CSC263 Week 7

Thursday

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Announcement

Pre-test office hour today at BA5287

11am~1pm, 2pm~4pm

PS5 out, due next Tuesday

Recap: Amortized analysis

- We do amortized analysis when we are interested in the total complexity of a **sequence** of operations.
  - Unlike in average-case analysis where we are interested in a **single** operation.

- The ***amortized sequence complexity*** is the "**average**" cost per operation **over the sequence**.
  - But unlike average-case analysis, there is **NO** probability or expectation involved.

For a sequence of m operations:

Amortized sequence complexity

$$= \frac{\text{worst-case } \textbf{sequence} \text{ complexity}}{m}$$

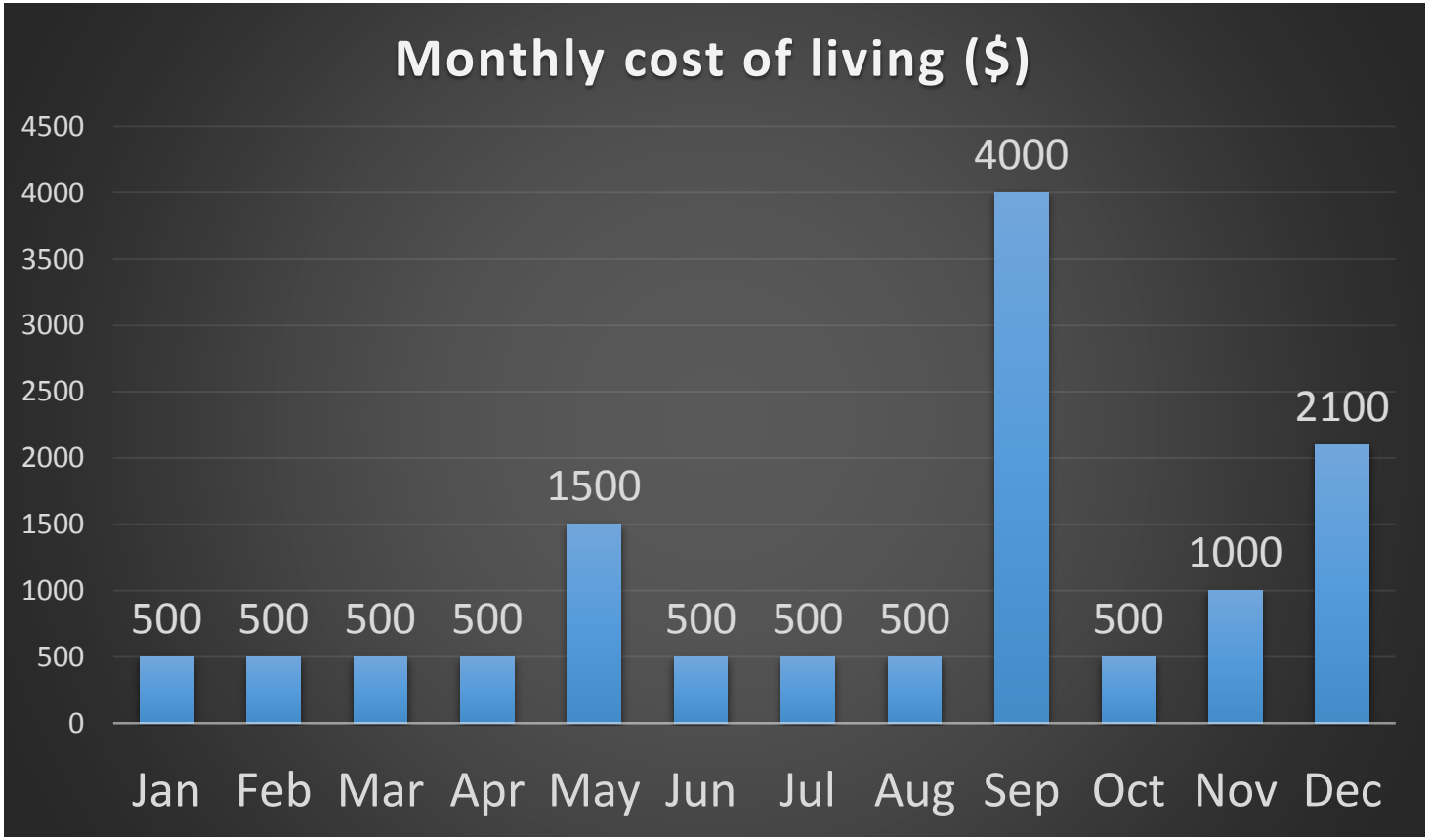
The MAXIMUM possible ***total*** cost of among all possible sequences of m operations

