CSC165 Week 9

Larry Zhang, November 4, 2014

today's outline

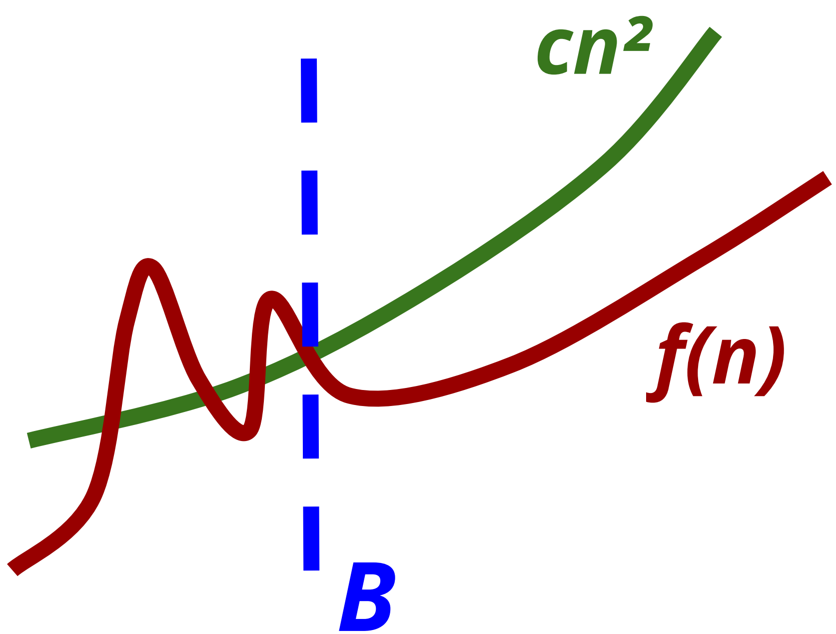
- ➔ exercises of big-Oh proofs
- ➔ prove big-Oh using limit techniques

Recap definition of $O(n^2)$

a function $f(n)$ is in $O(n^2)$ iff

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{such that } \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2$$

Beyond ***breakpoint B****,* ***f(n)*** *is upper-bounded by* ***cn²****, where* ***c*** *is some wisely chosen constant multiplier.*



Recap definition of O(n²)

functions that take in a **natural number** and return a **non-negative real number**

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

beyond breakpoint B, f(n) is upper-bounded by cn^2

set of all red functions which satisfy the green.

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Prove $3n^2 + 2n \in \mathcal{O}(n^2)$

thoughts: pick c and B

- **tip 1: c** should probably be larger than 3 (the constant factor of the highest-order term)
- **tip 2**: see what happens when n = 1
- if n = 1
  -

Proof $3n^2 + 2n \in \mathcal{O}(n^2)$

what if we add a constant?

Prove $3n^2 + 2n + 5 \in \mathcal{O}(n^2)$

thoughts: pick c and B

- **tip 1: c** should probably be larger than 3 (the constant factor of the highest-order term)
- **tip 2**: see what happens when n = 1
- if n = 1
  -

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Prove $7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$

thoughts:

- ➔ *assume $n \geq 1$*
- ➔ ***upper-bound** the left side by **overestimating***
- ➔ ***lower-bound** the right side by **underestimating***
- ➔ *choose a **c** that connects the two bounds*

Proof $7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$

how about disproving?

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Prove $n^3 \notin \mathcal{O}(3n^2)$

- ➔ first, negate it

$$\neg(\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow n^3 \leq c \cdot 3n^2)$$

- ➔ then, prove the negation

Prove $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^3 > c \cdot 3n^2$

thoughts

want to make $n^3 > c \cdot 3n^2$

Proof: $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^3 > c \cdot 3n^2$

so far all functions we talked about are polynomials

- ➔ between polynomials, it is fairly easy to figure out who is big-Oh of whom
- ➔ simply look at the highest-degree term
- ➔ ***f(n)*** is in ***O(g(n))*** exactly when the high-degree of ***f(n)*** is **no larger** than that of ***g(n)***

polynomials

$165n^{148} - 137n^{108} + 1130n^{11}$ is in $\mathcal{O}$ of...

