CSC165

Larry Zhang, September 30, 2014

Announcements

- **Assignment 1** due this Friday
- **Term test 1** next Tuesday in class
  - Time: 6:10pm -- 7pm
  - Location: BA1130
  - **Aid sheet: 8.5”x11”, both sides, handwritten**
- Old exam repository
  - https://exams-library-utoronto-ca.myaccess.library.utoronto.ca/simple-search?query=csc165*&submit=%EF%BF%BD%EF%8F%A5%E9%8A%B5%EF%BF%BD

Today’s agenda

- ➔ Bi-implications
- ➔ Transitivity
- ➔ Mix quantifiers

- ➔ Proofs

- ➔ Problem solving session

Lecture 4.1
bi-implication, transitivity, mixed quantifiers

Course Notes: Chapter 2

Review

P => Q is equivalent to

A. $\boxed{\neg P \lor Q}$

B. $P \lor \neg Q$

C. $P \land \neg Q$

D. $P \land Q$

E. None of the above

False only when P is true and Q is false

Review

$\neg(P \Rightarrow Q)$ is equivalent to

A. $\neg P \lor Q$

B. $P \lor \neg Q$

C. $\boxed{P \land \neg Q}$

D. $P \land Q$

E. None of the above

what a counter-example of P => Q needs to satisfy

Bi-implication

$$P \Leftrightarrow Q$$

Translate this into the conjunction of two disjunctions.

$(? \vee ?) \wedge (? \vee ?)$

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

$$(\neg P \vee Q) \wedge (\neg Q \vee P)$$

Bi-implication

$$P \Leftrightarrow Q \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$$

Translate this into the **disjunction** of two **conjunctions**.

$$(? \wedge ?) \vee (? \wedge ?)$$

$$(\neg P \vee Q) \wedge (\neg Q \vee P)$$

distributive

$$[(\neg P \vee Q) \wedge \neg Q] \vee [(\neg P \vee Q) \wedge P]$$

$$[(\neg P \wedge \neg Q) \vee (Q \wedge \neg Q)] \vee [(\neg P \wedge P) \vee (Q \wedge P)]$$

$$(\neg P \wedge \neg Q) \vee (Q \wedge P)$$

identity

Negation of bi-implication

$\neg(P \Leftrightarrow Q)$ turn into disjunction of two conjunctions

$\neg[(\neg P \vee Q) \wedge (\neg Q \vee P)]$

$\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P)$

$(P \wedge \neg Q) \vee (Q \wedge \neg P)$

De Morgan's

"Either P or Q is true, but not both true"

"P and Q must be different from each other"