Methods for amortized analysis

- Aggregate method
- Accounting method
- Potential method (skipped, read Chapter 17 if interested)

Recap: Amortized analysis

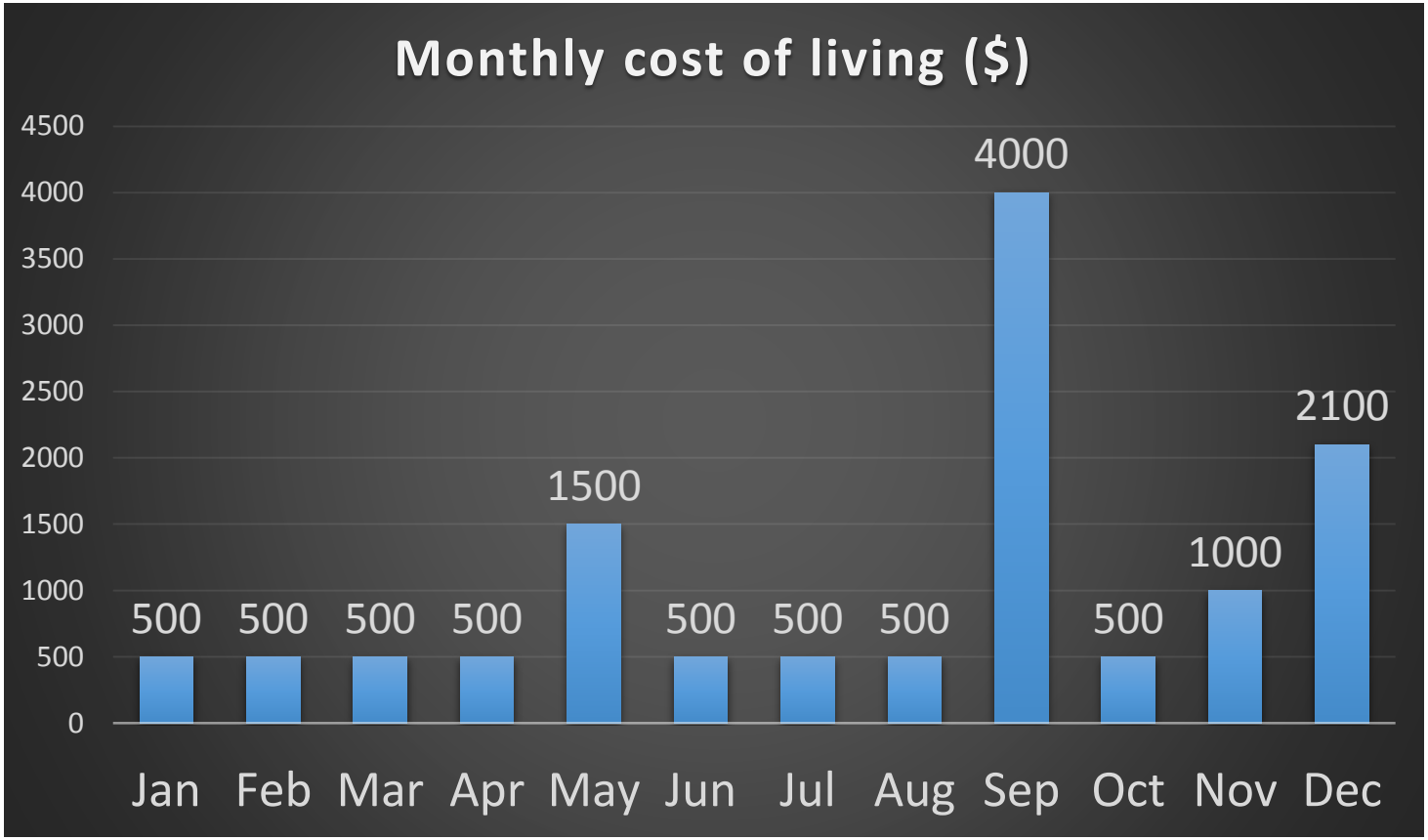
- Real-life intuition: Monthly cost of living, a sequence of 12 operations



Aggregate method

What is the amortized cost per month (operation)?

Just **sum up** the costs of all months (operations) and **divide** by the number of months (operations).