$A.\quad 0.0001n^{149} + 7n^{77} - 3n^{33} + 6n$

$B.\quad 10000n^{147} + 99999n^{146} + 473736743$

$C.\quad n^{148} + 2$

how about non-polynomials?

$$2^n \ ? \ \mathcal{O}(n^2)$$

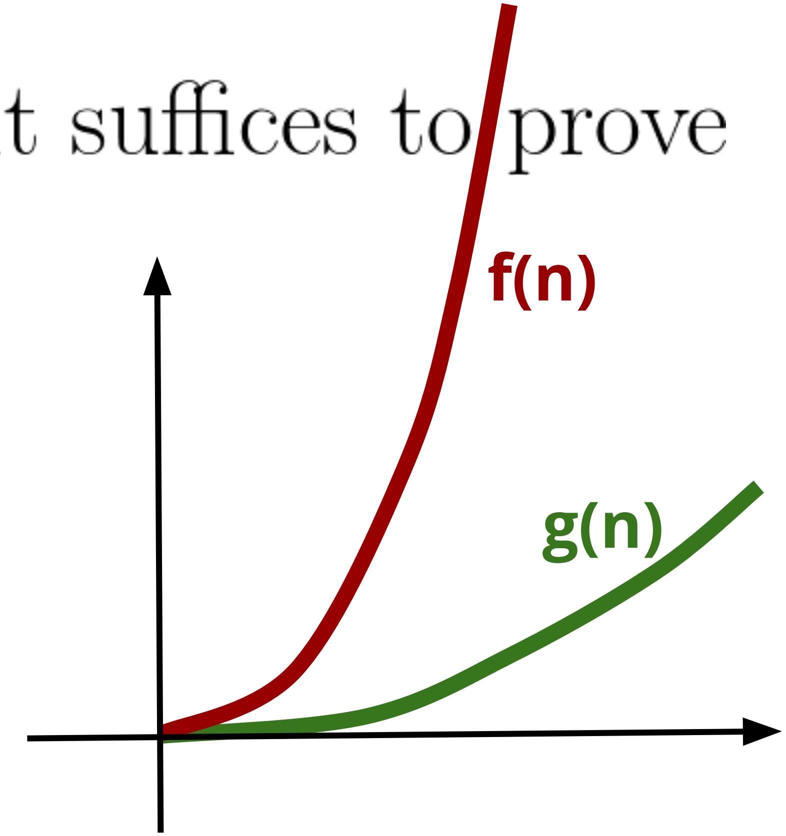
$$\text{probably} \quad 2^n \notin \mathcal{O}(n^2)$$

but how do we prove it?

➔

to prove $2^n \notin \mathcal{O}(n^2)$, it suffices to prove

$$\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$$



Intuition:

➔ *if the ratio* $\dfrac{f(n)}{g(n)}$ *approaches infinity when **n** becomes larger and larger, that means* ***f(n) grows faster*** *than* ***g(n)***

more precisely

$$\lim_{n \to \infty} \frac{2^n}{n^2} = \infty \quad \text{by definition means}$$

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

give me any big number

I can find a breakpoint

beyond which

the ratio is bigger than that big number

How to use limit to prove big-Oh

General steps:

1. prove $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$ using "some calculus"
2. translate the limit into its definition with c and n', i.e., $\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$
3. relate this definition to the definition of big-Oh, i.e., $2^n \notin \mathcal{O}(n^2)$ means

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > c \cdot n^2$$

Step 1. prove $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$ **using "some calculus"**

"some calculus": **L'Hopital's rule**

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

derivatives

l'hopital

$$\lim_{n\to\infty} \frac{2^n}{n^2}$$

$$= \infty$$

Step 2. translate the limit into its definition

we have proven $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$

then by definition of limit

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

give me any big number

I can find a breakpoint

beyond which

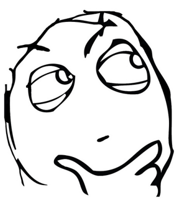
the ratio is bigger than that big number

Step 3. relate it to the definition of big-Oh

$$\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$$

what we have

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$



$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > c \cdot n^2$$

$$2^n \notin \mathcal{O}(n^2)$$

Proof: $2^n \notin \mathcal{O}(n^2)$