $A \cup B \supseteq [\ell]$.

Finally, let $A' = A \cap [\ell]$ and $B' = B \cap [\ell]$. We have $A' \cup B' = [\ell]$ and $|A' \cap B'| = |A \cap B| = t$, and so $|A'| + |B'| = |A' \cup B'| + |A' \cap B'| = \ell + t$. □

This suggests breaking down $X(\mathcal{F})$ according to the size of the intersection with $[\ell]$.

Definition 5.3. Let $\mathcal{F}$ be a family of sets on n points with $\ell = \ell(\mathcal{F})$. Its ith *boundary marginal* is given by

$$X_i(\mathcal{F}) = \{B \subseteq [n] \setminus [\ell + 1] : [i] \cup B \in X(\mathcal{F})\}. \quad ◯$$

The part played by the sets $[i]$ is arbitrary. Indeed, we have the following easy lemma.

Definition 5.4. For a set X and an integer i, we define

$$\binom{X}{i} = \{A \subseteq X : |A| = i\}. \quad ◯$$

Lemma 5.2. *Let $\mathcal{F}$ be a family of sets on n points with $\ell = \ell(\mathcal{F})$. Then*

$$X(\mathcal{F}) = \bigcup_{i=1}^{\ell} \binom{[\ell]}{i} \times X_i(\mathcal{F}).$$

Proof. If $A \in X(\mathcal{F})$ then $\mathbb{S}_{\ell+1 \leftarrow i}(A) \neq A$ for some $i \leq \ell$, and in particular $i \in A$. This shows that $X_0(\mathcal{F}) = \emptyset$. Also, clearly $\ell + 1 \notin A$ for all $A \in X(\mathcal{F})$. The lemma now follows directly from the ℓ-equivalence of $\mathcal{F}$. $\square$

We now present two different constructions that attempt to increase the μ_p-measure of a t-intersecting family. The first construction is the counterpart of Lemma 4.4.

Lemma 5.3. *Let $\mathcal{F}$ be a t-intersecting left-compressed family on n points with $\ell = \ell(\mathcal{F})$, and let $a + b = \ell + t$ for some non-negative integers $a \neq b$. Define*

$$\mathcal{G}_a = \mathcal{F} \setminus \binom{[\ell]}{b} \times X_b(\mathcal{F}) \cup \binom{[\ell]}{a-1} \times \{\ell+1\} \times X_a(\mathcal{F}),$$
$$\mathcal{G}_b = \mathcal{F} \setminus \binom{[\ell]}{a} \times X_a(\mathcal{F}) \cup \binom{[\ell]}{b-1} \times \{\ell+1\} \times X_b(\mathcal{F}).$$

The families $\mathcal{G}_a, \mathcal{G}_b$ are t-intersecting. Furthermore, if $G^*_a(\mathcal{F}) \neq \emptyset$ or $G^*_b(\mathcal{F}) \neq \emptyset$ and $t \geq 2$ then for all $p \in (0,1)$, $\max(\mu_p(\mathcal{G}_a), \mu_p(\mathcal{G}_b)) > \mu_p(\mathcal{F})$.

Proof. We start by showing that $\mathcal{G}_a$ is t-intersecting. Let $A, B \in \mathcal{G}_a$. If $A, B \notin \binom{[\ell]}{a-1} \times \{\ell+1\} \times X_a(\mathcal{F})$ then $A, B \in \mathcal{F}$ and so $|A \cap B| \geq t$, so assume that $A \in \binom{[\ell]}{a-1} \times \{\ell+1\} \times X_a(\mathcal{F})$. Pick some $x \in [\ell]$ such that $x \notin A$, and notice that $A' = A \setminus \{\ell+1\} \cup \{x\} \in \mathcal{F}$.