"exclusive OR", "XOR" $P \oplus Q$

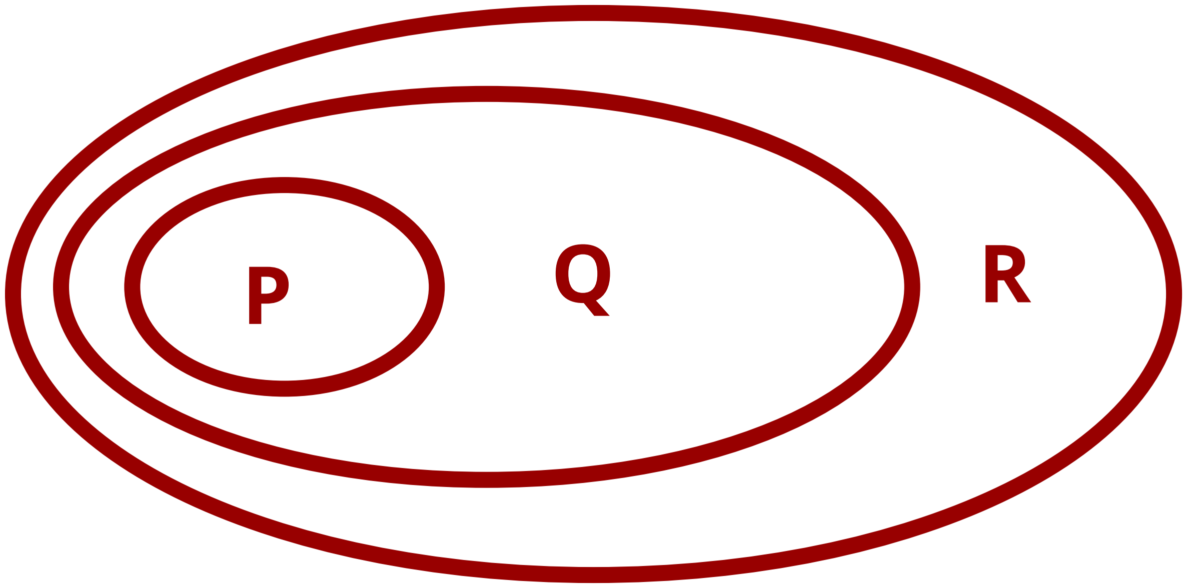
transitivity

Transitivity

$$\forall x \in X, (P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))$$

implies…

$$\forall x \in X, P(x) \Rightarrow R(x)$$



Transitivity

$$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$

NEG

$$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \wedge \neg(P \Rightarrow R)$$

$$((\neg P \vee Q) \wedge (\neg Q \vee R)) \wedge (P \wedge \neg R)$$

$$(\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \wedge P)$$

T T T T F T

Contradiction! So transitivity is always true!

mixed quantifiers

Different?

X: set of women, ***Y***: set of men
P(x, y): ***x*** and ***y*** are soul mates

$$\forall x \in X, \exists y \in Y, P(x, y)$$

For every woman, there is a man who is her soul mate.

$$\exists y \in Y, \forall x \in X, P(x, y)$$

There is a man, every woman is his soul mate.

Different?

$$\forall x \in X, \forall y \in Y, P(x, y)$$

$$\forall y \in Y, \forall x \in X, P(x, y)$$

Every man is every woman's soul mate.

Different?

$$\exists x \in X, \exists y \in Y, P(x, y)$$

$$\exists y \in Y, \exists x \in X, P(x, y)$$

There exist at least one pair of man and woman
who are soul mates of each other.

A more mathematical example

TRUE

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$$

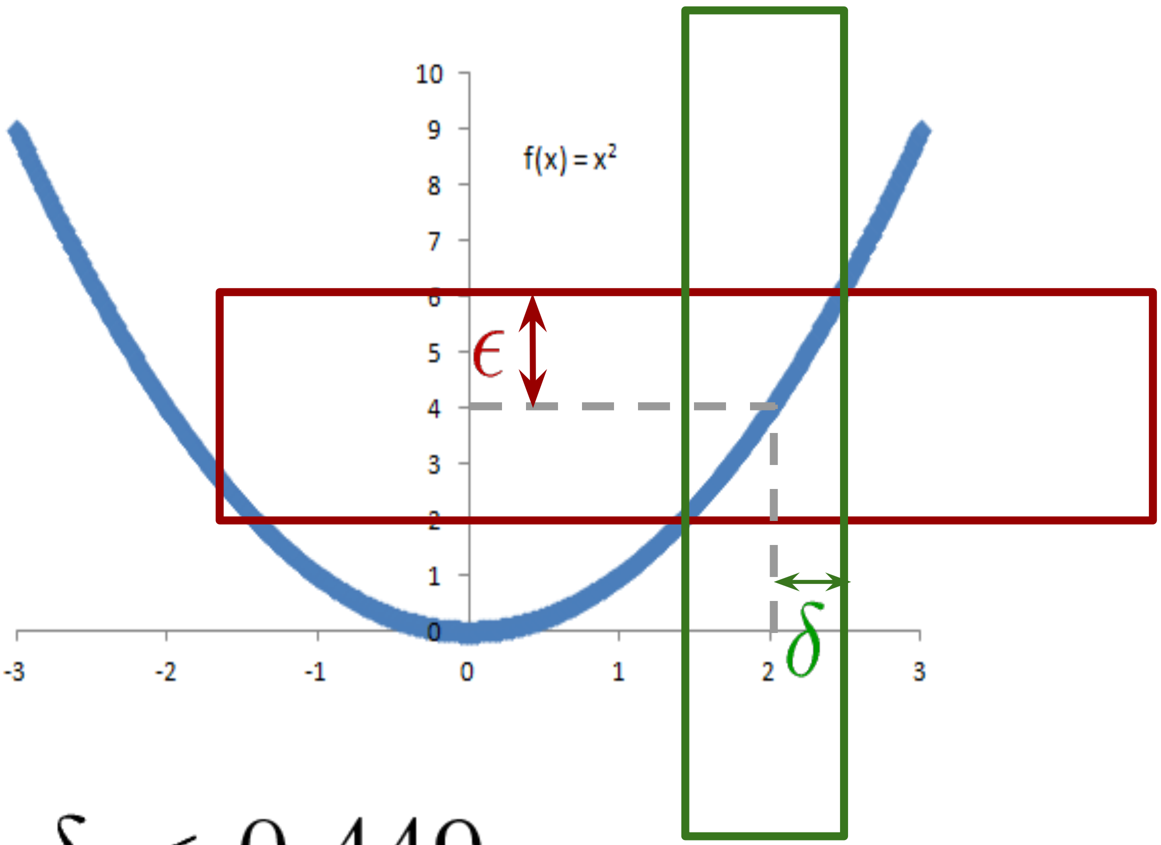
Let ε=2, how to choose δ ?