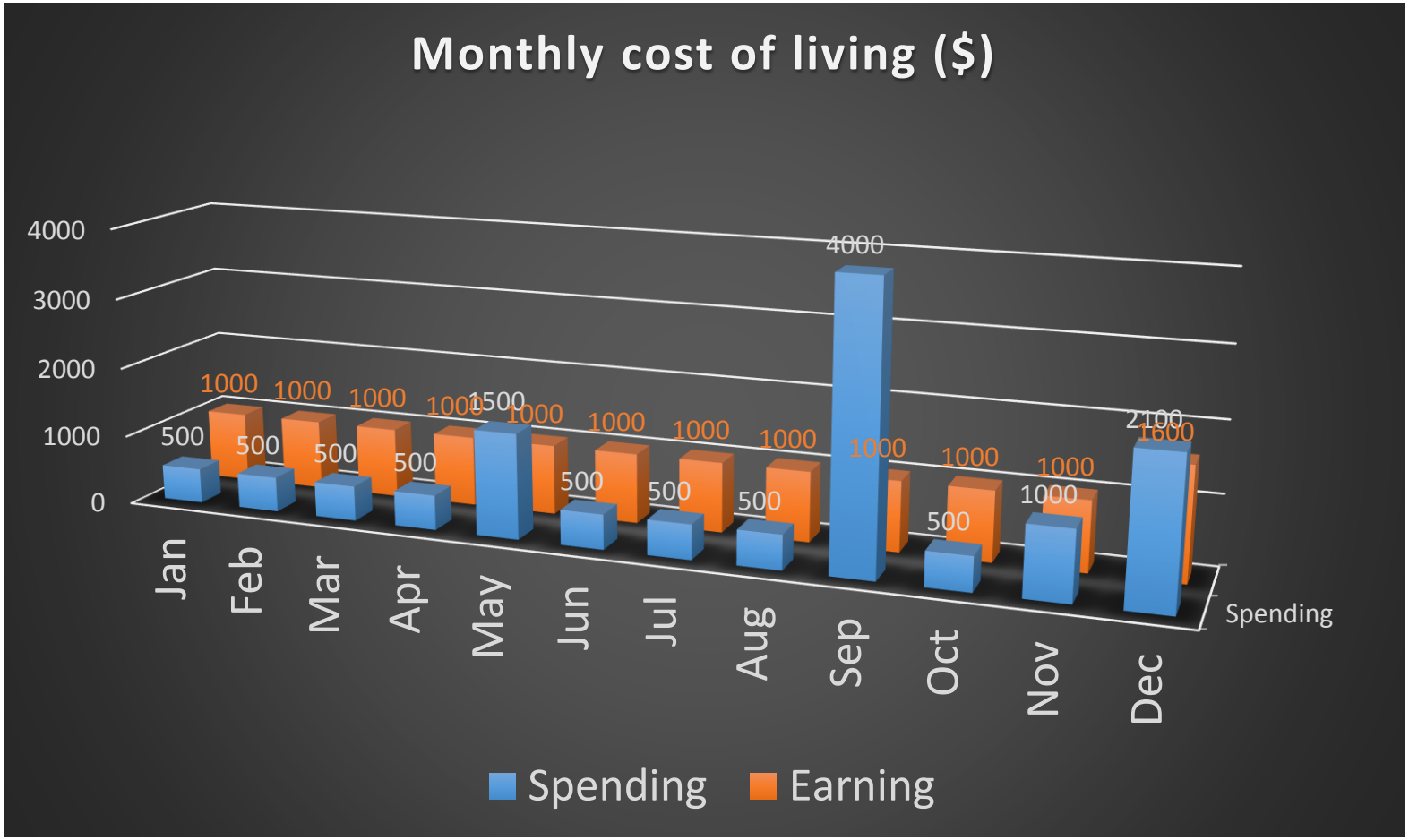
Aggregate method: sum of all months' spending is $126,00, divided by 12 months

– the amortized cost is $1,050 per month.

Accounting method

Instead of calculating the average spending, we think about the cost from a **different angle**, i.e.,

How much money do I need to **earn** each month in order to **keep living**? That is, be able to pay for the spending every month and **never become broke**.



Accounting method: if I **earn** $1,000 per month from Jan to Nov and earn $1,600 in December, I will never become broke (assuming earnings are paid at the beginning of month).

So the **amortized cost**: $1,000 from Jan to Nov and $1,600 in Dec.

Aggregate vs Accounting

- Aggregate method is easy to do when the cost of each operation in the sequence is concretely defined.