Suppose first that $B \in \mathcal{F}$. If $\ell+1 \in B$ or $x \notin B$ then $|A \cap B| \geq |A' \cap B| \geq t$, so suppose that $\ell + 1 \notin B$ and $x \in B$. If $B' = \mathbb{S}_{\ell+1 \leftarrow x}(B) \in \mathcal{F}$ then $|A \cap B| = |A' \cap B'| \geq t$. Otherwise, $B \in X(\mathcal{F})$ and since $\ell+1 \notin B$, $|B \cap [\ell]| \neq b$. Since $|A' \cap [\ell]| = a$, $|A' \cap [\ell]| + |B \cap [\ell]| \neq a + b = \ell + t$, and so Lemma 5.1 shows that $|A' \cap B| \geq t + 1$, which implies $|A \cap B| \geq |A' \cap B| - 1 \geq t$.

Finally, suppose that $A, B \notin \mathcal{F}$. Pick some $y \in [\ell]$ such that $y \notin B$, and notice that $B' = B \setminus \{\ell+1\} \cup \{y\} \in \mathcal{F}$. Since $|A' \cap [\ell]| + |B' \cap [\ell]| = 2a \neq a + b = \ell + t$, Lemma 5.1 shows that $|A' \cap B'| \geq t + 1$. Therefore $|A \cap B| = |[(A' \setminus \{x\}) \cap (B' \setminus \{y\})] \cup \{\ell+1\}| \geq t$. We conclude that $\mathcal{G}_a$ is t-intersecting.

It is straightforward to compute the μ_p-measures of $\mathcal{G}_a$ and $\mathcal{G}_b$:

$$\mu_p(\mathcal{G}_a) = \mu_p(\mathcal{F}) - \binom{\ell}{b} p^b (1-p)^{\ell+1-b} \mu_p^{[n]\setminus[\ell+1]}(X_b(\mathcal{F})) + \binom{\ell}{a-1} p^a (1-p)^{\ell+1-a} \mu_p^{[n]\setminus[\ell+1]}(X_a(\mathcal{F})),$$
$$\mu_p(\mathcal{G}_b) = \mu_p(\mathcal{F}) - \binom{\ell}{a} p^a (1-p)^{\ell+1-a} \mu_p^{[n]\setminus[\ell+1]}(X_a(\mathcal{F})) + \binom{\ell}{b-1} p^b (1-p)^{\ell+1-b} \mu_p^{[n]\setminus[\ell+1]}(X_b(\mathcal{F})).$$

These formulas become simpler if we put

$$\gamma_a = \binom{\ell}{a} p^a (1-p)^{\ell+1-a} \mu_p^{[n]\setminus[\ell+1]}(X_a(\mathcal{F})), \quad \gamma_b = \binom{\ell}{b} p^b (1-p)^{\ell+1-b} \mu_p^{[n]\setminus[\ell+1]}(X_b(\mathcal{F})).$$

By assumption, either $\gamma_a > 0$ or $\gamma_b > 0$. Substituting these variables, we get

$$\mu_p(\mathcal{G}_a) = \mu_p(\mathcal{F}) - \gamma_a + \frac{a}{\ell - a + 1}\gamma_b, \quad \mu_p(\mathcal{G}_b) = \mu_p(\mathcal{F}) - \gamma_b + \frac{b}{\ell - b + 1}\gamma_a.$$

Multiply the first equation by $\ell - a + 1$, the second equation by $\ell - b + 1$, and sum to get

$$(\ell-a+1)(\mu_p(\mathcal{G}_a)-\mu_p(\mathcal{F}))+(\ell-b+1)(\mu_p(\mathcal{G}_b)-\mu_p(\mathcal{F})) = (a+b-\ell-1)(\gamma_a+\gamma_b) = (t-1)(\gamma_a+\gamma_b) > 0.$$

We conclude that either $\mu_p(\mathcal{G}_a) > \mu_p(\mathcal{F})$ or $\mu_p(\mathcal{G}_b) > \mu_p(\mathcal{F})$. $\square$

The second construction, which is the counterpart of Lemma 4.5, concerns $a = (\ell + t)/2$, and works by adjoining a new element, which ensures that the resulting family is t-intersecting.

