CSC165 Week 8

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today's outline

- ➔ formal definition of O, Ω
- ➔ proving algorithm complexity

- ➔ problem solving session

formal definitions of O and Ω

recap $O(n^2)$

set of functions that **grow no faster** than n^2

- ➔ count the number of steps
- ➔ constant factors don't matter
- ➔ only highest-order term matter

These functions are in $O(n^2)$

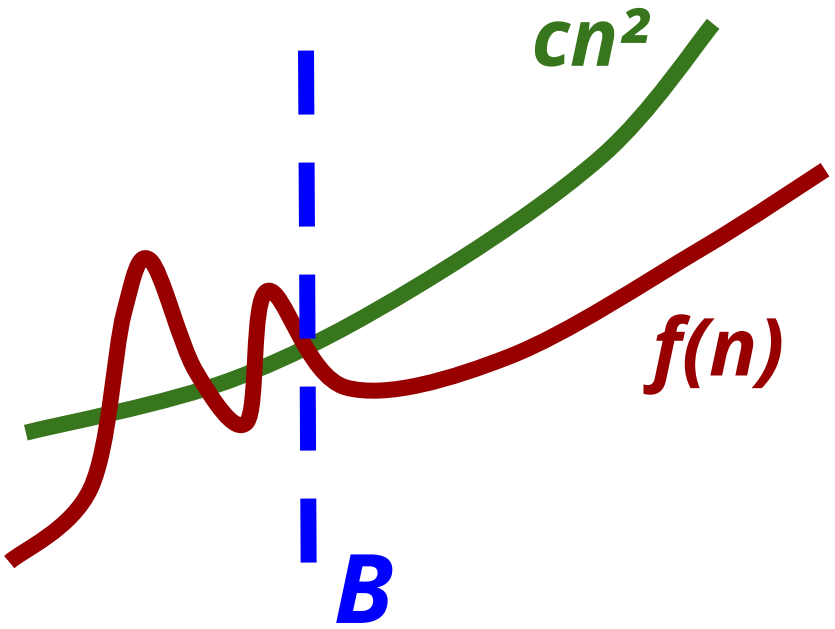
$$n^2 \qquad 2n^2 + 3n \qquad \frac{n^2}{165} + 1130n + 3.14159$$

the formal definition of O(n²)

a function $f(n)$ is in $O(n^2)$ iff

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{such that } \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2$$

Beyond ***breakpoint B****,* ***f(n)*** *is upper-bounded by* ***cn²****, where* ***c*** *is some wisely chosen constant multiplier.*



the formal definition of $O(n^2)$

a function $f(n)$ is in $O(n^2)$ iff

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{such that } \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2$$

Simple example: prove $700n^2 \in O(n^2)$

the formal definition of $\Omega(n^2)$

a function $f(n)$ is in $O(n^2)$ iff

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{such that } \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2$$

a function $f(n)$ is in $\Omega(n^2)$ iff

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{such that } \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq cn^2$$

$O(n^2)$: set of functions that **grow no faster** than **n^2**
$\Omega(n^2)$: set of functions that **grow no slower** than **n^2**
$\Theta(n^2)$: set of functions that are in **both $O(n^2)$ and $\Omega(n^2)$**
(functions growing **as fast as n^2**)

growth rate ranking of typical functions

$$f(n) = n^n$$

$$f(n) = 2^n$$

$$f(n) = n^3$$

$$f(n) = n^2$$

$$f(n) = n \log n$$

$$f(n) = n$$

$$f(n) = \sqrt{n}$$

$$f(n) = \log n$$

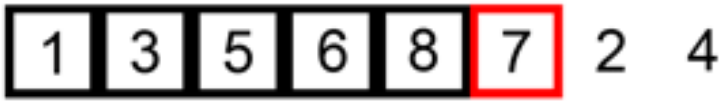
$$f(n) = 1$$

grow fast

grow slowly

analyse a sorting algorithm

insertion sort



- ➔ grow a sorted list inside an unsorted list
- ➔ in each iteration
  - ◆ remove an element from the unsorted part
  - ◆ insert it into the position it belongs to in the sorted part

see animation at: http://en.wikipedia.org/wiki/Insertion_sort

insertion sort

6 5 3 1 8 7 2 4

```
def IS(A):
'''sort the elements in A in
   non-decreasing order'''
```

n: size of A

```
i = 1
while i < len(A):
    t = A[i]   # take red square out
    j = i
    while j > 0 and A[j-1] > t:
        A[j] = A[j-1]  # shift
        j = j - 1
    A[j] = t    # put red square in
    i = i + 1   # next element to be red-squared
```

insert sort worst case runtime

$$W_{IS}(n) =$$

Prove the worst case complexity of insertion sort is O(n²)

$$W_{IS}(n) = \frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \mathcal{O}(n^2)$$

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow W_{IS}(n) \leq cn^2$$

Proof:

Prove $W_{IS}(n) = \frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Omega(n)$

$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow W_{IS}(n) \geq cn^2$

analyse another algorithm

maximum slice

- input: **L**, a list of numbers

- output: the maximum sum over slices of **L**

L = [-2, -3, <u>4, -1, 6</u>, -3]

max = 4 + (-1) + 6 = 9

```
def max_sum(L):
''' maximum sum over slices of L'''


max = 0
i = 0
while i < len(L):
    j = i + 1
    while j <= len(L):
        sum = 0
        k = i
        while k < j:
            sum = sum + L[k]
            k = k + 1
        if sum > max:
            max = sum
        j = j + 1
    i = i + 1
return max
```

```
def max_sum(L):
''' maximum sum over slices of L'''
```

upper bound $W_{MS}(n)$

```
1.  max = 0
2.  i = 0
3.  while i < len(L):
4.      j = i + 1
5.      while j <= len(L):
6.          sum = 0
7.          k = i
8.          while k < j:
9.              sum = sum + L[k]
10.             k = k + 1
11.         if sum > max:
12.             max = sum
13.         j = j + 1
14.     i = i + 1
15. return max
```

 # line 8-10, at most n iters

 # line 5-13, at most n iters

 # line 3-14, at most n iters

 # line 3-14, at most n iters

```
def max_sum(L):
''' maximum sum over slices of L'''

max = 0
i = 0
while i < len(L):
    j = i + 1
    while j <= len(L):
        sum = 0
        k = i
        while k < j:
            sum = sum + L[k]
            k = k + 1
        if sum > max:
            max = sum
        j = j + 1
    i = i + 1
return max
```

lower bound $W_{MS}(n)$

L:

 # first loop (i), at least $\left\lceil \frac{n}{3} \right\rceil$ iterations.
(pick i in red zone)

 # second loop (j), at least $\left\lceil \frac{n}{3} \right\rceil$ iterations
(pick j in blue zone)

 # third loop (k), at least $\left\lceil \frac{n}{3} \right\rceil$ iterations
(slice [i:j] is at least this long)

takeaway

- ➔ when finding **upper-bound**, it is OK to **over-estimate** the number of steps, as long as the over-estimated number is upper bounded.

- ➔ when finding **lower-bound**, it is OK to **under-estimate** the number of steps, as long as the under-estimated number is lower-bounded

problem solving: penny piles