$$y \in (2, 6)$$

$$x \in (\sqrt{2}, \sqrt{6})$$

$$\left|2 - \sqrt{2}\right| = 0.586$$

$$\left|2 - \sqrt{6}\right| = 0.449$$



$$\text{Choose any } \delta < 0.449$$

Another mathematical example

$$\exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > \delta \Rightarrow x^2 > \epsilon$$

FALSE

Given any $\delta > 0$

pick $x = 2\delta > \delta$

we have $x^2 = 4\delta^2$

pick $\epsilon = 5\delta^2$

then $x^2 < \epsilon$

We can find counter-examples for all δ

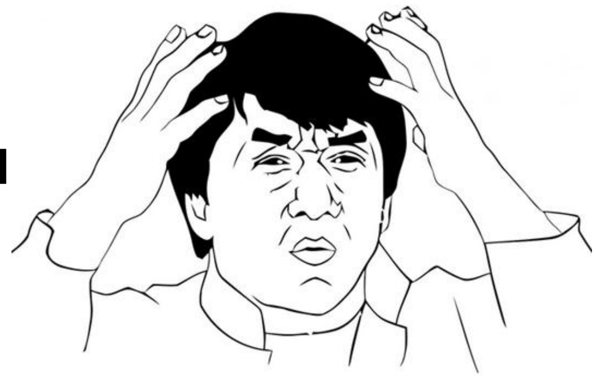
Summary

Language of math (logical notations)

- ➔ quantifiers, statements, predicates
- ➔ implications, equivalence
- ➔ conjunctions, disjunctions, negation
- ➔ Venn diagrams, truth tables
- ➔ manipulating laws

It's **not** about using **symbols**, it is about **understanding** and **expressing** in a **logical** way.

Back in Lecture 1.1, what made you



*Let **A**, **B**, and **C** be three statements.*

*The statement "**A** being true implying **B** being true implies **C** being true" is true if and only if either **A** is true and **B** is false or **C** is true.*

Is it true?

Prove that

$$(A \Rightarrow B) \Rightarrow C \;\Leftrightarrow\; (A \wedge \neg B) \vee C$$

Lecture 4.2 Proofs

Course Notes: Chapter 3

Why proofs?

Proofs are important for science.

➔ A mathematician / computer scientist believes nothing until it is proven.

➔ A physicist believes everything until it is proven wrong.

What is a proof

- A **proof** is an argument that convinces someone who is logical, careful and precise.
- You first **understand why** something is true, then you use a proof to **share your understanding** with others, to save them time and effort.
- **no understanding => no proof**
- **proof => understanding**

How to prove

1. **Find a proof**
   - ➔ **understand** why you believe the thing is true
   - ➔ requires **creativity** and multiple attempts
   - ➔ **lenient attitude**: discover, investigate
2. **Write up the proof**
   - ➔ **express** why you believe the thing is true
   - ➔ requires **carefulness** and **precision**
   - ➔ **skeptical** attitude: poke holes in the argument
   - ➔ sometimes need to go back to Step 1

What we will learn in CSC165

Learn several different **structures** for proofs, so that you can have ways to try when being creative to find the proof.

Be able to write up proofs in **structured** manners.

We learn **structures**.

