CSC165

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Today's outline

- ➔ proof using contrapositive
- ➔ proof using contradiction
- ➔ proof for existence
- ➔ proof about a sequence

Lecture 5.1
contrapositive, contradiction

Course Notes: Chapter 3

Last week

direct proof for universally quantified implication

$$\forall x \in X, P(x) \Rightarrow Q(x)$$

as example, we proved

$$\forall n \in \mathbb{N}, n \text{ is odd} \Rightarrow n^2 \text{ is odd}$$

$\forall n \in \mathbb{N}, n \text{ is odd} \Rightarrow n^2 \text{ is odd}$

The proof

write comments to explain each step

$\text{assume } n \in \mathbb{N}$ # n is a generic natural number

$\quad \text{assume } n \text{ is odd}$

$\quad\quad \text{then } \exists j \in \mathbb{N}, n = 2j + 1$ # definition of odd

$\quad\quad \text{then } n^2 = (2j+1)^2$ # square of both sides

$\quad\quad\quad = 4j^2 + 4j + 1$ # some algebra

$\quad\quad\quad = 2(2j^2 + 2j) + 1$ # some algebra

$\quad\quad \text{then } \exists k = 2j^2 + 2j \in \mathbb{N}, n^2 = 2k + 1$

$\quad\quad \text{then } n^2 \text{ is odd}$ # definition of odd

$\quad \text{then } n \text{ is odd} \Rightarrow n^2 \text{ is odd}$ # introduce =>

$\text{then } \forall n \in \mathbb{N}, n \text{ is odd} \Rightarrow n^2 \text{ is odd}$ # introduce $\forall$

Now, we want to prove...

$$\forall n \in \mathbb{N}, n^2 \text{ is even} \Rightarrow n \text{ is even}$$

The proof $\forall n \in \mathbb{N}, n^2 \text{ is even} \Rightarrow n \text{ is even}$

assume $n \in \mathbb{N}$ # n is a generic natural number

assume n is odd

then $\exists j \in \mathbb{N}, n = 2j + 1$ # definition of odd

$\vdots$ ***same proof for n odd => n² odd***

then n^2 is odd # definition of odd

then n is odd $\Rightarrow n^2$ is odd # introduce =>

then n^2 is even $\Rightarrow n$ is even **# contrapositive**

then $\forall n \in \mathbb{N}, n^2 \text{ is even} \Rightarrow n \text{ is even}$ # introduce $\forall$

Proof using contrapositive

Instead of proving **P => Q**,

prove **¬Q => ¬P** (equivalent to **P => Q**)

Chain of implication: $\neg Q \Rightarrow \cdots \Rightarrow \neg P$

When is this useful?

when the contrapositive is easier to prove than the original.

sometimes contrapositive is easier to prove

$$\forall n \in \mathbb{N}, \underbrace{n^2 \text{ is odd}}_{P} \Rightarrow \underbrace{n \text{ is odd}}_{Q}$$

Try the original direction P => Q

$$n^2 \text{ is odd}$$

$$\exists j \in \mathbb{N}, n^2 = 2j + 1$$

$$\exists j \in \mathbb{N}, n = \sqrt{2j + 1}$$

$$n \text{ is odd}$$

$$\forall n \in \mathbb{N}, n^2 \text{ is odd} \Rightarrow n \text{ is odd}$$

Try the contrapositive ¬Q => ¬P

$$n \text{ is even}$$

$$\exists j \in \mathbb{N}, n = 2j$$ # definition of even

$$\exists \in \mathbb{N}, n^2 = 4j^2$$ # algebra

$$\exists k \in \mathbb{N}, n^2 = 2k$$

$$n^2 \text{ is even}$$ # definition of even

Tip for finding proof

When it's not easy to prove P => Q,
try proving ¬Q => ¬P

contradiction

a special case of contrapositive

P => Q, with implicit P

sometimes it is not clear what **P** is, for example,

"There are infinitely many even natural numbers."

Q

$$P_1 \wedge P_2 \wedge \cdots \wedge P_m \Rightarrow Q$$

"If everything that we believe in is true, then Q"

contrapositively ...

De Morgan's

$$\neg Q \Rightarrow \neg P_1 \vee \neg P_2 \vee \cdots \vee \neg P_m$$

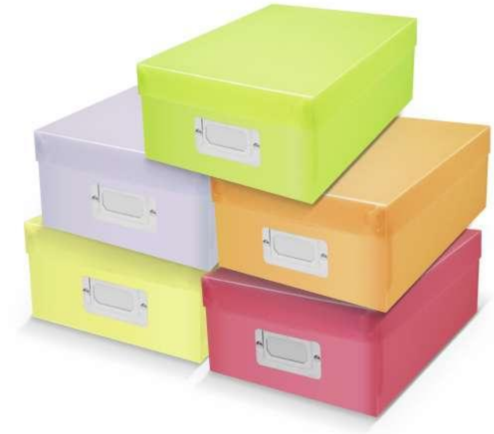
"if not Q, then something we believe in is false"

if not Q, then it will **contradict** with something we believe in

“There are infinitely many even natural numbers.”

Proof:

There are **5** boxes in which there are in total **51** balls. **Prove** that there is a box with at least **11** balls in it.



Proof:

getting ready for a more challenging one...

Prime number

A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself.

Prime numbers: 2, 3, 5, 7, 11, ..., 997, ...

Not prime numbers: 0, 1, 4, 6, 8, ..., 1000, ...

P : set of all prime numbers

$|P|$: the size of P

$S : \forall n \in \mathbb{N}, |P| > n$

Prove S

There are infinitely many prime numbers.

$$S : \forall n \in \mathbb{N}, |P| > n$$

Proof:

assume $\neg S: \ \exists n \in \mathbb{N}, |P| \leq n$ **# negation of S**

then we have a finite list, $p_1, \dots, p_k$ of elements in P

let $q = p_1 \times \cdots \times p_k$ **# q is the product of all primes**

then $q > 1$ **# q is at least 2 x 3 = 6**

then $q + 1 > 2$

then $\exists d \in P$ such that d divides $q + 1$ **# every integer > 2 has a prime divisor**

then d is one of $p_1, \dots, p_k$ **# p1...pk are all possible primes**

then d divides q **# q is the product of all primes**

then d divides $q + 1 - q$ **# d divides both q+1 and q**

then d divides 1

then $d = 1$ **# only 1 divides 1**

then $1 \in P$ **# contradiction! 1 is not a prime number!**

then S **# assume ¬S leads to contradiction**

Tip for finding proof

When the assumption is implicit,
try assuming $\neg Q$, and see whether it leads somewhere, hunting for a contradiction.

Lecture 5.2 existential, sequence

Course Notes: Chapter 3

direct proof of the existential

$$\exists x \in X, P(x)$$

How to prove: find a single example.

$$\exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12$$

Proof:

prove a claim about a sequence

Define a sequence a_n by:

$$\forall n \in \mathbb{N}, a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$