Lemma 5.4. *Let $\mathcal{F}$ be a t-intersecting left-compressed family on n points with $\ell = \ell(\mathcal{F})$, and let $a = (\ell + t)/2$ be integral. Define*

$$\mathcal{G} = \mathcal{F} \setminus \binom{[\ell]}{a} \times X_a(\mathcal{F}) \times 2^{\{n+1\}} \cup \binom{[\ell+1]}{a} \times X_a(\mathcal{F}) \times \{n+1\}.$$

Note that $\mathcal{G}$ is a family on $n+1$ points. The family $\mathcal{G}$ is t-intersecting. Moreover, if $X_a(\mathcal{F}) \neq \emptyset$, $t \geq 2$ and $r/(t + 2r - 1) < p < 1/2$, $\ell < t + 2r$ for some $r \geq 0$, then $\mu_p(\mathcal{G}) > \mu_p(\mathcal{F})$.

Proof. Put $\mathcal{F}' = \mathcal{G} \times 2^{\{n+1\}}$, and note that $\mathcal{F}'$ is t-intersecting and $\mu_p(\mathcal{F}') = \mu_p(\mathcal{F})$. We start by showing that $\mathcal{G}$ is t-intersecting. Let $A, B \in \mathcal{G}$. If $A, B \in \mathcal{F}'$ then clearly $|A \cap B| \geq t$, so suppose that $A \in \binom{[\ell+1]}{a} \times X_a(\mathcal{F}) \times \{n + 1\}$ and $\ell + 1 \in A$. Pick some $x \in [\ell]$ such that $x \notin A$, and notice that $A' = A \setminus \{\ell + 1, n + 1\} \cup \{x\} \in \mathcal{F}'$.

Suppose first that $B \in \mathcal{F}'$. If $\ell + 1 \in B$ or $x \notin B$ then $|A \cap B| \geq |A' \cap B| \geq t$, so suppose that $\ell + 1 \notin B$ and $x \in B$. If $B' = \mathbb{S}_{\ell+1 \leftarrow x}(B) \in \mathcal{F}'$ then $|A \cap B| = |A' \cap B'| \geq t$. Otherwise, $B \in X(\mathcal{F}')$. We distinguish between two cases. If $|B \cap [\ell]| \neq a$ then $|A' \cap [\ell]| + |B \cap [\ell]| \neq 2a = \ell + t$, and so Lemma 5.1 shows that $|A' \cap B| \geq t + 1$, which implies $|A \cap B| \geq |A' \cap B| - 1 \geq t$. If $|B \cap [\ell]| = a$ then necessarily $n + 1 \in B$, and so $B' = B \setminus \{n + 1\} \in \mathcal{F}'$. Therefore $|A \cap B| \geq |[(A' \setminus \{x\}) \cap B'] \cup \{n + 1\}| \geq |A' \cap B'| \geq t$.

Finally, suppose that $A, B \notin \mathcal{F}'$. Pick some $y \in [\ell]$ such that $y \notin B$, and notice that $B' = B \setminus \{\ell + 1, n + 1\} \cup \{y\} \in \mathcal{F}'$. We have $|A \cap B| = |[(A' \setminus \{x\}) \cap (B' \setminus \{y\})] \cup \{\ell + 1, n + 1\}| \geq |A' \cap B'| \geq t$. We conclude that $\mathcal{G}$ is t-intersecting.

It is straightforward to compute the μ_p-measure of $\mathcal{G}$:

$$\begin{aligned}\mu_p(\mathcal{G}) &= \mu_p(\mathcal{F}) - \binom{\ell}{a} p^a (1-p)^{\ell-a+1} \mu_p^{[n]\setminus[\ell+1]}(X_a(\mathcal{F})) + \binom{\ell+1}{a} p^{a+1} (1-p)^{\ell-a+1} \mu_p^{[n]\setminus[\ell+1]}(X_a(\mathcal{F})) \\ &= \mu_p(\mathcal{F}) + \left(-1 + \frac{\ell+1}{\ell-a+1} p\right) \binom{\ell}{a} p^a (1-p)^{\ell-a+1} \mu_p^{[n]\setminus[\ell+1]}(X_a(\mathcal{F})).\end{aligned}$$

