Today’s outline

➔ proof using contrapositive
➔ proof using contradiction
➔ proof for existence
➔ proof about a sequence
Lecture 5.1

contraposition, contradiction

Course Notes: Chapter 3
Last week

direct proof for universally quantified implication

\[ \forall x \in X, P(x) \implies Q(x) \]

as example, we proved

\[ \forall n \in \mathbb{N}, n \text{ is odd } \implies n^2 \text{ is odd} \]
∀n ∈ ℕ, n is odd ⇒ n² is odd

The proof

assume n ∈ ℕ            # n is a generic natural number

assume n is odd

then ∃j ∈ ℕ, n = 2j + 1  # definition of odd

then n² = (2j + 1)²      # square of both sides
        = 4j² + 4j + 1      # some algebra
        = 2(2j² + 2j) + 1  # some algebra

then ∃k = 2j² + 2j ∈ ℕ, n² = 2k + 1

then n² is odd            # definition of odd

then n is odd ⇒ n² is odd  # introduce =>

then ∀n ∈ ℕ, n is odd ⇒ n² is odd    # introduce ∀
Now, we want to prove...

$$\forall n \in \mathbb{N}, n^2 \text{ is even } \Rightarrow n \text{ is even}$$
The proof \( \forall n \in \mathbb{N}, n^2 \text{ is even} \Rightarrow n \text{ is even} \)

assume \( n \in \mathbb{N} \) \# n is a generic natural number

assume \( n \) is odd

then \( \exists j \in \mathbb{N}, n = 2j + 1 \) \# definition of odd

\[ \therefore \text{ same proof for } n \text{ odd } \Rightarrow n^2 \text{ odd} \]

then \( n^2 \) is odd \# definition of odd

then \( n \) is odd \( \Rightarrow n^2 \) is odd \# introduce \( \Rightarrow \)

then \( n^2 \) is even \( \Rightarrow n \) is even \# contrapositive

then \( \forall n \in \mathbb{N}, n^2 \text{ is even} \Rightarrow n \text{ is even} \) \# introduce \( \forall \)
Proof using contrapositive

Instead of proving $\mathbf{P} \Rightarrow \mathbf{Q}$,
prove $\neg \mathbf{Q} \Rightarrow \neg \mathbf{P}$ (equivalent to $\mathbf{P} \Rightarrow \mathbf{Q}$)

Chain of implication: $\neg \mathbf{Q} \Rightarrow \cdots \Rightarrow \neg \mathbf{P}$

When is this useful?

when the contrapositive is easier to prove than the original.
sometimes contrapositive is easier to prove

\[ \forall n \in \mathbb{N}, \, n^2 \text{ is odd } \Rightarrow \, n \text{ is odd} \]
Try the original direction $P \Rightarrow Q$

$n^2$ is odd

$\exists j \in \mathbb{N}, n^2 = 2j + 1$

$\exists j \in \mathbb{N}, n = \sqrt{2j + 1}$

$n$ is odd
\[ \forall n \in \mathbb{N}, n^2 \text{ is odd} \Rightarrow n \text{ is odd} \]

Try the contrapositive \( \neg Q \Rightarrow \neg P \)

\[ n \text{ is even} \]

\[ \exists j \in \mathbb{N}, n = 2j \]  \# definition of even

\[ \exists j \in \mathbb{N}, n^2 = 4j^2 \]  \# algebra

\[ \exists k \in \mathbb{N}, n^2 = 2k \]

\[ n^2 \text{ is even} \]  \# definition of even
Tip for finding proof

When it’s not easy to prove $P \implies Q$, try proving $\neg Q \implies \neg P$
contradiction

a special case of contrapositive
P => Q, with implicit P

sometimes it is not clear what P is, for example,

“There are infinitely many even natural numbers.”

\[ P_1 \land P_2 \land \cdots \land P_m \Rightarrow Q \]

“If everything that we believe in is true, then Q”

contrapositively ...

\[ \neg Q \Rightarrow \neg P_1 \lor \neg P_2 \lor \cdots \lor \neg P_m \]

“if not Q, then something we believe in is false”

if not Q, then it will contradict with something we believe in
“There are infinitely many even natural numbers.”

Proof:
There are 5 boxes in which there are in total 51 balls. **Prove** that there is a box with at least 11 balls in it.

*Proof:*
getting ready for a more challenging one...
Prime number

A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself.

Prime numbers: 2, 3, 5, 7, 11, ..., 997, ...

Not prime numbers: 0, 1, 4, 6, 8, ..., 1000, ...
There are infinitely many prime numbers.

\( P \) : set of all prime numbers

\( |P| \) : the size of \( P \)

\( S : \forall n \in \mathbb{N}, |P| > n \)

Prove \( S \)
Proof:

assume $\neg S : \exists n \in \mathbb{N}, |P| \leq n$ \hspace{1em} \# negation of $S$

then we have a finite list, $p_1, \ldots, p_k$ of elements in $P$

let $q = p_1 \times \cdots \times p_k$ \hspace{1em} \# $q$ is the product of all primes

then $q > 1$ \hspace{1em} \# $q$ is at least $2 \times 3 = 6$

then $q + 1 > 2$

then $\exists d \in P$ such that $d$ divides $q + 1$

then $d$ is one of $p_1, \ldots, p_k$ \hspace{1em} \# $p_1 \ldots p_k$ are all possible primes

then $d$ divides $q$ \hspace{1em} \# $q$ is the product of all primes

then $d$ divides $q + 1 - q$ \hspace{1em} \# $d$ divides both $q+1$ and $q$

then $d$ divides $1$

then $d = 1$ \hspace{1em} \# only 1 divides 1

then $1 \in P$ \hspace{1em} \# contradiction! 1 is not a prime number!

then $S$ \hspace{1em} \# assume $\neg S$ leads to contradiction
Tip for finding proof

When the assumption is implicit, try assuming \( \neg Q \), and see whether it leads somewhere, hunting for a contradiction.
Lecture 5.2 existential, sequence

Course Notes: Chapter 3
direct proof of the existential

$$\exists x \in X, P(x)$$

How to prove: find a single example.
∃x ∈ ℝ, x^3 + 3x^2 - 4x = 12

Proof:
prove a claim about a sequence
Define a sequence $a_n$ by:

$$\forall n \in \mathbb{N}, a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$