direct proof of
universally quantified implications

Find a proof for $\forall x \in X, P(x) \Rightarrow Q(x)$

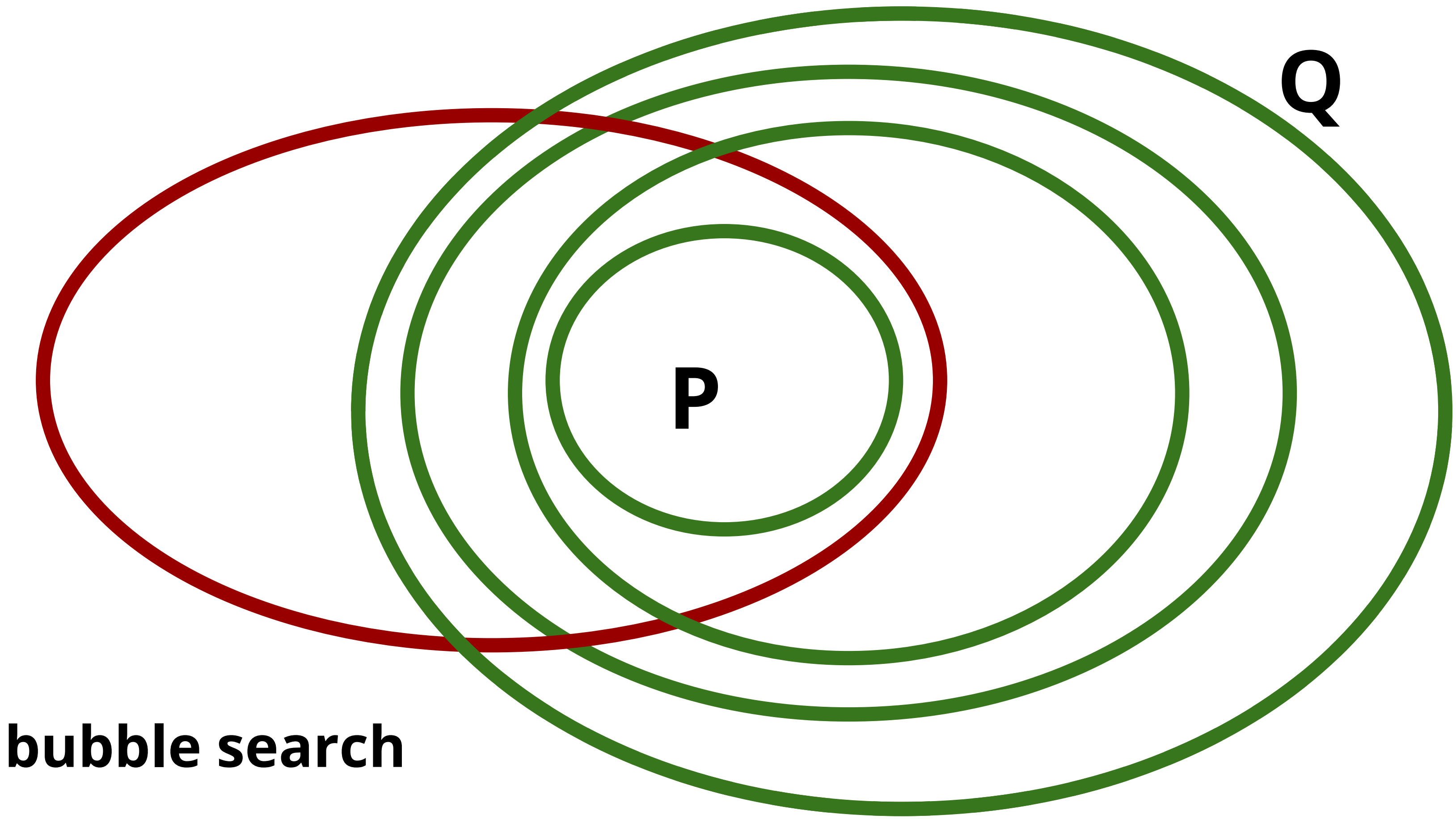
Ideally we would like to have the following.

$$
\begin{aligned}
\forall x \in X, P(x) &\Rightarrow R_1(x) \\
R_1(x) &\Rightarrow R_2(x) \\
&\cdots \\
R_{n-1}(x) &\Rightarrow R_n(x) \\
R_n(x) &\Rightarrow Q(x)
\end{aligned}
$$

Key: finding the "chain"