Since $X_a(\mathcal{F}) \neq \emptyset$, in order to complete the proof we need to show that the expression inside the parentheses is positive. Since $\ell < t + 2r$ and $\ell + t$ is even, $\ell \leq t + 2r - 2$. Clearly $a \leq \ell$ and so $\ell - a + 1 > 0$, hence the parenthesized expression is positive if the following expression is:

$$\begin{aligned}
2[(\ell+1)p - (\ell - a + 1)] &= 2a - 2(1-p)(\ell+1) \\
&= t - 1 - (1-2p)(\ell+1) \\
&\geq t - 1 - (1-2p)(t + 2r - 1) \\
&= 2p(t + 2r - 1) - 2r > 0,
\end{aligned}$$

using in the third line the assumption $p < 1/2$. □

Combining Lemma 5.3 and Lemma 5.4, we obtain the following result.

Lemma 5.5. *Let $\mathcal{F}$ be a left-compressed t-intersecting family on n points with $\ell = \ell(\mathcal{F}) < t + 2r$ for some $r \geq 0$. If $t \geq 2$ and $r/(t + 2r - 1) < p < 1/2$ then there exists a t-intersecting family $\mathcal{G}$ on $n + 1$ points satisfying $\mu_p(\mathcal{G}) > \mu_p(\mathcal{F})$.*

Proof. By definition, $X(\mathcal{F}) \neq \emptyset$, and so $X_a(\mathcal{F}) \neq \emptyset$ for some a. If $a \neq (\ell + t)/2$ then the result follows from Lemma 5.3, otherwise it follows from Lemma 5.4. □

Combining this result with Corollary 4.7, we can prove the Ahlswede–Khachatrian theorem for left-compressed families.

Theorem 5.6. *Let $\mathcal{F}$ be a left-compressed t-intersecting family on n points for $t \geq 2$. If $r/(t + 2r - 1) < p < (r + 1)/(t + 2r + 1)$ for some $r \geq 0$ then $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r})$, with equality if and only if $\mathcal{F} = U^n(\mathcal{F}_{t,r})$.*

If $p = (r+1)/(t+2r+1)$ for some $r \geq 0$ then $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r}) = \mu_p(\mathcal{F}_{t,r+1})$, with equality if and only if either $\mathcal{F} = U^n(\mathcal{F}_{t,r})$ or $\mathcal{F} = U^n(\mathcal{F}_{t,r+1})$.

Proof. Suppose first that $r/(t + 2r - 1) < p < (r + 1)/(t + 2r + 1)$ for some $r \geq 0$. Corollary 4.7 gives a monotone left-compressed t-intersecting family $\mathcal{F}^*$ on $t + 2r$ points such that $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}^*)$, with equality only if $m(\mathcal{F}) \leq t + 2r$. Lemma 5.5 shows that $\ell(\mathcal{F}^*) = t + 2r$, and so $\mathcal{F}^*$ must be of the form

$$\mathcal{F}^*_s = \{A \subseteq [t + 2r] : |A| \geq s\}$$

for some s. This family is t-intersecting for $s \geq t + r$, and the optimal choice $s = t + r$ shows that $\mathcal{F}^* = \mathcal{F}^*_{t+r} = \mathcal{F}_{t,r}$. The corollary and the lemma together show that $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}^*)$ is only possible if $m(\mathcal{F}) = \ell(\mathcal{F}) = t + 2r$, and so $\mathcal{F} = U^n(\mathcal{F}^*_s)$ for some s. This readily implies that $\mathcal{F} = U^n(\mathcal{F}^*)$.

