CSC165

Larry Zhang, October 14, 2014

Today's outline

- ➔ proof about non-boolean functions
- ➔ proof something false
- ➔ proof about limits

non-boolean functions

non-boolean functions

A function whose return value is not True/False.

non-boolean functions

$\lfloor x \rfloor$ "floor of ***x***"

the largest integer that ≤ ***x***

$\lfloor 3.5 \rfloor =$

$\lfloor 3.999 \rfloor =$

$\lfloor 4 \rfloor =$

what do you feel about

$$\forall \lfloor x \rfloor \in \mathbb{R}, \ldots$$

now prove

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

thoughts

Proof: $\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$

the definition of “floor” again

in symbolic form

$$\forall x \in \mathbb{R}, y = \lfloor x \rfloor \quad \Leftrightarrow \ldots$$

➔ ***y*** is an integer

➔ ***y*** is less than or equal to ***x***

➔ among all integers that are ≤ ***x***, ...
y is the largest one

now prove

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

the last one was...

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

thoughts

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

definition of floor

$$\underbrace{\lfloor x \rfloor \in \mathbb{Z}}_{\text{P1}} \wedge \underbrace{\lfloor x \rfloor \leq x}_{\text{P2}} \wedge \underbrace{\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq \lfloor x \rfloor}_{\text{P3}}$$

Proof:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

takeaway

- ➔ fully **understand the definition** of the function, since that's all you have
- ➔ manipulate expressions into forms that are **closer to those in the definition**, since the definition is all you have and you will have to use it

prove something false

disprove something

How to **disprove** a statement **S** ?

Just prove **¬S**.

disprove: all cars are red.

$$S : \forall x \in C, R(x)$$

Define a sequence

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

Now disprove

$$S : \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, (j > i) \Rightarrow (a_j = a_i)$$

thoughts

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

n	0	1	2	3	4	5	6	7	8
a_n									

$$S : \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, (j > i) \Rightarrow (a_j = a_i)$$

*"there exists an index **i**, such that all values after **i** are equal to **a_i** "*

thoughts

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

n	0	1	2	3	4	5	6	7	8
a_n									

$$\neg S : \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, (j > i) \wedge (a_j \neq a_i)$$

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

Proof: $\neg S : \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, (j > i) \wedge (a_j \neq a_i)$

takeaway

To disprove a statement

- ➔ first make sure to negate it correctly
- ➔ then prove the negated statement
- ➔ draw examples to obtain intuition