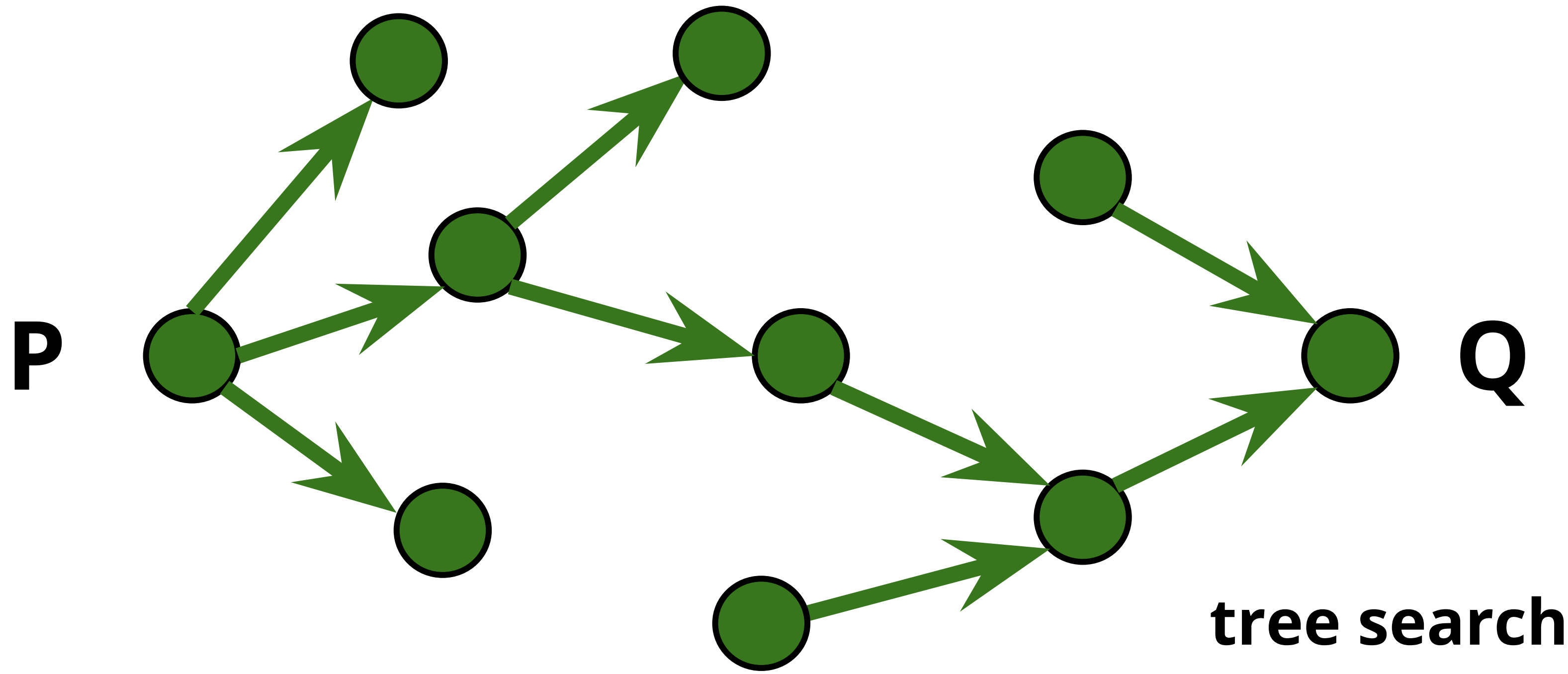
Find the “chain”

$$P \Rightarrow R_1 \Rightarrow R_2 \Rightarrow \cdots \Rightarrow R_{n-1} \Rightarrow R_n \Rightarrow Q$$



Find the "chain"

$$P \Rightarrow R_1 \Rightarrow R_2 \Rightarrow \cdots \Rightarrow R_{n-1} \Rightarrow R_n \Rightarrow Q$$



Search can go both forwards and backwards

Chains with $\wedge$ and $\vee$

$$(P \Rightarrow R_1) \wedge (P \Rightarrow R_2)$$
$$\Leftrightarrow (P \Rightarrow (R_1 \wedge R_2))$$

**If you work hard, you will get A+,
and if you work hard, you will be tired.**

$$(R_1 \Rightarrow Q) \wedge (R_2 \Rightarrow Q)$$
$$\Leftrightarrow (R_1 \vee R_2) \Rightarrow Q$$

**If you work hard, you will get A+,
and if you are a genius, you will get A+.**

Write the proof $\forall x \in X, P(x) \Rightarrow Q(x)$

Assume $x \in X$

  Assume $P(x)$

    Then $R_1(x)$

    Then $R_2(x)$

    …

    Then $R_n(x)$

    Then $Q(x)$

  Then $P(x) \Rightarrow Q(x)$

Then $\forall x \in X, P(x) \Rightarrow Q(x)$

Use indentation to present the scope of the assumption.

practice

Prove

$$\forall n \in \mathbb{N}, n \text{ is odd} \Rightarrow n^2 \text{ is odd}$$

NOTE: In computer science, natural numbers start from 0.