Suppose next that $p = (r + 1)/(t + 2r + 1)$ for some $r \geq 0$. Corollary 4.7 gives a monotone left-compressed t-intersecting family $\mathcal{F}^*$ on $t + 2r + 2$ points such that $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}^*)$, with equality only if $m(\mathcal{F}) \leq t + 2r + 2$. Since μ_p is continuous and there are finitely many families on $t + 2r + 2$ points, we see that $\mu_p(\mathcal{F}^*) = \mu_p(\mathcal{F}_{t,r}) = \mu_p(\mathcal{F}_{t,r+1})$. Corollary 4.7 and Lemma 5.5 show that

$\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}^*)$ is only possible if $m(\mathcal{F}) \leq t+2r+2$ and $\ell(\mathcal{F}) \geq t+2r$. Assume for simplicity that $n = t+2r+2$. The family $\mathcal{F}$ has the following general form:

$$\mathcal{F} = \mathcal{F}_a^* \cup \mathcal{F}_b^* \times \{t+2r+1\} \cup \mathcal{F}_c^* \times \{t+2r+2\} \cup \mathcal{F}_d^* \times \{t+2r+1, t+2r+2\}.$$

Some of these parts may be missing, in which case we use $\mathcal{F}_\infty^*$. Since $\mathcal{F}$ is t-intersecting, $d \geq t+r-1$. If $d = t+r-1$ then since $\mathcal{F}$ is t-intersecting, $a \geq t+r+1$ and $b, c \geq t+r$. Therefore $\mathcal{F} \subseteq \mathcal{F}_{t,r+1}$ and so $\mathcal{F} = \mathcal{F}_{t,r+1}$. Otherwise, $d \geq t+r$, and so monotonicity shows that $a, b, c \geq t+r$. Therefore $\mathcal{F} \subseteq U^n(\mathcal{F}_{t,r})$ and so $\mathcal{F} = U^n(\mathcal{F}_{t,r})$. $\square$

6 Culmination of the proof

Combined with Lemma 3.2, Theorem 5.6 already provides a tight upper bound on the μ_p-measure of arbitrary t-intersecting families. In order to complete the proof of the Ahlswede–Khachatrian theorem, it remains to prove uniqueness.

Recall that two families $\mathcal{F}, \mathcal{G}$ on n points are *equivalent* if they differ by a permutation of the coordinates. We start by showing that the families $\mathcal{F}_{t,r}$ are resilient to shifting in the case of t-intersecting families, using an argument from [1]. We need a preparatory lemma.

Lemma 6.1. *Let $t, r \geq 0$, and consider the following graph. The vertices are subsets of $[t+2r]$ of size $[t+r]$. Two subsets A, B are connected if $|A \cap B| = t$ (note that $|A \cap B| \geq t$). Then the graph is connected.*

Proof. If $r = 0$ then the graph contains a single vertex and there is nothing to prove, so suppose $r \geq 1$. We start by showing that $A = [t+r]$ and $B = [t+r]\Delta\{1, t+r+1\} = \{2, \ldots, t+r+1\}$ are connected. Let $C = [t] \cup \{t+r+1, \ldots, t+2r\}$. Then

$$|A \cap C| = |[t]| = t,$$
$$|B \cap C| = |\{2, \ldots, t\} \cup \{t+r+1\}| = t.$$

Hence A and B are connected via C. This shows that any two sets A, B with $|A \Delta B| = 2$ are connected, and so the graph is connected. $\square$

Now we can prove the desired result on shifting.

Lemma 6.2. *Let $\mathcal{F}$ be a t-intersecting family on n points, and suppose that for some $i, j \in [n]$, $\mathbb{S}_{i \leftarrow j}(\mathcal{F})$ is equivalent to $\mathcal{F}_{t,r}$. Then $\mathcal{F}$ is equivalent to $\mathcal{F}_{t,r}$.*