- Accounting method is more interesting
  - It works even when the sequence of operation is not concretely defined
  - It can obtain more refined amortized cost than aggregate method (different operations can have different amortized cost)

END OF RECAP

Amortized Analysis on **Dynamic Arrays**

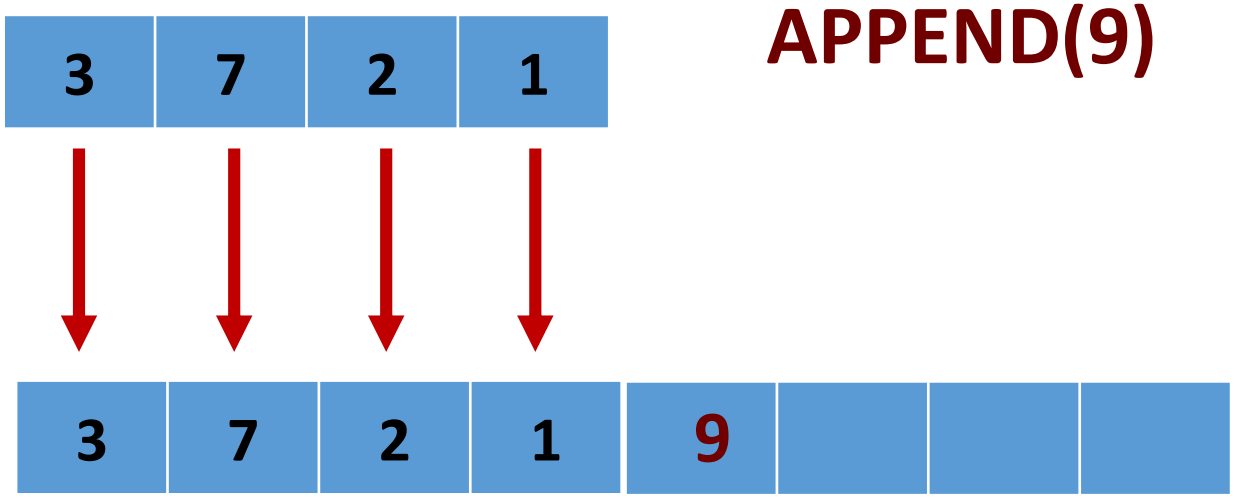
Problem description

- Think of an **array** initialized with a **fixed** number of slots, and supports **APPEND** and **DELETE** operations.
- When we APPEND too many elements, the array would be **full** and we need to **expand** the array (make the size larger).
- When we DELETE too many elements, we want to **shrink** to the array (make the size smaller).
- Requirement: the array must be using **one contiguous block** of memory all the time.

How do we do the **expanding** and **shrinking**?

One way to **expand**

- If the array is full when APPEND is called
  - Create a new array of **twice** the size
  - Copy the all the elements from old array to new array
  - Append the element



Amortized analysis of **expand**

Now consider a dynamic array initialized with size 1 and a sequence of ***m*** APPEND operations on it.

Analyze the amortized cost per operation

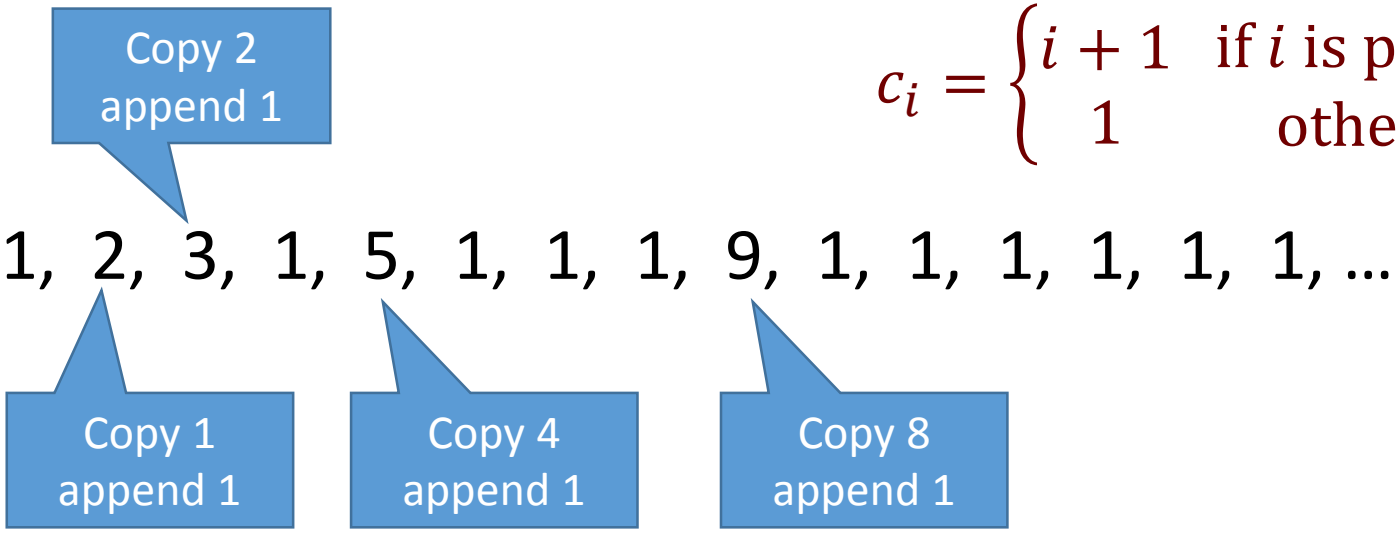
Assumption: only count array assignments, i.e.,
append *an element and* ***copy*** *an element*