Find a proof

n is odd

$$\exists j \in \mathbb{N}, n = 2j + 1$$

$$\begin{aligned} n^2 &= (2j+1)^2 \\ &= 4j^2 + 4j + 1 \\ &= 2\underline{(2j^2 + 2j)} + 1 \end{aligned}$$

$$\exists k \in \mathbb{N}, n^2 = 2\underline{k} + 1$$

n^2 is odd

Write the proof

$$
\begin{array}{l}
\text{assume } n \in \mathbb{N} \quad \# \text{ an arbitrary } n \in \mathbb{N} \\
\qquad \text{assume } n \text{ is odd} \\
\qquad\qquad \text{then } \exists j \in \mathbb{N}, n = 2j + 1 \\
\qquad\qquad \text{then } n^2 = (2j+1)^2 \\
\qquad\qquad\qquad\quad = 4j^2 + 4j + 1 \\
\qquad\qquad\qquad\quad = 2(2j^2 + 2j) + 1 \\
\qquad\qquad \text{then } \exists k = 2j^2 + 2j \in \mathbb{N}, n^2 = 2k + 1 \\
\qquad\qquad \text{then } n^2 \text{ is odd} \\
\qquad \text{then } n \text{ is odd} \Rightarrow n^2 \text{ is odd} \\
\text{then } \forall n \in \mathbb{N}, n \text{ is odd} \Rightarrow n^2 \text{ is odd}
\end{array}
$$

Prove that for every pair of non-negative real numbers (x, y), if x is greather than y, then the geometric mean, $\sqrt{xy}$ is less than the arithmetic mean, $(x + y)/2$.

Find a proof

$$x \geq 0, y \geq 0, x > y$$

$$\frac{1}{2}(\sqrt{x} - \sqrt{y})^2 > 0$$

$$\frac{1}{2}(x + y - 2\sqrt{xy}) > 0$$

$$\frac{x+y}{2} - \sqrt{xy} > 0$$

$$\frac{x+y}{2} > \sqrt{xy}$$

Proofs found by searching backwards look clever!

Write the proof

assume $x, y \in \mathbb{R}, x \geq 0, y \geq 0, x > y$

$$\text{then } \frac{1}{2}(\sqrt{x} - \sqrt{y})^2 > 0$$

$$\text{then } \frac{x+y}{2} - \sqrt{xy} > 0$$

$$\text{then } \frac{x+y}{2} > \sqrt{xy}$$

then $\forall x, y \in \mathbb{R}, x \geq 0, y \geq 0, x > y, \dfrac{x+y}{2} > \sqrt{xy}$

a real-life proof

*The judge tells a condemned prisoner that he will be hanged at noon on one weekday (Monday~Friday) in the following week, and the execution will be a **surprise** to the prisoner, i.e., the prisoner will not know the day of the hanging until the executioner knocks on his cell door.*

***Prove** that the prisoner will **not** be hanged.*

Proof

It cannot be on Friday, since after Thursday noon it would not be a surprise anymore.

Assuming it is not Friday, then it cannot be on Thursday, since after Wednesday noon, it would not be a surprise anymore.

Assuming it is not Friday or Thursday, it cannot be on Wednesday. For the same reason, it cannot be Tuesday or Monday either.

Therefore, the hanging will **not** happen.

The prisoner thanks you and joyfully goes back to his cell being confident that the hanging will not happen.

The next Monday at noon, the executioner knocks on the prisoner's door and hangs him.

*It is a **surprise**.*

"unexpected hanging paradox"

Summary

- Why proofs and what is a proof
- How to prove
  - find a proof, **search forwards and backwards**
  - write up a proof, be precise.
- We learn **structures**
  - today: direct proof of universally quantified.
  - will learn more ...

Lecture 4.3 problem solving session

Do **NOT** turn to back of the sheet, which contains severe spoiler.

Why

➔ We have been trained to solve problems like

$$\text{Given } x^2 - 5x + 3 = 0, \text{find } x.$$

➔ Real-life problems aren't that well defined, and the methods used for solving them aren't that clear.

➔ We would like to learn the skills for attacking real-life, challenging, open problems.

➔ Solving good problems is fun!

http://www.cdf.toronto.edu/~heap/165/F14/folding.pdf

What to do

- Follow Polya's problem solving scheme
  - understand the problem
  - devise a plan
  - carry out the plan
  - look back
  - acknowledge when, and how, you're stuck
- Don't need to have a solution in class, the process is more important.
- Can keep working on it after class in the **Problem Solving Wiki** (see link in handout)
- Can write a SLOG about it.