Proof. We can assume that $\mathbb{S}_{i \leftarrow j}(\mathcal{F}) = U^n(\mathcal{F}_{t,r})$. If $j \in [t+2r]$ then since $\mathbb{S}_{i \leftarrow j}(\mathcal{F})$ depends only on the first $t+2r$ coordinates, necessarily $i \in [t+2r]$ and so $\mathbb{S}_{i \leftarrow j}(\mathcal{F}) = \mathcal{F}$. Similarly, if $i \notin [t+2r]$ then necessarily $j \notin [t+2r]$ and again $\mathbb{S}_{i \leftarrow j}(\mathcal{F}) = \mathcal{F}$. In both cases the lemma trivially holds. So without loss of generality, suppose that $n = t+2r+1$, $i = t+2r$ and $j = t+2r+1$. The following two subfamilies are involved in the shift:

$$\mathcal{F}_1 = \{A \in \mathcal{F} : j \in A, i \notin A, A\Delta\{i,j\} \notin \mathcal{F}\},$$
$$\mathcal{F}_2 = \{A \in \mathcal{F} : i \in A, j \notin A, A\Delta\{i,j\} \notin \mathcal{F}\}.$$

We have

$$\mathbb{S}_{i\leftarrow j}(\mathcal{F}) = \mathcal{F} \setminus \mathcal{F}_1 \cup \{A\Delta\{i,j\} : A \in \mathcal{F}_1\}.$$

If $\mathcal{F}_1 = \emptyset$ then $\mathbb{S}_{i\leftarrow j}(\mathcal{F}) = \mathcal{F}$, and the lemma clearly holds. If $\mathcal{F}_2 = \emptyset$ then $\mathbb{S}_{i\leftarrow j}(\mathcal{F})$ results from $\mathcal{F}$ by switching the coordinates i and j, and again the lemma holds. It remains to consider the case $\mathcal{F}_1, \mathcal{F}_2 \neq \emptyset$. Consider the family

$$\mathcal{G} = \{A \subseteq [t + 2r - 1] : |A| = t + r - 1\}.$$

For every $A \in \mathcal{G}$, $A \cup \{i\} = A \cup \{t + 2r\} \in \mathcal{F}_{t,r}$, and so either $A \cup \{i\} \in \mathcal{F}_2$ or $A \cup \{j\} \in \mathcal{F}_1$ (but not both). Form a graph whose vertices are the sets in $\mathcal{G}$, and two sets A, B are connected if $|A \cap B| = t - 1$. Color a vertex A with 1 if $A \cup \{j\} \in \mathcal{F}_1$, and with 2 if $A \cup \{i\} \in \mathcal{F}_2$. Since $\mathcal{F}_1, \mathcal{F}_2 \neq \emptyset$, the coloring is not monochromatic. Lemma 6.1 shows that the graph is connected, and so there is some bichromatic edge (A, B), say $A' = A \cup \{j\} \in \mathcal{F}_1$ and $B' = B \cup \{i\} \in \mathcal{F}_2$. However, $|A' \cap B'| = |A \cap B| = t - 1$, contradicting the fact that $\mathcal{F}$ is t-intersecting. We conclude that either $\mathcal{F}_1 = \emptyset$ or $\mathcal{F}_2 = \emptyset$. $\square$

The Ahlswede–Khachatrian theorem is an easy corollary.

Theorem 1.3. *Let $\mathcal{F}$ be a t-intersecting family on n points for $t \geq 2$. If $r/(t + 2r - 1) < p < (r + 1)/(t + 2r + 1)$ for some $r \geq 0$ then $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r})$, with equality if and only if $\mathcal{F}$ is equivalent to $U^n(\mathcal{F}_{t,r})$.*

If $p = (r+1)/(t+2r+1)$ for some $r \geq 0$ then $\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{F}_{t,r}) = \mu_p(\mathcal{F}_{t,r+1})$, with equality if and only if $\mathcal{F}$ is equivalent to either $U^n(\mathcal{F}_{t,r})$ or $U^n(\mathcal{F}_{t,r+1})$.

Proof. Let $\mathcal{G}$ be the left-compressed family satisfying $\mu_p(\mathcal{G}) = \mu_p(\mathcal{F})$ given by Lemma 3.2. Theorem 5.6 implies the upper bounds. Together with Lemma 6.2, the theorem implies the cases of equality. $\square$

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