proof about limits

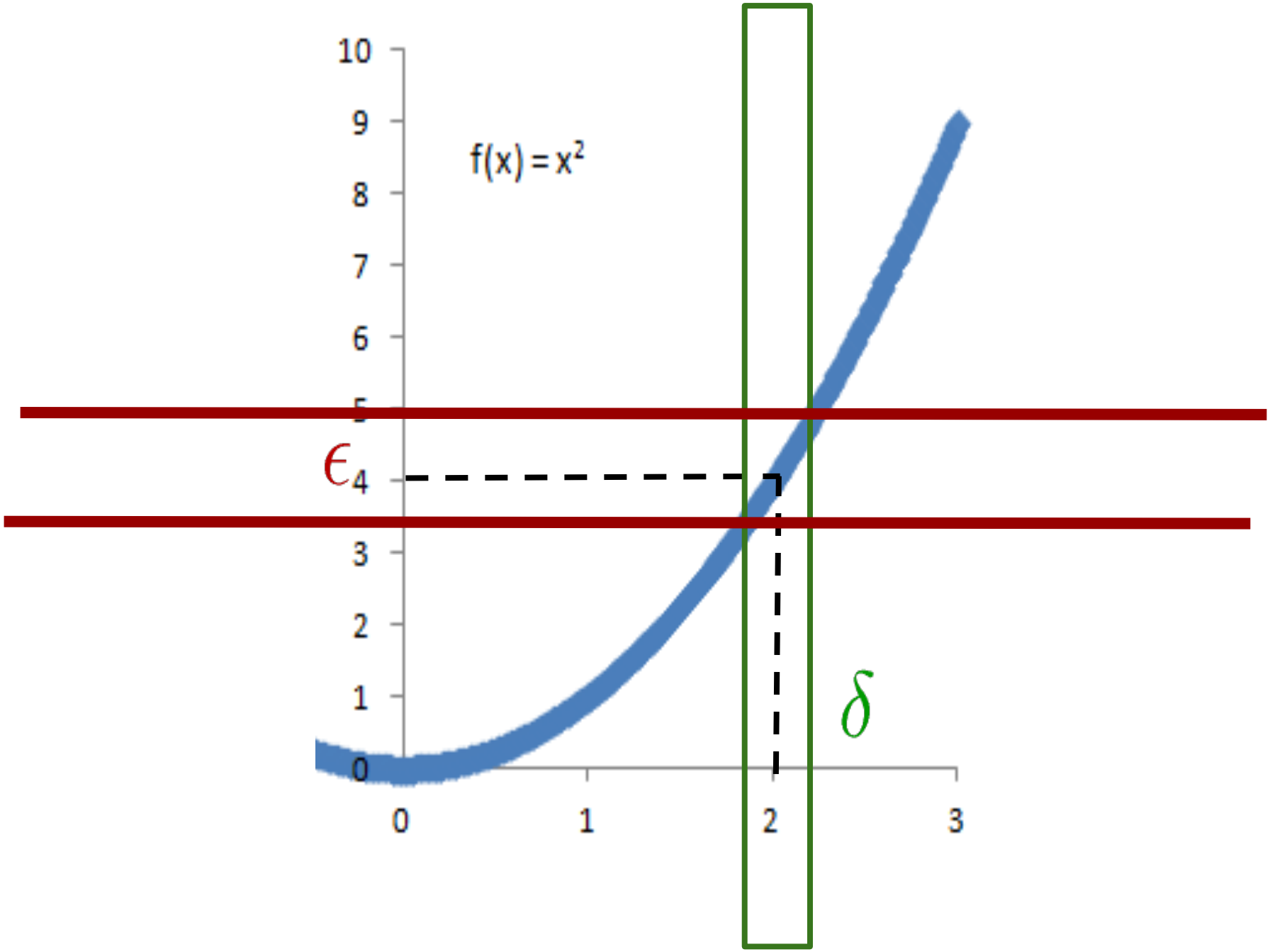
about limit

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, \underline{|x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon}$$

Intuition:

if *x* gets really really close to 2,
then *x²* gets really really close to 4.

$$\lim_{x \to 2} x^2 = 4$$


10
9
8
7
6
5
4
3
2
1
0
0
1
2
3
f(x) = x²
ϵ
δ

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$$

Proof: (only the structure for now)

assume $\epsilon > 0$ # a generic ε

pick $\delta = ???$ # some smart choice of δ

then $\delta > 0$ # satisfy the quantification

assume $x \in \mathbb{R}$ # a generic x

assume $|x-2| < \delta$ # antecedent of the =>

$\vdots$

then $|x^2 - 4| < \epsilon$

then $|x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$

then $\forall x \in \mathbb{R}, |x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$

then $\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x-2| < \delta \Rightarrow |x^2 - 4| <$

what needs to be done

find a δ such that $|x^2 - 4| < \epsilon$ (assuming $|x - 2| < \delta$)

let's try a backward search

$$|x^2 - 4| < \epsilon$$

$$|x - 2| \cdot |x + 2| \quad \textit{\# factorization}$$

$$< \delta \cdot |x + 2|$$

$$= \delta \cdot |(x - 2) + 4| \quad \textit{\# } |a+b| \leq |a| + |b|$$

$$\leq \delta \cdot (|x - 2| + 4)$$

$$= \delta \cdot (\delta + 4) \quad \textbf{\# make } \delta \leq 1$$

$$\leq \delta \cdot 5 \leq \epsilon$$

$$\delta \leq \frac{\epsilon}{5}$$

$$\boxed{\delta = \min\left(1, \frac{\epsilon}{5}\right)}$$

Keep in mind, we can make δ as small as we want, as long as it's > 0.

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$$

Proof: (fill in the content)

assume $\epsilon > 0$ # a generic ε

pick $\delta = \min\left(1, \frac{\epsilon}{5}\right)$ # smart choice of δ

then $\delta > 0$ # satisfy the quantification

assume $x \in \mathbb{R}$ # a generic x

assume $|x-2| < \delta$ # antecedent of the =>

$\vdots$

then $|x^2 - 4| < \epsilon$

then $|x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$

then $\forall x \in \mathbb{R}, |x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$

then $\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x-2| < \delta \Rightarrow |x^2 - 4| <$

takeaway

➔ smart choices are usually found by backward search (like the choice of δ)

next week

- ➔ finish up proofs
- ➔ (maybe) start talking about algorithm runtime