Use the **aggregate** method

The cost sequence would be like:

Assume Index starts from 0

$$c_i = \begin{cases} i+1 & \text{if } i \text{ is power of 2} \\ 1 & \text{otherwise} \end{cases}$$



Cost sequence concretely defined, sum-and-divide can be done, but we want to do something more interesting…

Use the **accounting** method!

*How much money do we need to **earn** at each operation, so that all future costs can be paid for?*

*How much money to earn for **each APPEND'ed element** ?*

$1 ?

$2 ?

$3 ?

$log m ?

$m ?

Earn **$1** for each appended element

This $1 (the “append-dollar”) is spent when appending the element.

But, when we need to copy this element to a new array (when expanding the array), we don’t any money to pay for it --

BROKE!

Earn **$2** for each appended element

$1 (the “append-dollar”) will be spent when appending the element

$1 (the “copy-dollar”) will be spent when copying the element to a new array

What if the element is copied for a **second** time (when expanding the array for a second time)?

BROKE!

Earn $3 for each appended element

$1 (the “append-dollar”) will be spent when appending the element

$1 (the “copy-dollar”) will be spent when copying the element to a new array

$1 (the “recharge-dollar”) is used to **recharge** the old elements that have spent their “copy-dollars”.

NEVER BROKE!

$1 (the “recharge-dollar”) is used to **recharge** the old elements that have used their “copy-dollar”.

Old elements who have used their “copy-dollars”

New elements each of whom **spares** $1 for recharging one old element’s “copy-dollar”.

There will be enough new elements who will spare **enough money** for **all** the old elements, because the way we expand – **TWICE the size**

So, in summary

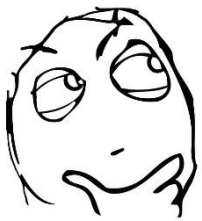
If we earn $3 upon each APPEND it is enough money to pay for all costs in the sequence of APPEND operations.

In other words, for a sequence of ***m*** APPEND operations, the amortized cost per operations is **3**, which is in O(1).

In a regular worst-case analysis (non-amortized), what is the worst-case runtime of an APPEND operation on an array with m elements?

By performing the amortized analysis, we showed that "**double the size when full**" is a good strategy for expanding a dynamic array, since it's amortized cost per operation is in O(1).

In contrast, "**increase size by 100 when full**" would not be a good strategy. **Why?**



Takeaway

Amortized analysis provides us valuable insights into what is the proper strategy of expanding dynamic arrays.

Shrinking dynamic arrays

A bit trickier…

First that comes to mind…

When the array is **½** full after DELETE, create a new array of half of the size, and copy all the elements.

Consider the following sequence of operations performed on a **full** array with ***n*** element…

APPEND, DELETE, APPEND, DELETE, APPEND, …

Θ(n) amortized cost per operation since every APPEND or DELETE causes allocation of new array.

NO GOOD!

The right way of shrinking

When the array is ¼ full after DELETE, create a new array of ½ of the size, and copy all the elements.

Earning $3 per APPEND and $3 per DELETE would be enough for paying all the cost.

- 1 append/delete-dollar
- 1 copy-dollar
- 1 recharge-dollar

The array, after shrinking…

Elements who just spent their copy-dollars

Array is half-empty

Before the **next expansion**, we need to **fill** the empty half, which will spare enough money for copying the green part.

Before the **next shrinking**, we need to **empty** half of the green part, which will spare enough money for copying what's left.

So, overall

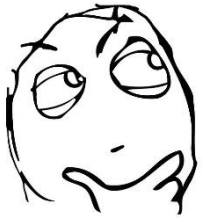
In a dynamic array, if we expand and shrink the array as discussed (double on full, halve on ¼ full)…

For any sequence of APPEND or DELETE operations, earning $3 per operation is enough money to pay for all costs in the sequence,…

Therefore the amortized cost per operation of any sequence is upper-bounded by 3, i.e., O(1).

Next week

Graphs!



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