Announcements

➔ Assignment 1 due this Friday

➔ Term test 1 next Tuesday in class
   ◆ Time: 6:10pm -- 7pm
   ◆ Location: BA1130

➔ Old exam repository
   ◆ https://exams-library-utoronto-ca.myaccess.library.utoronto.ca/simple-search?query=csc165*&submit=%EF%BF%BD%EF%8F%A5%E9%8A%B5%EF%BF%BD
Today’s agenda

➔ Bi-implications
➔ Transitivity
➔ Mix quantifiers

➔ Proofs

➔ Problem solving session
Lecture 4.1
bi-implication, transitivity, mixed quantifiers

Course Notes: Chapter 2
Review

\( P \implies Q \) is equivalent to

A. \( \neg P \lor Q \)
B. \( P \lor \neg Q \)
C. \( P \land \neg Q \)
D. \( P \land Q \)
E. None of the above
Bi-implication

\[ P \iff Q \]

Translate this into the conjunction of two disjunctions.

\[ (? \lor ?) \land (? \lor ?) \]
Bi-implication

\[ P \iff Q \equiv (\neg P \lor Q) \land (\neg Q \lor P) \]

Translate this into the **disjunction** of two **conjunctions**.

\[ (? \land ?) \lor (? \land ?) \]
Negation of bi-implication
\[ \neg(P \iff Q) \]
transitivity
Transitivity

\[ \forall x \in X, (P(x) \Rightarrow Q(x)) \land (Q(x) \Rightarrow R(x)) \]

implies...
Transitivity

\[ ((P \Rightarrow Q) \land (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R) \]
mixed quantifiers
Different?

\[ \forall x \in X, \exists y \in Y, P(x, y) \]

\[ \exists y \in Y, \forall x \in X, P(x, y) \]
Different?

\[ \forall x \in X, \forall y \in Y, P(x, y) \]

\[ \forall y \in Y, \forall x \in X, P(x, y) \]
Different?

\[ \exists x \in X, \exists y \in Y, P(x, y) \]

\[ \exists y \in Y, \exists x \in X, P(x, y) \]
A more mathematical example

\[ \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - 2| < \delta \Rightarrow |x^2 - 4| < \varepsilon \]
Another mathematical example

\[ \exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > \delta \Rightarrow x^2 > \epsilon \]
Summary

**Language of math** (logical notations)

- quantifiers, statements, predicates
- implications, equivalence
- conjunctions, disjunctions, negation
- Venn diagrams, truth tables
- manipulating laws

It’s **not** about using **symbols**, it is about **understanding** and **expressing** in a **logical** way.
Lecture 4.2  Proofs

Course Notes: Chapter 3
Why proofs?

Proofs are important for science.

→ A mathematician / computer scientist believes nothing until it is proven.
What is a proof

A proof is an argument that convinces someone who is logical, careful and precise.

You first understand why something is true, then you use a proof to share your understanding with others, to save them time and effort.

no understanding => no proof
proof => understanding
How to prove (high level)

1. Find a proof
   - understand why you believe the thing is true
   - requires creativity and multiple attempts
   - lenient attitude: discover, investigate

2. Write up the proof
   - express why you believe the thing is true
   - requires carefulness and precision
   - skeptical attitude: poke holes in the argument
   - sometimes need to go back to Step 1
What we will learn in CSC165

Learn several different structures for proofs, so that you can have more ways to try when being creative to find the proof.

Be able to write up proofs in structured manners.

We learn structures.
direct proof of universally quantified implications
Find a proof for $\forall x \in X, P(x) \Rightarrow Q(x)$

Ideally we would like to have the following.

$$\forall x \in X, P(x) \Rightarrow R_1(x)$$

$$R_1(x) \Rightarrow R_2(x)$$

$$\ldots$$

$$R_{n-1}(x) \Rightarrow R_n(x)$$

$$R_n(x) \Rightarrow Q(x)$$

Key: finding the “chain”
Find the “chain”

\[ P \Rightarrow R_1 \Rightarrow R_2 \Rightarrow \cdots \Rightarrow R_{n-1} \Rightarrow R_n \Rightarrow Q \]
Find the “chain”

\[ P \Rightarrow R_1 \Rightarrow R_2 \Rightarrow \cdots \Rightarrow R_{n-1} \Rightarrow R_n \Rightarrow Q \]

Search can go both forwards and backwards
Chains with $\land$ and $\lor$

$$(P \Rightarrow R_1) \land (P \Rightarrow R_2)$$

$$(R_1 \Rightarrow Q) \land (R_2 \Rightarrow Q)$$
Write the proof \( \forall x \in X, P(x) \Rightarrow Q(x) \)

Assume \( x \in X \)

Assume \( P(x) \)

Then \( R_1(x) \)

Then \( R_2(x) \)

\[ \cdots \]

Then \( R_n(x) \)

Then \( Q(x) \)

Then \( P(x) \Rightarrow Q(x) \)

Then \( \forall x \in X, P(x) \Rightarrow Q(x) \)

Use indentation to present the scope of the assumption.
exercise time
Prove

\[ \forall n \in \mathbb{N}, \text{n is odd} \Rightarrow n^2 \text{ is odd} \]
Prove that for every pair of non-negative real numbers \((x, y)\), if \(x\) is greater than \(y\), then the geometric mean, \(\sqrt{xy}\) is less than the arithmetic mean, \((x + y)/2\).
Lecture 4.3  problem solving session